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On Totally Geodesic Affine Immersion in Locally Product Riemannian Manifolds

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Abstract : In this paper the totally geodesic affine immersions $f : (M, \nabla) \to (\overline{M}, \overline{\nabla})$ are studied in the case when $(\overline{M}, \overline{\nabla})$ is an affine locally product manifold of recurrent curvature. It is proved that (M, ∇) is flat or of recurrent curvature.

1. Preliminaries

Let (M, ∇) and $(\overline{M}, \overline{\nabla})$ be connected differentiable manifolds with torsion free affine connection ∇ and $\overline{\nabla}$ with a Riemannian metric g and $\overline{g}$ respectively. Then Gauss and Wiengarten formulae given by

$$\text{(1.1)} \qquad \text{(a)} \quad \overline{\nabla}_X Y = \nabla_X Y + B(X, Y), \qquad \text{(b)} \; \overline{\nabla}_X V = -A_V X + D_X V$$

for all $x, y \in TM$ and $V \in T^1M$, where $\overline{\nabla}$, ∇ and D are respectively the Riemannian, induced Riemannian and induced connections in $\overline{M}$, M and the normal bundle of T^1M of M respectively. B is the second fundamental form related to A by $g(B(X, Y), U) = g(A_U X, Y)$.

The submanifold M of $\overline{M}$ is known to be

(i) totally geodesic in $\overline{M}$ if $B = 0$.

(ii) minimal if μ = Trace (B) / Dim $(M) = 0$, and

(iii) totally umblical if $B(X, Y) = g(X, Y)\mu$, $X, Y \in TM$.

Fundamental Gauss and Codazzi equations for the affine immersion can be written as follows :

$$\text{(1.2)} \qquad \overline{R}(X, Y)Z = R(X, Y)Z + A_{B(X,Z)}Y - A_{B(Y,Z)}X + \left(\nabla_X B\right)(Y, Z) - \left(\nabla_Y B\right)(X, Z),$$

$$(1.3)\qquad \bar{R}(X,Y)V = \left(\nabla_Y A\right)_V X - \left(\nabla_X A\right)_V Y + B\left(A_V X, Y\right) - B\left(X, A_V Y\right) + R^1(X,Y)V$$

for vector fields X, Y and Z tangent to M. Taking the normal component of (1.1a) we obtain the equation of Codazzi as

$$(1.4)\qquad (\bar{R}(X,Y)Z)^1 = \left(\bar{\nabla}_X B\right)(Y,Z) - \left(\bar{\nabla}_Y B\right)(X,Z).$$

For a submanifold M of a locally product Riemannian manifold $\bar{M}$ we put

$$FX = tX + fX \quad \text{and} \quad FV = hV + sV$$

where tX is the tangential part of FX and fX the normal part of FX. Then t is an endomorphism of the tangent bundle TM and f is a normal bundle value 1-form on the tangent bundle. In this case

$$(1.5)\qquad t^2X = X - hfX,\ ftX + sfX = 0,$$

$$(1.6)\qquad s^2V = V - fhV,\ thV + hsV = 0.$$

The covariant derivatives $\nabla_X B$ and $\nabla_X A$ are defined by

$$(1.7)\qquad \nabla_X B(Y,Z) = D_X(B(Y,Z)) - B\left(\nabla_X Y, Z\right) - B\left(Y, \nabla_X Z\right)$$

$$(1.8)\qquad \left(\nabla_X A\right)_V Y = \nabla_X A_V Y - A_V \nabla_X Y - A_{D_X V} Y.$$

2. Riemannian Product Immersion

Let $\bar{M}^m$ and $\bar{M}^n$ be Riemannian manifolds of dimension m and n respectively. We consider the product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$ of dimension $m+n$, then $\bar{M}$ admits the product structure tensor field F such that $F^2 = I$, where I the identity tensor and $g(FX, Y) = g(X, FY)$ for any vector field X and Y on $\bar{M}$.

Let M be a k-dimensional submanifold of $\bar{M}$. If $F\,T_x(M) \subset T_x(M)$ for each point x of M, then M is said to be an F-invariant in $\bar{M}$. Let $\bar{M}$ be a locally decomposable Riemannian manifold, *i.e.* $\bar{\nabla}_X F = 0$. If M is an F-invariant submanifold of a locally decomposable Riemannian manifold $\bar{M}$, then $(\nabla_X F)V = 0$ and $sB(X,Y) = B(X, fY)$. Then we have

Theorem 2.1 : *Let M be an F-invariant submanifold of a Riemannian product manifold $\overline{M} = \overline{M}^m \times \overline{M}^n$. Then M is a Riemannian product manifold $M^p \times M^q$ where M^p is a submanifold of $\overline{M}^m$ and M^q is a submanifold of $\overline{M}^n$, M^p and M^q being both totally geodesic in $\overline{M}$.*

We denote by the same F the almost product structure on M, we now define the curvature tensor R^1 of the normal bundle of M by

$$(2.1) \qquad R^1(X, Y) = D_X D_Y - D_Y D_X - D_{[X, Y]}.$$

If $R^1 = 0$, the normal connexion of M is said to be flat. It is well known that $R^1 = 0$ if and only if we can choose an orthonormal frame $\{V_a\}$ of the normal bundle TM^1 such that $D_{V_a} = 0$ for all a.

Lemma 2.2 : *Let M be an F invariant submanifold of a locally Riemannian product manifold $\overline{M} = \overline{M}^m \times \overline{M}^n$. If the normal connection of M is flat, then the normal connection of M^p* in *$\overline{M}^m$ and that of M^q* in *$\overline{M}^n$ are both flat, where $M = M^p \times M^q$.*

Proof : Let V be a vector field in $TM^{p\,1}$ in $\overline{M}^m$. We can suppose that

$$T_X\left(\overline{M}^m\right) = \left\{X \in T_X(\overline{M}) : FX = X\right\}.$$

For any vector field X tangent to M, we have

$$F D_X V = F \overline{\nabla}_X V + F A_V X = \overline{\nabla}_X F V + F A_V X$$

$$= - A_{FV} X + D_X FV + FA_V X = D_X V$$

becuase $FV = V$. Therefore, if $V \in TM^{p\,1}$, then $D_X V = TM^{p\,1}$ which means that $TM^{p\,1}$ is parallel. From this we see that the normal connection of M^p in $\overline{M}^m$ is flat. Similarly, we can see that the normal connection of M^q in $\overline{M}^n$ is also flat. We assume that $\overline{M}^m$ and $\overline{M}^n$ are complex space forms with constant sectional curvature c_1 and c_2 and denote them by $\overline{M}^m(c_1)$ and $\overline{M}^n(c_2)$ respectively. Let M be an F-invariant submanifold of $\overline{M} = \overline{M}^m(c_1) \times \overline{M}^n(c_2)$. We denote by R the Riemannian curvature tensor of M. Then the Gauss equation of M is given by

$$(2.2) \qquad R(X, Y)Z = \frac{1}{16}\left(c_1 + c_2\right)[g(Y, Z)X - g(X, Z)Y + g(tY, Z)tX$$

$$- g(tX, Z)tY + 2g(X, tY)tZ + g(FY, Z)FX$$

$$- g(FX, Z)FY + g(FtY, Z)FtX - g(FtX, Z)FtY$$

$$+ 2g(FX, tY)FtZ] + \frac{1}{16}\left(c_1 - c_2\right)[g(FY, Z)X$$

$$- g(FX, Z)Y + g(Y, Z)FX - g(X, Z)FY + g(FtX, Z)tX$$

$$- g(FtY, Z)tY + g(tY, Z)FtX - g(tX, Z)FtY + 2g(FX, tY)tZ$$

$$+ 2g(X, tY)tfZ] + A_B(Y, Z)^{X-A}B(X, Z)^{Y}.$$

and the Codazzi equation by

$$\left(\nabla_X B\right)(Y, Z) - \left(\nabla_Y B\right)(X, Z) \tag{2.3}$$

$$= \frac{1}{16}\left(c_1 + c_2\right)[g(tY, Z)fX - g(tX, Z)fY + 2g(X, tY)fZ$$

$$+ g(FtY, Z)FfX - g(FtX, Z)FfY + 2g(FX, tY)FfZ]$$

$$+ \frac{1}{16}\left(c_1 - c_2\right)[g(FtY, Z)fX - g(FtX, Z)fY + g(tY, Z)FfX$$

$$- g(tX, Z)FfY + 2g(FX, tY)fZ + 2g(X, tY)fFZ].$$

3. Totally Geodesic Immersion

Since for a totally geodesic immersion $\overline{\nabla}_X Y = \nabla_X Y$, the Gauss equation becomes

$$R(X, Y)Z = \frac{1}{16}\left(c_1 + c_2\right)[g(Y, Z)X - g(X, Z)Y \tag{3.1}$$

$$+ g(tY, Z)tX - g(tX, Z)tY + 2g(X, tY)tZ + g(FY, Z)FX$$

$$- g(FX, Z)FY + g(FtY, Z)FtX - g(FtX, Z)FtY$$

$$+ 2g(FX, tY)FtZ] + \frac{1}{16}\left(c_1 - c_2\right)[g(FY, Z)X - g(FX, Z)Y$$

$$+ g(Y, Z)FX - g(X, Z)FY + g(FtX, Z)tX - g(FtY, Z)tY$$

$$+ g(tY, Z)FtX - g(tX, Z)FtY + 2g(FX, tY)tZ$$

$$+ 2g(X, tY)tfZ].$$

In this case the Ricci Tensor S of M is given by

$$S(X, Y) = \frac{1}{16}\left(c_1 + c_2\right)[(k-2)g(X, Y) + g(FX, Y)\,TrF + 6g(tX, tY)] + \frac{1}{16}\left(c_1 - c_2\right)[(k-2)g(FX, Y) + g(X, Y)\,TrF + 6g(FtX, tY)]. \tag{3.2}$$

Assume that f is an affine immersion. We define covariant derivative $\nabla^2 A$ by

$$\left(\nabla^2_{XY} A\right)_V W = \left(\nabla_X\left(\nabla_Y A\right)\right)_V W - \left(\nabla_{\nabla_X Y} A\right)_V W \tag{3.3}$$

for arbitrary vector fields X, Y, Z, W tangent to M and V a normal vector field. After a simple calculation we have

$$\left(\nabla^2_{XY} A\right)_V W - \left(\nabla^2_{YX} A\right)_V W = (R(X, Y)A)_V W \tag{3.4}$$

$$\begin{aligned}\left(\nabla^2_{XY} A\right)_V W = {} & \nabla_X \nabla_Y A_V W - \nabla_W A_V \nabla_Y W \\ & - \nabla_X A_{D_Y V} W - \nabla_Y A_V \nabla_X W + A_V \nabla_Y \nabla_X W \\ & + A_{D_Y V} \nabla_X W - \nabla_Y A_{D_X V} W + A_{D_X V} \nabla_Y W + A_V \nabla_{\nabla_X Y} W \\ & + A_{D_Y D_X V} W - \nabla_{\nabla_X Y} A_V W + A_{D_{\nabla_X Y} V} W.\end{aligned} \tag{3.5}$$

In consequence of (3.5) we have

$$(R(X, Y)A)_V W = R(X, Y)A_V W - A_V R(X, Y) W - A_{R^1(X, Y)V} W \tag{3.6}$$

Theorem 3.1 : *We have*

$$\overline{R}(X, Y)Z = R(X, Y)Z. \tag{3.7}$$

$$\overline{R}(X, Y)V = -\left(\nabla_X A\right)_V Y + \left(\nabla_Y A\right)_V X + R^1(X, Y)V. \tag{3.8}$$

Theorem 3.2 : *For a totally geodesic immersion*

$$\left(\overline{\nabla}_W \overline{R}\right)(X, Y)Z = \left(\nabla_W R\right)(X, Y)Z \tag{3.9}$$

$$(3.10)\qquad \left(\overline{\nabla}_W \overline{R}\right)(X, Y)V = (R(X, Y)A)_V W + A_V R(X, Y)W$$

$$- \left(\nabla^2_{WX} A\right)_V Y + \left(\nabla^2_{WY} A\right)_V X + \left(\nabla_W R^1\right)(X, Y)V.$$

Proof : The relation is a direct consequence of formulae and

$$\left(\overline{\nabla}_W \overline{R}\right)(X, Y)Z = \overline{\nabla}_W \overline{R}(X, Y)Z - \overline{R}\left(\overline{\nabla}_W X, Y\right)Z$$

$$- \overline{R}\left(X, \overline{\nabla}_W Y\right)W - \overline{R}(X, Y)\overline{\nabla}_W Z$$

$$= \nabla_W R(X, Y)Z - R\left(\nabla_W X, Y\right)Z - R\left(X, \nabla_W Y\right)W$$

$$- R(X, Y)\nabla_W Z = \left(\nabla_W R\right)(X, Y)Z.$$

Using $\overline{\nabla}_X Y = \nabla_X Y$, (3.4) and (1.1b), we get

$$\overline{\nabla}_W \overline{R}(X, Y)V = \overline{\nabla}_W\left(\left(\nabla_Y A\right)_V X - \left(\nabla_X A\right)_V Y + R^1(X, Y)V\right)$$

$$= \nabla_W\left(\nabla_Y A\right)_V X - \nabla_W\left(\nabla_X A\right)_V Y + \overline{\nabla}_W\left(R^1(X, Y)V\right)$$

$$= \nabla_W\left(\nabla_Y A\right)_V X - \nabla_W\left(\nabla_X A\right)_V Y - A_{R^1(X,Y)V} W$$

$$+ D_W R^1(X, Y)V$$

and

$$R\left(\overline{\nabla}_W X, Y\right)V = \overline{R}\left(\nabla_W X, Y\right)V$$

$$= \left(\nabla_Y A\right)_V \nabla_W X - \left(\nabla_{\nabla_W X} A\right)_V Y + R^1\left(\nabla_W X, Y\right)V$$

$$\overline{R}\left(X, \overline{\nabla}_W Y\right)V = \overline{R}\left(X, \nabla_W Y\right)V$$

$$= \left(\nabla_{\nabla_W Y} A\right)_V X - \left(\nabla_X A\right)_V \nabla_W Y + R^1\left(X, \nabla_W Y\right)V$$

$$\overline{R}(X, Y)\overline{\nabla}_W V = \overline{R}(X, Y)\nabla_W V$$

$$= \overline{R}(X, Y)\left(-A_V W + D_W V\right) \qquad \text{by (1.2)}$$

$$= -\overline{R}(X, Y)A_V W + \overline{R}(X, Y)D_W V$$

$$= -\overline{R}(X, Y)A_V W + \left(\nabla_Y A\right)D_W V^X$$

$$-\left(\nabla_X A\right)D_W V^Y + R^1(X, Y)D_W V$$

Applying the above and (3.4) to the formulae

$$\left(\overline{\nabla}_W \overline{R}\right)(X, Y)V = \overline{\nabla}_W \overline{R}(X, Y)V - \overline{R}\left(\overline{\nabla}_W X, Y\right)V - \overline{R}\left(X, \overline{\nabla}_W Y\right)V$$

$$-\overline{R}(X, Y)\overline{\nabla}_W V = \nabla_W\left(\nabla_Y A\right)_V X - \nabla_W\left(\nabla_X A\right)_V Y$$

$$-A_{R^1}\left(\nabla_W X, Y\right) + D_W R^1(X, Y)V - \left(\nabla_Y A\right)_V D_W X$$

$$+\left(\nabla_{\nabla_W X} A\right)_V Y - R^1\left(\nabla_W X, Y\right)V - \left(\nabla_{\nabla_W Y} A\right)_V X$$

$$+\left(\nabla_X A\right)_V \nabla_W Y - R^1\left(X, \nabla_W A\right)V + R(X, Y)A_V W$$

$$-\left(\nabla_Y A\right)D_X V^X + \left(\nabla_X A\right)D_X V^{Y-R^1}(X, Y)D_X V$$

$$= R(X, Y)A_V W - \left(\nabla^2_{WX} A\right)_V Y + \left(\nabla^2_{WY} A\right)_V X$$

$$-\dot{A}_{R^1(X,Y)V} W + \left(D_W R^1\right)(X, Y)V.$$

$$= (R(X, Y)A)_V W + A_V R(X, Y)W - \left(\nabla^2_{WX} A\right)_V Y$$

$$+\left(\nabla^2_{WY} A\right)_V X + \left(\nabla_W R^1\right)(X, Y)V \qquad \text{by (3.6)}$$

Theorem 3.3 : *Assume that* $f : (M, \nabla) \to (\overline{M}, \overline{\nabla})$ *is a totally geodesic affine immersion and* $(\overline{M}, \overline{\nabla})$ *is an affine locally decomposable Riemannian manifold of recurrent curvature say* $\overline{\nabla}\,\overline{R} = \overline{\phi} \times \overline{R}$ *then* (M, ∇) *is (a) flat or (b) of recurrent curvature, precisely* $\nabla R = \phi \times R$, ϕ *being the pull back of the recurrence* $\overline{\phi}$ *onto M.*

Proof is obvious.

Let f be an affine immersion. For a 1-form ρ on the normal bundle $N(M)$ and its first and second covariant derivatives with respect to the connection D are defined by

$$\left(D_X \ \rho\right)(V) = X(\rho(V)) - \rho\left(D_X V\right),$$

$$\left(D^2_{XY} \ \rho\right) = D_X\left(D_Y \ \rho\right) - D_{\nabla_X Y} f$$

respectively. Assuming $R^1(X, Y) = D^2 XY\rho - D^2 YX\rho$, we obviously have :

Theorem 3.4 : *If the second derivative of the normal connection is symmetric, then the curvature tensor of the normal connection of M vanish identically.*

If f is umblical i.e., $A(V) = \rho(V) I$ for certain 1-form ρ, then

$$\left(\nabla_X \ A\right)_V Y = \left(D_X \ \rho\right)(V) Y, \left(\nabla^2_{XY} \ A\right)_V Z = \left(D^2_{XY} \ \rho\right)(V) Z$$

and

$$(R(X, Y) A)_V Z = \left(R^1(X, Y) \ \rho\right)(V) Z.$$

Proposition 3.1 : *Let* $f : (M, \nabla) \to (\overline{M}, \overline{\nabla})$ *be a totally geodesic affine immersion, where* $(\overline{M}, \nabla)$ *is an affine locally product Riemannian manifold of recurrent curvature, say* $\overline{\nabla}\,\overline{R} = \overline{\phi} \otimes \overline{R}$, *then we have*

(3.11) $$A_V R(X, Y) W = -(R(X, Y) A)_V W - \left(\nabla^2_{WY} \ A\right)_V X + \left(\nabla^2_{WX} \ A\right)_V Y + \phi(W)\left(\left(\nabla_Y \ A\right)_V X - \left(\nabla_X \ A\right)_V Y\right)$$

(3.12) $$\left(D_W R^1\right)(X, Y) V = \phi(W) R^1(X, Y) V$$

In particular when f is additionally umblical then

(3.13) $$\rho(V) R(X, Y) W = -\left(R^1(X, Y)\rho\right)(V)(W) - \left(\left(D^2_{WY} \ \rho\right)(V)\right) - \phi(W)\left(D_Y \ \rho\right)(V) X + \left(\left(D^2_{WX} \ \rho\right)(V)\right) - \phi(W)\left(D_X \ \rho\right)(V) Y$$

Proof : (3.11) and (3.12) are consequences of the formulae $\overline{\nabla}_X Y = \nabla_X Y$, (3.6) and the assumption $\overline{\nabla}\,\overline{R} = \overline{\phi} \otimes \overline{R}$. In this case $(R(X, Y)A)_V Z = (R^1(X, Y)\rho)(V)Z$ becomes (3.13).

We shall study the existence of a certain class of f invariant submanifold in a complex space form of non-null holomorphic sectional curvature.

A proper F invariant submanifold M of a locally product Riemannian manifold M is a F invariant with both distributions ∇ and ∇^T of non-null dimensions. Also M is totally umblical if there exists a normal vector field L such that the second fundamental form B satisfies $B(X, Y) = g(X, Y)L$, for any vector fields X, Y tangent to M.

Now we propose :

Theorem 3.5 : *There exists no totally umblical proper F invariant submanifolds of an elliptic or hyperbolic complex space.*

Proof : Suppose there exists a totally umblical proper F-invariant submanifold M of a complex space form $M(c_1 \neq 0, c_2 \neq 0)$. Let X and Y be two non-null vector field, from ∇ and D respectively then, for the normal part of $\overline{R}(X, FX)Y$, we get $[\overline{R}(X, FX)Y]^1 \neq 0$. On the other hand, since M is totally umblical, the Codazzi equations give $[\overline{R}(X, FX)Y]^1 = g(FX, Y)D_X L - g(X, Y)D_{FX}L = 0$. Thus, we get a contradiction. This completes the proof.

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