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On C^h- birecurrent Finsler Spaces

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Abstract : Makoto Matsumoto[1] characterized his C^h-recurrent Finsler space by the condition

(A) $\quad C^i_{jk|h} = \lambda_h C^i_{jk}$

where C^i_{jk} is $(h)hv$-torsion tensor and $C^i_{jk|m}$ is the h-covariant derivative of the tensor C^i_{jk}. He studied the properties of such space specially when it is h-isotropic. In this paper we study a Finsler space satisfying the condition

(B) $\quad C^i_{jk|h|m} = a_{hm} C^i_{jk}$

which is more general than the previous one. This space will be called as C^h-birecurrent space. We also study C^h-birecurrent spaces of scalar curvature and h- isotropic C^h-birecurrent spaces.

1. Preliminaries

Let us consider an n-dimensional Finsler space F^n equipped with the Cartan's Euclidean connection. Let us denote the fundamental function and the components of the metric tensor by $F(x^i, y^i)$ and g_{ij} respectively. If T^i_j be an arbitrary tensor of type (1,1), then its h-covariant derivative and v-covariant derivative are defined by

$$T^i_{j|k} = \partial_k T^i_j - \left(\dot{\partial}_r T^i_j\right)\Gamma^{*r}_{sk} y^s + T^r_j \Gamma^{*i}_{rk} - T^i_r \Gamma^{*r}_{jk}, \quad \partial_k \equiv \frac{\partial}{\partial x^k}, \; \dot{\partial}_k \equiv \frac{\partial}{\partial y^k}. \tag{1.1}$$

$$T^i_j|_k = \dot{\partial}_k T^i_j + T^r_j C^i_{rk} - T^i_r C^r_{jk} \tag{1.2}$$

respectively, $\partial_k \equiv \dfrac{\partial}{\partial x^k}$, $\dot{\partial}_k \equiv \dfrac{\partial}{\partial y^k}$. The Ricci identities for the vector X^i are given by

$$X^i_{|k|h} - X^i_{|h|k} = X^r R^i_{rkh} - X^i|_r H^r_{kh} \tag{1.3}$$

where R^i_{rkh} are components of h-curvature tensor and H^r_{kh} are components of $(v)h$-torsion tensor. These two tensors are connected by

$$R^i_{rkh}\, y^r = H^i_{kh}. \tag{1.4}$$

The $(v)h$-torsion tensor and the deviation tensor of L. Berwald are connected by

$$H^{\ i}_{jk}\, y^j = H^{\ i}_{k} = -\, H^{\ i}_{kj}\, y^j \tag{1.5}$$

and

$$H^{\ i}_{h\,k} = \frac{1}{3}\left(\dot{\partial}_h H^{\ i}_{k} - \dot{\partial}_k H^{\ i}_{h}\right). \tag{1.6}$$

The $(v)h$-torsion tensor and Berwald's curvature tensor satisfy

$$\text{(a)}\quad \dot{\partial}_r H^{\ i}_{h\,k} = H^{\ i}_{rh\,k}, \qquad \text{(b)}\quad H^{\ i}_{rh\,k}\, y^r = H^{\ i}_{h\,k}. \tag{1.7}$$

The $(v)hv$-torsion tensor is given by

$$P^{\ i}_{jk\,h}\, y^j = P^{\ i}_{k\,h} = C^{\ i}_{kh\,|\,r}\, y^r \tag{1.8}$$

where $P^{\ i}_{jk\,h}$ are components of hv-curvature tensor.

2. C^h-birecurrent Spaces

Let us consider a C^h-recurrent Finsler space characterized by the condition (A). The h-covariant derivative of (A) gives

$$C^{\ i}_{jk\,|h\,|m} = a_{h\,m}\; C^{\ i}_{jk} \tag{2.1}$$

where $a_{h\,m} = \lambda_{h\,|\,m} + \lambda_h\, \lambda_m$. Thus, we see that a C^h-recurrent Finsler space satisfies (B). Therefore the space is C^h-birecurrent. Since the metric tensor g_{ij} *is* h-covariant constant, the equation (2.1) is equivalent to

$$C_{ijk\,|h\,|m} = a_{h\,m}\; C_{ijk}. \tag{2.2}$$

Conversely, if we assume the condition (2.2) which is equivalent to the characterizing condition (B) of C^h-birecurrent space, it does not imply the condition (A) in general. Therefore the condition (B) is more general than (A). In this case, the recurrence tensor need not be of the form $\lambda_{h\,|\,m} + \lambda_h\, \lambda_m$.

Let us consider a C^h-birecurrent space characterized by (2.2) or equivalently by the condition (B).

The transvection of (B) by y^h, in view of (1.8), gives

$$P^{\ i}_{jk\,|\,m} = a_{h\,m}\, y^h\, C^{\ i}_{j\,k}. \tag{2.3}$$

Now we propose :

Theorem 2.1 : *A C^h-birecurrent space is C^h- recurrent if it is a* *P-*Finsler space and* $a_{h\,m}\, y^h - \lambda_{|\,m} \neq 0$, *where λ is defined by* $P^{i}_{j\,k} = \lambda C^{\ i}_{j\,k}$.

Proof : Let us consider a C^h-birecurrent space characterized by (2.2), which is also a *P-Finsler space. For such space, we have the conditions $P^i_{jk} = \lambda C^i_{jk}$ and (2.3). From these two conditions, we get

$$C^i_{jk|m} = \frac{a_{hm}\, y^h - \lambda_{|m}}{\lambda}\, C^i_{jk} \tag{2.4}$$

which shows that the space is C^h-recurrent provided $a_{hm}\, y^h - \lambda_{|m} \neq 0$.

Taking skew-symmetric part of (B) with respect to the indices h and m and using the commutation formula (1.3), we have

$$A_{hm}\, C^i_{jk} = C^r_{jk} R^i_{rhm} - C^i_{rk} R^r_{jhm} - C^i_{jr} R^r_{khm} - C^i_{jk}|_r H^r_{hm} \tag{2.5}$$

where $A_{hm} = a_{hm} - a_{mh}$. Contracting the indices i and j in (2.5) and putting C_k for C^r_{rk}, we have

$$A_{hm}\, C_k = -\, C_r R^r_{khm} - C_k|_r H^r_{hm}. \tag{2.6}$$

Due to skew-symmetry of R_{rkhm} in its first two indices, we have

$$C_r R^r_{khm} C^k = R_{rkhm}\, C^r C^k = 0 \tag{2.7}$$

where $C^k = g^{jk} C_j$. Transvecting (2.6) by C^k and using (2.7), we have

$$A_{hm}\, C_k C^k = -\, C_k|_r C^k H^r_{hm}. \tag{2.8}$$

Suppose that there exists a non-null covariant vector field λ_m such that

$$\lambda_m H^i_{lh} + \lambda_h H^i_{ml} + \lambda_l H^i_{hm} = 0. \tag{2.9}$$

Transvecting (2.8) by λ_l, taking skew- symmetric part with respect to the indices l, h and m, and then using (2.9), we get

$$\left(\lambda_l A_{hm} + \lambda_m A_{lh} + \lambda_h A_{ml}\right) C_k C^k = 0. \tag{2.10}$$

This implies at least one of the following

$$\text{(2.11)} \qquad \text{(a)}\ \ C_k C^k = 0\,, \qquad \text{(b)}\ \ \lambda_l A_{hm} + \lambda_m A_{lh} + \lambda_h A_{ml} = 0.$$

The condition (2.11a) implies $C_k = 0$, which in view of Deicke's theorem[3] implies that the space is Riemannian. Thus, we see that if a C^h-birecurrent space admits the identity (2.9), the space is either Riemannian or it admits (2.11b). Thus, we have :

Theorem 2.2 : *If a C^h-birecurrent space admits the identity* (2.9), *the space is either Riemannian or it admits* (2.11b).

Since an R^h-recurrent space, a K^h-recurrent space and H-recurrent space [4-9] admit the identity (2.9), we may conclude

Corollary 2.1 : *A C^h-birecurrent space is either Riemannian or it admits (2.11b) provided it satisfies any one of the following*

(i) *it is an R^h-recurrent Finsler space,*

(ii) *it is a K^h-recurrent Finsler space,*

(iii) *it is an H-recurrent Finsler space.*

If a C^h-birecurrent space satisfies the condition[11] $T = 0$ i.e.

$$T = F^2 C_{hij}|_k + C_{kij} y_h + C_{hkj} y_i + C_{ihk} y_j + C_{jih} y_k = 0 \,, \tag{2.12}$$

the transvection of (2.12) by g^{ij}, and use of $C_{hij} g^{ij} = C_h$ and $C_{hkj} y_i g^{ij} = C_{hkj} y^j = 0$ give

$$F^2 C_h|_k = -\left(C_k\, y_h + C_h\, y_k\right). \tag{2.13}$$

This reduces (2.6) to

$$A_{hm} C_k = -\, C_r R^r_{kh\,m} + \left(C_r\, y_k\right) H^r_{h\,m} / F^2 \tag{2.14}$$

for $y_r H^r_{h\,m} = 0$. Transvecting (2.14) by C^k and using (2.7) and $C^k y_k = 0$, we have $A_{hm} C_k C^k = 0$, which gives either $A_{hm} = 0$ or $C_k C^k = 0$. $C_k C^k = 0$ implies $C_k = 0$; which in view of Deicke's theorem[3] shows that the Finsler space is essentially Riemannian. If the space is not Riemannian, we have $A_{hm} = a_{hm} - a_{mh} = 0$. Hence (2.14) becomes

$$-\, C_r R^r_{k\,h\,m} + \frac{y_k}{F^2}\, C_r H^r_{h\,m} = 0. \tag{2.15}$$

Thus, we have :

Theorem 2.3 : *A C^h-birecurrent space satisfying $T = 0$ is either Riemannian or its recurrence tensor is symmetric.*

If the deviation tensor H^i_h of a C^h-birecurrent space vanishes identically, in view of (1.6), (2.8) reduces to $A_{hm} C_k C^k = 0$. This implies that either the space is Riemannian or its recurrence tensor is symmetric. In the latter case, (2.6) reduces to

$$C_r R^r_{kh\,m} = 0. \tag{2.16}$$

Thus, we have :

Theorem 2.4 : *A C^h-birecurrent space with vanishing deviation tensor is either Riemannian or its recurrence tensor is symmetric and the curvature tensor $R^{\,i}_{jk\,h}$ satisfies* (2.16).

The condition $H^{\,i}_{h} = 0$ implies $H^{\,i}_{k\,h} = 0$, $H^{\,i}_{jk\,h} = 0$ and $H_{jik\,h} = 0$. Therefore the well-known identity

$$H_{h\,ijk} + H_{ih\,jk} = 2\left(P_{h\,ij\,|\,k} - P_{h\,ik\,|\,j}\right) - 2\,C_{h\,im}\,H^{\,m}_{j\,k}$$

reduces to

$$P_{h\,ij\,|\,k} - P_{h\,ik\,|\,j} = 0; \tag{2.17}$$

which, in view of (2.3), gives

$$a_{r\,k}\,y^{r}\,C_{h\,i\,j} - a_{r\,j}\,y^{r}\,C_{h\,i\,k} = 0. \tag{2.18}$$

Transvecting (2.18) by y^{k} and using $C_{\,h\,ik}\;y^{k} = 0$, we have $a_{rk}\,y^{r}\,y^{k}\,C_{hi\,j} = 0$; which shows that either the space is Riemannian or $a_{rk}\,y^{r}\,y^{k} = 0$. If the space considered is not Riemannian and $\beta_{k} \equiv a_{r\,k}\,y^{r} \neq 0$; (2.18) implies $C_{h\,ij} = \beta_{h}\,b_{ij}$, where b_{ij} is a non-null symmetric tensor. Due to symmetry of C_{hij} in h and i, we have $\beta_{h}\,b_{i\,j} = \beta_{i}\,b_{h\,j}$. This gives $b_{h\,i} = \beta_{h}\,b_{i}$, which due to symmetry of b_{hi} implies $b_{h\,i} = b\,\beta_{h}\,\beta_{i}$, where b is some non-zero scalar. Thus, we have

$$C_{ij\,k} = b\,\beta_{i}\,\beta_{j}\,\beta_{k}\,. \tag{2.19}$$

Transvecting (2.19) by g^{ij} and putting $g^{\,ij}\,\beta_{i}\,\beta_{j} = \beta^{2}$, we have $C_{k} = b\,\beta^{2}\beta_{k}$. In view of this fact, (2.19) may be written as $C_{ij\,k} = \Phi\;C_{i}\,C_{j}\,C_{k}$ where $\Phi = b^{-2}\,\beta^{-6}$. Transvection of $C_{ij\,k} = \Phi\;C_{i}\,C_{j}\,C_{k}$ by $g^{\,ij}$ gives $\Phi\,C^{\,i}\,C_{i} = 1$. Putting $C_{i}\,C^{\,i} = C^{2}$ we get $\Phi = \dfrac{1}{C^{2}}$, and hence

$$C^{2}\,C_{i\,j\,k} = C_{i}\,C_{j}\,C_{k}\,. \tag{2.20}$$

A Finsler space $F_n(n > 2)$ for which the tensor C_{ijk} satisfies (2.20) is called a $C2$-like Finsler space[12]. Thus we have :

Theorem 2.5 : *If the deviation tensor of a C^h- birecurrent Finsler space vanishes, the space is either Riemannian or $a_{h\,m}\,y^{h}\,y^{m} = 0$. If this space is not Riemannian and $a_{h\,m}\,y^{h} \neq 0$, the space is $C\,2$-like.*

From theorem (2.5) we observe that for a non-Riemannian space, the recurrence tensor a_{hm} satifies $a_{h\,m}\,y^{h}\,y^{m} = 0$. Partial differentiation of $a_{h\,m}\,y^{h}\,y^{m} = 0$ with respect to y^{k} gives

$$\left(\dot{\partial}_{k}\,a_{h\,m}\right)y^{h}\,y^{m} + a_{k\,m}\,y^{m} + a_{h\,k}\,y^{h} = 0. \tag{2.21}$$

Differentiating (2.21) partially with respect to y^j we have

$$(2.22)\quad \dot{\partial}_j\left\{\left(\dot{\partial}_k a_{hm}\right)y^h y^m\right\} + \left(\dot{\partial}_j a_{km}\right)y^m + \left(\dot{\partial}_j a_{hk}\right)y^h + a_{jk} + a_{kj} = 0.$$

If the recurrence tensor a_{hm} is independent of y^i, (2.22) gives $a_{jk} + a_{kj} = 0$, i.e. the recurrence tensor is skew-symmetric; which contradicts theorem 2.4.

Thus, we have :

Corollary 2.2 : *A C^h-birecurrent space whose deviation tensor vanishes and recurrence tensor is a function of positional coordinates only is necessarily Riemannian.*

3. C^h-birecurrent Spaces of Scalar Curvature

Let us consider a C^h-birecurrent space $F_n\,(n>2)$ of scalar curvature. This space satisfies equations (2.1) to (2.8) together with the condition

$$(3.1)\qquad H^i_h = F^2K(\delta^i_h - l^i l_h)$$

which characterizes a Finsler space $F_n\,(n>2)$ of scalar curvature. From (3.1) we may derive

$$(3.2)\qquad H^i_{kh} = \frac{F^2}{3}\left\{\left(\delta^i_h - l^i l_h\right)\dot{\partial}_k K - \left(\delta^i_k - l^i l_k\right)\dot{\partial}_h K\right\} + FK\left\{l_k\delta^i_h - l_h\delta^i_k\right\}.$$

Transvecting (2.8) by y^h and using (1.5), (3.1), $l^i = y^i / F$, (1.2) and the fact that the tensor C_k is positively homogeneous of degree -1 in y^i, we have

$$(3.3)\qquad C_{k|m}C^k = -\,C_k C^k\,\frac{(A_{hm}y^h + Ky_m)}{F^2K}.$$

From (2.8) and (3.3) we have

$$C_k C^k\left(F^2KA_{hm} - A_{sr}y^s H^r_{hm}\right) = 0\,.$$

This gives either of the two conditions

$$(3.4)\qquad \text{(a)}\quad C_k C^k = 0, \qquad\qquad \text{(b)}\quad F^2KA_{hm} = A_{sr}y^s H^r_{hm}\,.$$

The condition (3.4a) implies $C_k = 0$, which due to Deicke's theorem implies that the space is Riemannian. From (3.4b) and (3.2) we get

$$(3.5)\quad 3K\left(F^2A_{hm} + A_{sh}y^s y_m + A_{ms}y^s y_h\right) = F^2\left(A_{sm}y^s\dot{\partial}_h K - A_{sh}y^s\dot{\partial}_m K\right)$$

Thus, we conclude :

Theorem 3.1 : *A C^h-birecurrent Finsler space $F_n\,(n>2)$ of scalar curvature is either Riemannian or satisfies* (3.5).

If the space considered is of constant curvature i.e. $\dot{\partial}_h K = 0$, (3.5) reduces to $3K\left(F^2 A_{hm} + A_{sh} y^s y_m + A_{ms} y^s y_h\right) = 0$. This gives at least one of the conditions

$$\text{(3.6)} \qquad \text{(a) } K = 0 \qquad \text{(b) } F^2 A_{hm} + A_{sh} y^s y_m + A_{ms} y^s y_h = 0$$

If (3.6a) holds, (3.1) becomes $H^i_h = 0$; which in view of theorems 2.4 and 2.5 shows that either the space is Riemannian or the recurrence tensor is symmetric, the curvature tensor satisfies (2.14) and if $a_{rk} y^r \neq 0$, we have (2.20). Thus, we conclude :

Theorem 3.2 : *A C^h-birecurrent Finsler space F_n $(n > 2)$ of constant curvature is either Riemannian or it satisfies at least one of the conditions*

(i) $K = 0$, $H^i_h = 0$, $a_{hm} = a_{mh}$, $a_{mh} y^m y^h = 0$, $C_r R^r_{khm} = 0$ and $C^2 C_{ijk} = C_i C_j C_k$ (provided $a_{rk} y^r \neq 0$),

(ii) $F^2 A_{hm} + A_{sh} y^s y_m + A_{ms} y^s y_h = 0$.

Let us consider a C^h-birecurrent space F_n $(n > 2)$ satisfying the condition

$$R^i_{jkh} = K\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right). \tag{3.7}$$

The space characterized by (3.7) is called h-isotropic. It is to be noted that the concept of h-isotropy does not coincide with that of constant curvature due to Berwald[10]. For an h-isotropic space F_n $(n > 2)$, K is constant (due to Akbar-Zadeh). Therefore K is constant for the space considered. *Transvecting (3.7) by y^j, we get*

$$H^i_{kh} = K\left(y_k\delta^i_h - y_h\delta^i_k\right). \tag{3.8}$$

Differentiating (3.8) partially with respect to y^j and using (1.7) and $g_{ij} = \dot{\partial}_i y_j$, we get $H^i_{jkh} = K\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right)$, which shows that the space is of constant curvature. Thus, an h-isotropic Finsler space F_n $(n > 2)$ is of constant curvature. In view of this fact and theorem 3.2, we find that an h-isotropic C^h-birecurrent Finsler space is either Riemannian space of constant curvature or it satisfies any one of the two conditions (i) and (ii) of theorem 3.2. Now we shall show that the condition (ii) implies $K = 0$. In view of (3.7) and (3.8), (2.6) becomes

$$A_{hm} C_k = K\left\{-C_m g_{kh} + C_h g_{km} - C_k|_m y_h + C_k|_h y_m\right\}. \tag{3.9}$$

Transvecting (3.9) by y^h, and using $C_k|_h y^h = -C_k$, we get

$$K C_k|_m = \frac{-k(C_m y_k + C_k y_m) - A_{hm} y^h C_k}{F^2}. \tag{3.10}$$

Substituting (3.10) in (3.9), and using the condition (ii) of the theorem 3.2 we find either $K = 0$ or

$$C_h\left(g_{km} - l_k l_m\right) - C_m\left(g_{kh} - l_k l_h\right) = 0. \tag{3.11}$$

where $l_k = y_k/F$. Transvecting this equation by g^{km}, we get $(n-2)\,C_h = 0$; which implies $n = 2$ for the space is non-Riemannian. This gives a contradiction to the fact $n > 2$. Thus, we find that the condition (ii) of theorem 3.2 implies $K = 0$. Substituting $K = 0$ in (3.7) and (3.8), we have

$$\text{(a)}\quad R^i_{jkh} = 0, \qquad \text{(b)}\quad H^i_{kh} = 0. \tag{3.12}$$

Summarizing the above discussion, we have

Theorem 3.3 : *An h-isotropic and C^h-birecurrent Finsler space F_n $(n > 2)$ is either a Riemannian space of constant curvature or it admits the following*

$$K = 0, \quad H^i_h = 0, \quad a_{mh} = a_{hm}, \quad a_{mh}y^m y^h = 0 \text{ and } R^i_{jkh} = 0\,.$$

If the recurrence tensor a_{hm} of a non-Riemannian h- isotropic C^h-birecurrent Finsler space satisfies $a_{hm}y^h \neq 0$, we have $C^2 C_{ijk} = C_i C_j C_k$, i.e. the space is C 2-like.

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