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Fixed Point Theorem on Complete Cone Metric Space*

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Abstract: In this paper, two common fixed point theorems in cone metric space have been obtained for mappings satisfying a new contraction type condition.

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1. Introduction

Huang and Zhang[1] recently introduced the concept of cone metric space and established some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, several other authors[2,3,4] studied the existence of fixed points and common fixed points of mappings satisfying a contractive type condition on a normal cone metric space. In the present paper, we prove some common fixed point theorems in complete cone metric space.

2. Preliminaries

The following notions have been used to prove the main result.

Definition 2.1: Let E be a real Banach Space. A subset P of E is called cone[1] if and only if

(i) P is closed, non empty and $P \neq \{o\}$.
(ii) $0 \leq a, b \in R$, $a, b \geq 0$, and $x, y \in P \Rightarrow ax + by \in P$.
(iii) $x \in P$ and $-x \in P \Rightarrow x = o$.

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Definition 2.2: A cone P is called normal[1] if there is a number $\lambda > o$ such that for all $x, y \in E$, the inequality

$$0 \leq x \leq y \Rightarrow \|x\| \leq \lambda\|y\|.$$

The least positive number λ satisfying the above inequality is called the normal constant of P.

In this paper we always suppose that E is a real Banach space and P is a cone in E with int.$P \neq \Phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.3: Let X be a non empty set. Suppose that the mapping $\rho: X \times X \rightarrow E$ Satisfies:

(i) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$.

(ii) $\rho(x, y) = \rho(y, x)$ for $x, y \in X$.

(iii)$\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$ and for all $x, y, z \in X$.

Then ρ is called a cone metric on X and (X, ρ) is called cone metric space[2].

Lemma[1] 2.4: *Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x then $\{x_n\}$ is a Cauchy sequence.*

Lemma[1] 2.5: *Let (X, d) be a cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \rightarrow 0$ as $m, n \rightarrow \propto$.*

3. Common Fixed Point Theorems

In this section we shall prove two common fixed point theorems.

Theorem 3.1: *Let (X, d) be a complete cone metric space and P be a normal cone with normal constantλ. Let f and g are two self mappings satisfying the condition,*

$$d(fx, gy) \leq k\,[d(x, fx) + d(y, gy)] \text{ for all } x, y \in X,\ k \in [0, 1/2).$$

Then f and g has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X and $\{x_{2n}\}$ be a sequence in X. We define the mappings f and g in X such that

$$x_{2n-1} = fx_{2n-2};\ n \in N / \{0\} \text{ and } x_{2n} = gx_{2n-1}\ ;\ n \in N \cup \{0\}.$$

Puting $x = x_{2n-2}$, $y = y_{2n-1}$ in equation (1), we get

$$d(x_{2n-1}, x_{2n}) = d(fx_{2n-2}, gx_{2n-1}) \leq k\,[d(x_{2n-2}, fx_{2n-2}) + d(x_{2n-1}, gx_{2n-1})]$$

$$\leq k\,[d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})].$$

Thus $d(x_{2n-1}, x_{2n}) \leq \frac{k}{1-k} d(x_{2n-2}, x_{2n-1})$.

Again putting $x = x_{2n-3}$, $y = y_{2n-2}$ in equation (1)

$$d(x_{2n-2}, x_{2n-1}) = d(fx_{2n-3}, gx_{2n-2}) \leq k\,[d(x_{2n-3}, fx_{2n-3}) + d(x_{2n-2}, gx_{2n-2})]$$
$$\leq k\,[d(x_{2n-3}, x_{2n-2}) + d(x_{2n-2}, x_{2n-1})].$$

Thus $d(x_{2n-2}, x_{2n-1}) \leq \frac{k}{1-k} d(x_{2n-3}, x_{2n-2})$.

So by induction, $d(x_{2n-1}, x_{2n}) \leq \left(\frac{k}{1-k}\right)^n d(x_0, x_1)$

$$\Rightarrow d(x_{2n-1}, x_{2n}) \leq h^n\, d(x_0, x_1), \text{ where } h = \left(\frac{k}{1-k}\right) < 1.$$

Now for $n > p$ we have

$$d(x_{2n}, x_{2n+p}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+p})$$
$$\leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \ldots\ldots\ldots + d(x_{2n+p-1}, x_{2n+p})$$
$$\leq h^n\, d(x_0, x_1) + h^{n+1}\, d(x_0, x_1) + \ldots\ldots\ldots + h^{n+p}\, d(x_0, x_1)$$
$$\leq h^n\,[1 + h + h^2 + \ldots\ldots\ldots\ldots\ldots h^p]\, d(x_0, x_1) \leq \frac{h^n}{1-h} d(x_0, x_1).$$

Thus we get, $\|d(x_{2n}, x_{2n+p})\| \leq \frac{h^n}{1-h} \lambda \|d(x_0, x_1)\|$.

This implies that $\|d(x_{2n}, x_{2n+p})\| \to 0$ as $n \to \propto$.

Hence $\{x_{2n}\}$ is a Cauchy sequence. By the completeness of X there is $x \in X$ such that $x_{2n} \to x$ $(n \to \propto)$.

Now we show that $fx \to x$ i. e. fx has a fixed point in X. Since

$$d(fx, x) = d(fx_{2n-1}, x_{2n-1}) = d(fx_{2n-1}, gx_{2n-2})$$
$$\leq k\,[d(x_{2n-1}, fx_{2n-1}) + d(x_{2n-2}, gx_{2n-2})] \leq k\,[d(x_{2n-1}, x_{2n-2}) + d(x_{2n-2}, x_{2n-1})]$$
$$\leq 2k\, d(x_{2n-2}, x_{2n-1}),$$

So $\|d(fx, x)\| \leq 2k\lambda \|d(x_{2n-2}, x_{2n-1})\| \to 0$ as $n \to \propto$.

Hence $\|d(fx, x)\| = 0$. This implies fx = x i. e. x is the fixed point of f. Similarly we can show that g has the fixed point x.

Now we show that x is unique. For suppose x′ is another fixed point in x.

Then fx = x = gx and fx′ = x′ = gx′. Now

$d(x, x') = d(fx, gx') \leq k [d(x, fx') + d(x', gx')] = k [d(x, x) + d(x', x')] = 0.$

Hence $d(x, x') = 0$, i. e. x = x′. Thus x is the unique fixed point of f and g in X.
This completes the proof of the theorem.

Theorem 3.2: *Let (X, d) be a complete cone metric space and P be a normal cone with normal constantλ. Let f and g are two self mappings satisfies the condition,*

$$d(fx, gy) \leq k \left[\frac{d(x,y)+d(x,fx)}{2} + d(y, gy)\right] \text{ for all } x, y \in X,\ k \in [0, 1/2).$$

Then f and g has a unique fixed point in X.

Proof: Proof is similar to theorem (3.1),s so omitted.

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