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Reduced Multi Objective Linear Fractional Programming Problem

Tarni Mandal and Ajit Ghosh
Department of Mathematics
National Institute of Technology, Jamshedpur- 831014.
Email: tmandal_nitjsr@rediffmail.com
Email: agritjsr@gmail.com

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Abstract: The fractional programming problem usually defined by the problems in which one has to maximize a ratio of two functions subject to given conditions, arise in various decision making problems. For example fractional programming problem is used in the field of production planning, traffic planning, network flows and game theory. A numerical example is is worked out to explain the method developed which shows improvement in the computational complexities.

1. Introduction

Mostly income planning problem, government policy-making problems and socioeconomic problems require multi-objective fractional programming problems. Public/Govt policy decision making is generally based on economic criterion on the consideration of social equality. The objective function can be expressed as a set of fractional programming problem which are usually ratio of two linear function. Mathematically Multi Objective Linear Fractional Programming (MOLFP) can be stated as,

(1.1) $\text{Max}\{Z_1(X), Z_2(X), \ldots\ldots\ldots\ldots Z_k(X)\}$

(1.2) Subject to $Ax \le b$

where $Z_i(X) = \dfrac{N_i(x)}{D_i(x)}$, $i = 1, 2 \ldots\ldots k$, A is m x n matrix, $b \in R^m$, $x \in R^n$

(1.3) $\text{Let } S^l = \{x \in R^n : Ax \le b\}$

An efficient solution of MOLFP is the set of all solution of 1.1, 1.2 and 1.3. The normal definition of solution efficiency does not allow Simplex based algorithm to be used for solving MOLFP, we categories the efficient solution into two types, weak efficiency and strong efficiency referred to as w-and s-efficiency.

a) Definition of s-efficient

A point $\overline{x} \in S'$ *is defined to be s-efficient if and only if there does not exist another point* $x \in S^{l}$ *such that* $z_i(x) \geq z_i(x)$ *for all i, and* $z_i(x) \geq z_i(\overline{x})$ *for at least one i. In other words there is no other point in* S^{l} *which weakly dominates* $\overline{x}$.

b) Definition of w-efficient

A point $\overline{x} \in S$ *is defined to be w-efficient if and only if there does not exist another point x* $\in$ *s such that* $z_i(\overline{x}) \geq z_i(x)$ *for all i. In other words these is no other point in S which strongly dominates* $\overline{x}$.

Remark

It may be noted that w-efficiency is a more generalised version of the standard definition of strong efficiency.

c) Definition of θ_{max}

We consider a w-efficient basis at a given point x $\in$ S^{l}, the set of all efficient solution. Let x_i be a non basic variable associated with an initially w-efficient edge emanating from x_i. Let θ be the value of x_i which produce a change of basis then there exists a value of θ such that the movement along the initially w-efficient edge emanating from x_i become w in efficient. We refer this value of θ as θ_{max}.

2. Formulation

We begin by referring to Ecker and Kouda[1] Eans and Stuer[2] that the solution of the problem (1.1) is set of all efficient solution because of complication i.e. restriction imposed on the efficient solution simplex like algorithm could not be established, however, by defining two types of efficiency namely weak efficiency (W.E) and strong efficiency (S.E). We may develop simplex like procedure.

The solution procedure then systematically pursues all w-efficient vertices. By pursing a vertex, we mean that

i) All emanating edes are tested for initial w-efficiency.
ii) All initially w-efficient edges found in (i) are tested for break points.
iii) Each break point detected in (ii) is analyzed to determine which level curves cause the point in question to be a break point.
iv) θ_{max} cutting plane are added to the MOLFP for the level curves of the objectives that.

a) Intersect the break point,
b) form part (or all) of the boundary of (SE), and
c) do not support S^{I} at the break point.

3. Reduced Multiobjetive Linear Fractional Programm (Rmolfp)

The MOLFP (1.1) becomes,

(3.1) $$\text{Max}\{Z_{1R}(v), Z_{2R}(v), \ldots\ldots\ldots\ldots Z_{kR}(v)\}$$

(3.2) where, $$Z_{iR} = \frac{N_{i_R}(v)}{D_{i_R}(v)},$$

where $N_{iR}(v) = C_{iR}^{T}V + C_{iR_o}$ and

(3.3) $D_{iR}(v) = D_{iR}^{T}V + D_{iR_o}$, i = 1, 2 …….k

where,

(3.4) $$C_{iR}^{T} = (C_{iv}^{T} - C_{iu}^{T}B^{-1}N)$$

(3.5) $$D_{iR}^{T} = (D_{iv}^{T} - D_{iu}^{T}B^{-1}N)$$

(3.6) $$C_{iR_o} = (C_o + C_{iu}^{T}B^{-1}b)$$

(3.7) $$D_{iR_o} = (D_o + D_{iu}^{T}B^{-1}b)$$

4 The Augmented Rmolfp

The augmented RMOLFP for

$$\text{Max}\{Z_1(v), Z_2(v), \ldots\ldots\ldots\ldots Z_k(v)\}$$

(4.1) where $Z_i\ (v) = \dfrac{C_R^{iT} v + (C_{R_o}^{i})}{D_R^{iT} v + D_{R_0}^{i}}$

Subject to,

$V \in R^{n-m}$, deviational variables d_j^{+} & d_j^{-1} as defined by

(4.2) $(C_R^{jT} - Z_R D_R^{j})V - d_j^{+} + d_j^{-} = -(D_j^{R} - \overline{Z}_R C_j^{R}),\ \ d_j^{+}, d_j^{-} > 0$

The augmented constraint set does not restrict the original feasible region S this is because the deviational variable d_j^{+} and d_j^{-} add additional dimensionality to the feasible region. Thus by adding reduced θ_{max} cutting planes, we are able to argument the reduced feasible region by "etching" additional edges into and on to R as needed when pursueing the starting w-efficient extreme point of S. It is likely that one or more of its emanating edges will be initially w-efficient. These will then lead to adjacent w-efficient vertices. (Which may not correspond to extreme point of S) which are as yet unpursued. All unpursued vertices are stored in a LIST (denoted by L^{ω}) until the algorithm can get to move them eventually, we will get to the point at which all w-efficient vertices have been pursued and no un pupursned w-efficient vertices remain in L^{ω}. At this point the algorithm terminates because all w-efficient vertices have been found.

4.1 Test for initially W-efficient Edge

Consider the vector maximum problem

(4.1.1) $$\begin{aligned}&\text{Eff}\{\overline{T}(\overline{V}).v | v \in R^{n-m}\}\\ &\text{Let } v_1 = (\theta \max) = 1_j(\theta \max)/m_i(\theta \max)\end{aligned}$$

where the ith row of the k x n matrix. $T(\overline{v})$ is the local gradient of its objective in an RMOLFP problem evaluated at $v \in R^{n-m}$

It is sufficient to define the ith raw of $T(\overline{v})$ as

(4.1.2) $(D_R^{i} v + D_o^{j})C_R^{i} - (C_R^{j} v + C_o)D_R^{j}$

From Tanion[3] we know that $\bar{v}$ is S (strong) efficient in an RMOLFPP iff $\bar{v}$ is R^{n-m} efficient in (4.1.1). As an immediate Corollary to Tanion theorem, $\bar{v}$ w-efficient in an RMOLFP iff $\bar{v}$ is w-efficient in (4.1.1). Using perturbation of the $T(\bar{v})$ matrix we can detect initially w-efficient direction as they emanate from a w-efficient point $\bar{v} \in R^{n-m}$. This method is similar to the way R^n efficient edges are detected as they emanate from s-efficient extreme points in a vector-maximum problem.

Let us now review how w-efficient edges are detected in vector maximum problem. Let B_R be a w-efficient basis of extreme point $\bar{v} \in R^{n-m}$. To define a w-efficient basis

(4.1.3) Let $\tilde{T}(\bar{v}) = T_B(\bar{v})B^{-1}N - T(\bar{v})$

be the reduced cost matrix corresponding to B at $\bar{v}$. Then B is w-efficient if and only if there exists a $\lambda \in R^k$, such that the system

$$(4.1.4) \qquad \begin{cases} \lambda^T \bar{T}(v) \geq O \\ \lambda^T e = 1 \\ \lambda \geq O \end{cases}$$

is consistent. Clearly, any extreme point corresponding to a w-efficient basis is w-efficient. Now, by reworking Lemma in Evans and Steuer[2] in terms of w-efficiency, We have the following result. For B a w-efficient basis at $\bar{v}$, and $\tilde{T}_j(\bar{v})$. The jth column of the reduced cost matrix $\tilde{T}(\bar{v})$, the edge emanating from $\bar{v}$ pertaining to the introduction of the ith non basic variable is initially w-efficient. Eff the sub-problem.

$$(4.1.5) \qquad \begin{aligned} &\max(r \in R^{n-m}) \\ &\text{St. } (\bar{T}_j(\bar{v})u - \tilde{T}_j(\bar{v}_n)w + re \leq O, \\ &O \leq u \in R^{n-m} \\ &O \leq w \in R^{n-m} \\ &r \text{ unrestricted,} \end{aligned}$$

has bounded objective function value $(e \in R^k)$ is the sum vector and $(\bar{n})$ is the number or non basic variable in the vector maximum formulation

$$(4.1.6) \qquad \text{Eff}\{T(\bar{V}).v|v \in R^{n-m}\}$$

In RMOLFP the objective function matrix and the reduced cost matrix are constants as we move along an efficient edge, thus having identified an edge, that is initially efficient in RMOLFP, it will remain efficient upto the next extreme point. As we have seen in RMOLFP edges can be broken. Further, more, T ($\bar{v}$), the linear objective function matrix is not constant as we move along a w-efficient edge. For this reason, in RMOLFP an edge may start out w-efficient and become w-inefficient before reaching the extreme point at the other end. The most extreme case of this 'breaking' will be the case in which the test (4.1.5) suggests that an edge is initially w-efficient but that the break point occurs immediately (for θ > 0). To avoid such situation, the sub-problem test (4.1.5), must be modified so as to detect edges which have a non empty w-efficient segment (4.1.5) is modified as follows. Let B be a w-efficient basis at $\bar{v}$, and let $\hat{v}$ denote the extreme point that would result by an acceptable pivot choice when introducing the jth nonbasic variable. Then edge r ($\bar{v}, \hat{v}$) is initially w-efficient (as it emanates from $\bar{v}$) if there exists a [$K \in (0,1)$] such that

(4.1.7)

$$\begin{aligned}
&\text{Max } (r \in R^{n-m})\\
&\text{s t } (\tilde{T}_j(\bar{v}+\delta_j)u - \tilde{T}_j(\bar{v}\ +\delta_j)w + re \le O.\\
&O \le u \in R^n,\\
&O \le w \in R^n,\\
&r \text{ unrestricted}
\end{aligned}$$

has a bounded objective functions value for

$$\delta_j = D.K(\hat{v} - \bar{v}_n) \text{ for all } \alpha \in (0, 1)$$

i.e. there is some finite portion of the edge as it initially emanates from $\bar{v}$ that is w-efficient.

To implement the revised sub problem test, it has been found convenient to solve (4.1.7) only once, using a δ_j that is based upon k value of 0.01 i.e. set δ_j = 0.01 $(\hat{v} - \bar{v}\ )$. The k value 0.01 is chosen (as a tolerance) to be large enough to allow the gradients to change by a numerically significant amount when going from $\bar{v}$ to $(\hat{v} + \delta_j)$, yet small enough to minimize the probability of overshooting a break point (beyond which the edge becomes w-inefficient).

4.2 Detecting of Break Points W-efficient Edges

Consider a w-efficient basis B at $\bar{v} \in R^{n-m}$ let v_j be a non basic variable associated with on initially w-efficient edge emanating from $\bar{v}$. Let $\hat{\theta}$ be the value of v_i that would produce a basis change at ($\hat{v}$). We want to determine If there exists a value of θ (that we will refer to as θ_{max}) beyond which movement along the initially w-efficient edge associated with v_i becomes w-inefficient (or would become w-inefficient if it were feasible.
If

a) $Q_{max} < \hat{\theta}$, we have detected a broken edge whose break point occurs at the point along with the edge at which $v_i = \theta_{max}$. passing through this break point is at least one level curve of an objective that forms part of the boundary of the w-efficient set E^w. Such level curves will be called θ_{max} - cutting planes. Identification and insertion of the Q_{max} - cutting planes is considered in (4.3).

b) $\theta_{max} = \hat{\theta}$, the edge is co-efficient up to the next vertex but is neverthless a broken edge. Beyond this vertex the edge (if it were feasible) would become w-in efficient. The significance of $\theta_{max} = \hat{\theta}$ is that there exists a θ_{max} cutting plane that goes through the adjustment vertex.

c) $Q_{max} > \hat{\theta}$, the entire edge is w-efficient and is not interdicted by any θ_{max} -cutting planes.

Before discussing how θ_{max} is calculated, let us study the nature of the reduced cost matrix $\tilde{T}(\bar{v})$ at B and $\bar{v}$. Because the gradient of the objectives in RMOLFPP are not constant $T(\bar{v})$ can very as we emanate from $\bar{v}$ along an edge. If we are moving along the edge associated with v_j $\tilde{T}(\bar{v})$ can be expressed as a function of θ as follows.

(4.2.1) Let $z_i(\bar{v}) = \dfrac{l_i(v)}{m_i(v)}$ where

(4.1.2)
$$l_j(v) = C_R^1 v + C_{R_o}$$
$$m_j(v) = D_R^1 v + D_{R_o}$$

As we move along the edge $(\bar{v}, \hat{v})$

we have

$$l_j\,(\theta) = l\ (\bar{v}_n) - \bar{C}_R\theta$$

(4.2.3) $$m_j\,(\theta) = m\ (\bar{v}_n) - \bar{D}_{Rj}\theta$$

where C_{Rj} *and* D_{Rj} are the ith columns of the numerator and denominator reduced cost matrices and

where $$\bar{C}_R = C_B B^{-1} N - C_o$$

(4.2.4) $$\bar{D}_R = B_B B^{-1} N - D_o$$

Therefore

$$\begin{aligned} T_R(\theta) &= m_R(\theta) \circ C - l_R(\theta) \circ D \\ &= (m_R(\bar{v}) \circ C - l_R(\bar{v}v \circ D) + (\theta)\,(D_{jR} \circ C^R - C_{jR} \circ D^R) \\ &= T(\bar{v}) + \theta\, C\bar{D}_{jR} \circ C \ \ - C_{jR} \circ D) \end{aligned}$$

It follows that

(4.2.5) $$\tilde{T}(\theta) = \tilde{T}(\bar{v}) + \theta H(\bar{v}, j)$$

where $H(\bar{v}, j)$ is given by ta matrix

(4.2.6) $$H(\bar{v}, j) = (\,(\bar{D}_j \circ C_B - \bar{C}_j \circ D_B) B^{-1} N - (\bar{D}_j \circ C_N - \bar{C}_j \circ D_o)$$

where $H(\bar{v}, j)$ is a constant matrix the same size as $\tilde{T}(\bar{v})$ that represents the rate at which the reduced cost matrix value along the edge associated with B and v_j. Therefore, the maximum value of θ for which test system

(4.2.7) $$\begin{cases} \lambda^T \left[\tilde{T}(\bar{v}) + \theta H(\bar{v}, j) = \lambda^T \tilde{T}(\theta) \geq O\right] \\ \lambda^T e = 1 \\ O \leq \lambda \in R^k \end{cases}$$

remains consistent in θ_{max}

To calculate θ_{max} sub problem (4.2.8) is solved for p = 1,2,…..

(4.2.8)
$$\begin{cases} \max(S \in S^n) \\ \text{s.t } \lambda_R^T \theta_R^p - Se \geq O \\ \lambda_R^T e = 1 \\ O \leq \lambda \in R^k \\ O \leq S \in R^k \quad O \leq S \in R, \end{cases}$$

where $P^p = T(\overline{v})$

Once we have identified that R^k (the optimal value of S in (4.2.8) is less than some prespecified tolerance, we can use standard extrapolation techniques based on the method of Williams[5] to get an 'accurate' evaluation of θ_{max}. An example is presented in table (4.2.5) and (4.2.6).

A tighter form for the θ_{max} test is given by requiring that the redused cost of v_i be zero as we move along the edge $\gamma(\overline{v}, \hat{v})$. This is done by adding the requirement 1 $\lambda^T \theta_j^P = O$ to the constraints. Set of (4.2.8). The calculation of θ_{max} corresponds to Block 1(c).

4.3 Identification and Insertion of The θ_{Max}-Cutting Planes

We have shown how the value for θ_{max} is calculated. Given θ_{max}, the cutting planes are identified as follows:

a) Identify all alternative solutions λ_{Ri} for (4.2.7) for which $\theta = \theta_{max}$.

b) Let $1=\{i | \lambda_i > O\}$. The set I is the set of objectives whose levels curves pass through the point $\theta = \theta_{max}$ and lead to alternate (additional) w-efficient vertices.

c) Ignore any cutting plane whose gradient is a linear combination of gradients of already identified cutting planes.

Having identified which planes need to be inserted, the feasible region is augmented as follows.

i) Assume that the current (as yet pursued) vertex is $\overline{v}$. Introducing non basic $v_j = o$ leads along an initially w-efficient edge to $\theta = \theta_{max}$.

ii) Let $v_1(\theta\max) = 1_j(\theta\max)/m_i(\theta\max)$ be the value of z_i at $v_1 = (\theta\max)$ and let C^{-j} and d^{-j} be the updated rows of the numerator and denominator functions.

iii) For linear objective we use

(4.3.1) $$Cv - d_i^+ + d_i^- = z_i(\theta_{\max}) - z_i(\bar{V})$$

iv) For fractional objectives we use

(4.3.2) $$C^i - z_i(\theta_{\max})d_{v^i}) - d_i^+ + d_i^- = ZR_i(\theta_{\max}) - mR_i(\bar{V}) - lR_i((\bar{V}))$$

5. Algorithm

We begin by referring to Ecker and Kouda[1] & Eans and Steure[2] that the solution of the problem (1) is set of all efficient solution because of complication i.e. restriction imposed on the efficient solution simplex like algorithm could not be established, however, by defining two types of efficiency namely weak efficiency (W.E) and strong efficiency (S.E).

The solution procedure the systematically peruses all w-efficient vertices. By perusing a vertex, we mean that

Step-0

Procedure ‘initial w-efficient’

All emonating edes one tested for initial w-efficiency.

Step-1

Procedure ‘w-efficient edges’

All initially w-efficient edges found in (a) are tested for break points.

Step-2

Procedure ‘break point’

Each break point detected in (b) is analyzed to determine which level curves cause the point in equations to be a break point.

Step-3

Procedure ‘cutting plane’

Max-cutting plane are added to the RMOLFP for the level curves of the objectives that.

Step-4
Procedure 'inter section'
Intersect the break point,

Step-5
Procedure 'boundary'
Form part (or all) of the boundary of E^R, and

Step-6
Procedure 'optimality'

6. Numerical Example of Molfp

We consider the following example of MOLFP in which there are three objectives in a polyhedral feasible region.

Example 6.1

$$\text{Max } \left\{ \frac{x_1 + 2x_2 + x_3 - x_4 - 4}{-x_1 + x_3 + x_4 + 3} = z_1(x) \right\}$$

$$\text{Max } \left\{ \frac{-x_1 + x_2 + 2x_3 + x_4 + 4}{x_2 + 2x_3 + x_4 + 1} = z_2(x) \right\}$$

$$\text{Max } \left\{ \frac{-x_1 + x_2 + x_3 - x_4}{-x_1 + x_2 + 2x_3 + 2x_4} = z_3(x) \right\}$$

Subject to the constraints

$$E^S = E^W - [\gamma(x^3, x^4) - x^2) - [\gamma(x^4, x^5) - x^4]$$

$$\text{Max}\{Z_1(v), Z_2(v), \ldots\ldots\ldots. Z_k(v)\}$$

$$\text{Eff}\{T(v).v | v \in R^{n-m}\}$$

unrestricted.

$$\max(r \in R^{n-m})$$

$$d_{1^+}, d_{1^-} \geq 0$$

We shall use the symbol γ for convex combination operator in the sense that the set of all convex combinations of the N-points x^1, x^2………x^n is written as.

$$\gamma(x^1, x^2 \ldots\ldots. x^n)$$

then the set of all w-efficient and all s-efficient solutions are given by

$$E^W = \gamma(x^1, x^2, x^3) \cup \gamma(x^2, x^3, x^4, x^5) \cup \gamma(x^5, x^6)$$

$$E^S = E^W - [\gamma(x^3, x^4) - x^2) - [\gamma(x^4, x^5) - x^4]$$

Then the augmented feasible region gives the following

$$E_a^W = \{x^1, x^2, x^3, x^4, x^5, x^6\}$$
$$E_v^S = E_a^W - \{x^3, x^4\}$$
$$E_x^W = \{x^1, x^4, x^6\}$$
$$E_x^S = E_x^W - \{x^4\}$$
$$z_i(x) \geq z_i(x)$$

Step-1

First w-efficient basis **(Table 6.1)**

	x_1	x_2	x_3	x_4	x_5	x_6	rhs
x_5	1	2	2	0	1	0	12
x_6	12	3	0	4	0	1	24

Step 2 to step 7 determined the following

2) x_1 is the edge initially w-efficient
3) x_5 is the edge initially w-efficient
4) along the x_5 edge introduce cutting plane $\theta_{max} > 0$ $(\hat{\theta} = 0)$
5) Enter the basis & (x_5, x_6) in L^W
6) Persuing the designated basis
7) Crash to the basis (x_5, x_6)

(Table 6.2)

	x_1	x_2	x_3	x_4	x_5	x_6	rhs

x_2	½	1	1	0	½	0	6
x_6	21/2	0	-3	1	-1/2	1	6

Step 8 to step 12 carried out the following operations.

8) x_1 edge initially efficient
9) along x_1 edge $\theta_{max} = 4$ $(\hat{\theta} = 4)$
10) enter the basis
11) defining the level curve $z_1 = 12$

12) Inserted the cutting plane θ_{max} in MOLFP and augmented with the following constraints.

$$x_1 - d_{1^+} + d_{1^-} = 4$$

$$d_{1^+}, d_{1^-} \geq 0$$

(Table 6.3)

	x_1	x_2	x_3	x_4	x_5	x_6	d_1	d_2	rhs
x_2	½	1	1	0	½	0	0	0	6
x_6	21/2	0	-3	1	-1/2	1	0	0	6
d_1	1	0	0	0	0	0	-1	1	4

Steps 13 to step 33 carried out to give the optimal solution.

$$\begin{Bmatrix} x_1^* = 2 \\ x_2^* = 6 \\ x_3^* = 0 \\ x_4^* = 0 \end{Bmatrix} \quad \begin{Bmatrix} z_1^{opt} = 10 \\ z_2^{opt} = 8/7 \\ z_3^{opt} = 1 \end{Bmatrix}$$

7. Conclusion

In this chapter we have presented a multi-objective linear fractional programming problem in the reduced frame work. We have devised a simplex based solution procedure for the RMOLFPP by a shift from the usual notation and defined a new kind of efficiency called w-efficient base. The method has important advantages of allowing the model builder and decision maker to include fractional objective.

This increases the possibility of application of multiple objective fractional programming to a wide variety of problems. The geometric properties presented in the set of efficient and weekly efficient solution of multiple linear fractional programming problem have been investigated and we recommend that the set of efficient point should be always closed and should have strong connectedness property.

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