ISSN 0974 - 9373

Vol. 15(2011) Special Issue 1 **Journal of International Academy of Physical Sciences** pp. 17-22

Common Fixed Point and Best Simultaneous Approximations Theorems Under Releaxed Conditions*

H. K. Pathak
School of Studies in Mathematics
Pt. Ravishankar Shukla University, Raipur, India
E-mail: hkpathak05@gmail.com

Satyaj Tiwari
Department of Mathematics, Shri Shankaracharya Institute of Professional Management and Technology, Raipur, India
Email: tsatyaj@yahoo.co.in

(Received June 14, 2011)

Abstract: In this paper ,we obtain some common fixed point theorems for P-operator pair and occasionally weakly compatible mapping on a set X together with the function $d: X \times X \to [0,\infty)$ without using the triangle inequality and assuming symmetry only on the set of points of coincidence. As an application, we have established best simultaneous approximation result. Our theorems generalize recent results existing in the literature.

Keywords: Common fixed point, Best simultaneous approximation, P- operator, Occasionally weakly compatible map

AMS Subject Classification : 47H10; 54H25

1. Introduction

Fixed point theorems for pairs of mappings involved additional hypothesis, in addition to contractive condition. First hypothesis is given a space X , containment of range of one mapping into the range of another mapping involved. Second hypothesis is some kind of commutivity condition. Sessa weakend the notion of commutivity as "weakly commutivity'' Which was generalised by Jungck[1] as "compitable mappings". More recently Althagafi and Shahzed[2] introduced two new classes of non-commuting mappings namely "occasionally weakly compatible" and "ultraoccasionally weakly compatible" mappings established common fixed point and invariant approximation theorems.

*Paper presented in CONIAPS XIII at UPES, Dehradun during June 14-16, 2011.

In this paper, we prove some common fixed point theorems and existence of best simultaneous approximation results under relaxed condition on d, assuming symmetry only on points of coincidence without using triangle inequality.

2. Preliminaries and Definition

Definition 2.1. Two self mappings f and g of a metric space (X,d) are said to be weakly commuting if $d(fgx,gfx) \leq d(fx,gx)$ for all $x \in X$.
It is clear that two commuting mappings are weakly commuting, but the converse is not true[3].

Definition 2.2 Two self mappings f and g of a metric space (X,d) are said to be compatible[1] if $lim_{n\to\infty} d(fgx_n, gfx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 2.3 Two self mappings f and g of a metric space (X,d) are said to be weakly compatible[4] if they commute at their coincidence points, i.e., $fx = gx$ for some $x \in X$, then $fgx = gfx$.
It is easy to see that two compatible maps are weakly compatible.

Definition 2.4 Let X be a set and let f, g be two self-mappings of X. A point x in X is called a coincidence point[5] of f and g iff $fx = gx$. we shall call $u = fx = gx$ a point of coincidence of f and g.

Definition 2.5 Two self map f and g of a set X are occasionally weakly compatible(OWC)[2] iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

Definition 2.6 Let (T, f) denote the set of coincident points of the pair (T, f). The pair (T, f) is called $P-$operator pair[6] if

$$d(u,Tu) \leq diam C(T, f), \quad \text{for some} \quad u \in C(T, f).$$

Clearly, Occasionally weakly compatible maps T and f are $P-$operator. If the self-maps f and T of X satisfy $T(C(T, f)) \subseteq C(T, f)$, then (T, f) is a weakly compatible pair and hence is $P-$operator pair.

Definition 2.7 Let X be a non-empty set and d be a function $d{:}X \times X \to [0,\infty)$ such that

(1) $d(x,y)=0$ *iff* $x=y$, *for all* $x,y \in X,$

A topology $\tau(d)$ is given by $U \in \tau(d)$ if and only if for each $x \in U$, $B(x,\varepsilon) \subset U$ for some $\varepsilon > 0$, where $B(x,\varepsilon) = \{y \in X{:}d(x,y) < \varepsilon\}$.

Definition 2.8 Let A,G are subsets of X, then where set A is bounded, we write

$$\mathrm{r}_{\mathrm{G}}(\mathrm{A}) = \inf_{\mathrm{g} \in \mathrm{G}} \sup_{\mathrm{a} \in \mathrm{A}} \mathrm{d}(\mathrm{a}, \mathrm{g}),$$

$$\mathrm{cent}_{\mathrm{G}}(\mathrm{A}) = \{\mathrm{g}_0 \in \mathrm{G}{:}\sup_{\mathrm{a} \in \mathrm{A}} \mathrm{d}(\mathrm{a}, \mathrm{g}_0) = \mathrm{r}_{\mathrm{G}}(\mathrm{A})\}.$$

The number $r_G(A)$ is called the *Chebyshev radius* of A w.r.t G and an element $y_0 \in cent_G(A)$ is called a *best simultaneous approximation* of A w.r.t G. If $A = \{u\}$, then $r_G(A) = d(u,G)$ and $cent_G(A)$ is the set of all best approximations, $P_G(u)$, of u out of G.
We also refer the reader to Milman[7], Sahney and Singh[3] and Vijayaraju[8] for further details.

3. Main Results

In this section, we prove some common fixed point theorems for a pair of P-operator and OWC mappings on space X, d, satisfying only condition (1). Let $\psi,\varphi{:}[0,\infty) \to [0,\infty)$ are both continuous and monotone decreasing function with $\psi(t)=0=\varphi(t)$ if and only if $t=0$.

Theorem 3.1. *Let X be a non empty set and $d{:}X \times X \to [0,\infty)$ be a function satisfying the condition (1). Suppose that T and f are occasionally weakly compatible(owc) self-mapping of X and satisfy the following condition:*

(2) $$\psi(d(Tx,Ty)) \le \psi(m(x,y)) - \phi(m(x,y)),$$

for all $x,y \in X$, where

$$m(x,y)=\max\Big\{d(fx,fy),d(fx,Tx),d(fy,Ty),\ \frac{1}{2}[d(fy,Tx),d(f\,x,Ty)]\Big\},$$

and $\psi,\varphi:\to[0,\infty)$ *are both continuous and monotone decreasing with* $\psi(t)=0=\varphi(t)$ *if and only if* $t=0$. *Then* T *and* f *have a common fixed point.*

Proof : Since T *and* f are owc, there exist a point u in X such that $Tu=fu$, $Tfu=fTu$. We claim that Tu is the unique common fixed point of T *and* f. We first assert that Tu is a fixed point of T. For, if $TTu\neq Tu$, then from (2) we get

(3) $$\psi(d(Tu,TTu))\le\psi(m(u,Tu)-\phi(m(u,Tu)),$$

where

$$m(u,Tu)=max\Big\{d(fu,fTu),d(fx,Tu),d(fTu,TTu),\ \frac{1}{2}[d(fTu,Tu)+d(fu,TTu)]\Big\},$$

$$=max\ \{d(Tu,TTu),d(Tx,Tu),d(TTu,TTu),\frac{1}{2}[d(TTu,Tu)+d(Tu,TTu)]\}$$

$$=max\ \{d(Tu,TTu),d(Tx,Tu),d(TTu,TTu),\frac{1}{2}[d(TTu,Tu)+d(TTu,Tu)]\}$$

$$=maxd\ (Tu,TTu),d(TTu,Tu)$$
$$=d(Tu,TTu).$$

From (3), we get

$$\psi(d(Tu,TTu))\le\psi(Tu,TTu)-\phi(Tu,TTu),$$

$$\phi(d(Tu,TTu))\le 0,$$

$$\phi(d(Tu,TTu))=0.$$

Hence $Tu=TTu$ and $TTu=Tfu=fTu=Tu$. Thus, Tu is a common fixed point of T and f.

Remark 3.1. By taking $\psi(t)=t$ and $\varphi(t)=1-\psi(t)$ in theorem 3.1, we obtain Theorem[9] 2.1.

Theorem 3.2. *Let* X *be a non empty set and* $d:X\times X\to[0,\infty)$ *be a function satisfying the condition* (1). *Suppose that* f,g,S,T *are self mapping of* X *and that the pair* (f,S) *and* (g,T) *are each* $P-$*operator, if*

(4) $d(u,w) = d(w,u)$,

where u and w are point of coincidence of (f,S) and (g,T) respectively, and,

(5) $\psi(d(fx, gy)) \le \psi(m(x, y)) - \varphi(m(x, y))$,

for all $x,y \in X$ *, where*

$$m(x,y) = max\ \{d(Sx,Ty), d(Sx,fx), d(Ty,gy), \frac{1}{2}[d(Sx,gy), d(Ty,fx)]\},$$

for all $x,y \in X$ *for which* $fx \ne gx$*, Then* f,g,S,T *have a unique common fixed point.*

Proof : Since the pair *(f,S)* and *(g,T)* are each $P-$operator, hence there exist x, y in X such that $fx = Sx = w$ and $gy = Ty = z$, we claim that $fx = gy$. For, otherwise, by (5)

$$\begin{aligned} m(x,y) &= max\ \{d(Sx,Ty), d(Sx,fx), d(Ty,gy), \frac{1}{2}[d(Sx,gy), d(Ty,fx)]\}, \\ &= max\ \{d(w,z), d(w,w), d(z,z), \frac{1}{2}[d(w,z), d(z,w)]\} \\ &= max\ \{d(w,z), 0, 0, \frac{1}{2}[d(w,z), d(z,w)]\} \\ &= d(w,z) = d(fx,gy). \end{aligned}$$

Hence

$$\begin{aligned} &\psi(d(fx,gy)) \le \psi(fx,gy) - \phi(fx,gy), \\ &\phi(d(fx,gy)) \le 0, \\ &\phi(d(fx,gy)) = 0. \end{aligned}$$

Hence, $fx = gy$; i.e., $fx = Sx = gy = Ty$.

Four self mappings defined in T heoerm 3.2 may be called generalised (ψ,φ) -weak contraction. The effectiveness of the generalization with respect to previous results may be seen from the fact trhat in the case $\psi(t) = t$, Theorem 3.2 still holds, while relation 13 of Theorem[9] 3.4 is useless.

Application to best simultaneous approximation

Theorem 3.3. *Let* X *be a non empty set and* $d{:}X \times X \to [0,\infty)$ *be a function satisfying the condition* (1). *Let* G *and* A *be nonempty subset of* X *such that* $G \cap A = \varphi$. *Suppose that there exist* g^{*} *in* G *and* a^{*} *in* A *such that* a^{*} *is an* $d-$*farthest point of* g^{*} *from* A *and* g^{*} *is an* $d-$*nearest point of* a^{*} *from K. Then* $cent_G(A) = d(g\ \ ,a\ \ )$ *and* $g\ \ \in cent_G(A)$

Proof : Let $g \in G$ then

$$d_A(g) = sup\{d(g,a) : a \in A\} \geq d(g,a^{*}) \geq d(g^{*},a^{*}) = d_A(g^{*}).$$

Hence $d_A(G) = inf\left\{d_A(g){:}g \in G\right\} = d_A(g\ \ )$ and $g \in cent_G(A)$.

References

1. G. Jungck, Compitable mappings and common fixed points, *Int. J. Math. Math. Sci*., **9** (1986) 771-779.
2. M. A. Al-Thagafi, N.Shahzad,Generalised I-nonexpansive selfmaps and Invarient approximation, *Acta mathematica Sinica, English series*, **24(5)** (2008) 867-876.
3. B. N. Sahney and S. P. Singh, On best simultaneous approximation, Approximation Theory III, *Academic Press*, (1980) 783-789.
4. G. Jungck, Common fixed points for noncontinuous nonself maps on non metric spaces, *Far East J. Math. Sci*., **4**(1996) 199-215.
5. G. Jungck and B. E.Rhoades, Fixed point theorem for occassionaly weakly compatible mappings, *Fixed Point Theory*, **7**(2006) 286-296.
6. H. K.Pathak and N. Hussain, Common Fixed Points for $P-$Operator pair With Applications, to appear in A.M.C.
7. P. D. Milman, On best simultaneous approximation in normed linear spaces, *J. Approximation Theory*, **20**(1977) 223-238.
8. P. Vijayraju, Applications of fixed point theorem to best simultaneous approximations, *Indian J. pure appl. Math*., **24(1) (**1993) 21-26.
9. A. Bhatt, H. Chandra and D. R. Sahu, Common fixed points theorems for occasionnaly weakly compitable mappings under relaxed conditions, Nonlinear Analysis, **73** (2010) 176-182.
10. M. A. Al-Thagafi, N. Shahzad, Noncommuting selfmaps and invarient approximations, *Nonlinear Anal*., **64** (2006) 2778-2786.
11. R. Kannan, Some results on fixed points, *Bull.Calcutta Math. Soc*., **60** (1968) 71-76.
12. G. Jungck, Commuting mappings and fixed points, *Amer. Math. Monthly*, **83** (1976) 261-263.
13. S. Sessa, On weak comuutative condiyion of mappings in fixed point considerations, *Publ. Inst. Math. N. S*., **32(46)** (1982) 149-153.