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Common Fixed Points in Fuzzy Metric Spaces Involving Conditionally Reciprocally Continuous Self Mappings

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Abstract: The aim of this paper is to consider a new approach for obtaining common fixed point theorems in Fuzzy metric spaces by subjecting new continuity condition, conditionally reciprocally continuous selfmappings introduced by Pant[1], which is independent of the known continuity definitions. We give examples and initiate the application of, conditionally reciprocally continuous selfmappings for investigating fixed points of mappings in fuzzy metric space.

Key Words: Fuzzy Metric Space, Common Fixed Point, Conditionally Reciprocally Continuous Self mappings, property (E.A)

Mathematics Subject Classification: 47 H 10, 54 H 25

1. Introduction

The introduction of the notion of fuzzy sets by Zadeh[24] in 1965 laid down the foundation of fuzzy mathematics and since then the discipline of fuzzy mathematics has witnessed tremendous growth in diverse directions. Kramosil and Michalek[9] in 1975 introduced the notion of fuzzy metric space. George and Veeramani[6] modified the definition of fuzzy metric space due to Kramosil and Michalek[9] with a view to introduce a Husdorff topology on fuzzy metric spaces. Grabiec[7] in 1988 extended the well known Banach contraction theorem to fuzzy metric spaces. In recent years, interesting generalizations of the Grabiec Fuzzy Contraction Theorem[7] have been obtained by Cho *et al*[4], Balasubramaniam *et al*[2], Sharma[18], Singh and Chauhan[19] and Vasuki[22]. Sharma *et al*[17] studied fixed points of fuzzy mappings in linear metric spaces and Sharma[18] have also discussed the investigations into the possible connection between existence and unicity of fixed points and functioning of biological systems. In an interesting development, Vasuki[23] in 1999 extended the well known Boyd and Wong[3] fixed point theorem to fuzzy metric spaces by introducing a fuzzy analogue of the ϕ-contractive condition of Boyd and Wong[3]. Recently, Chugh and Kumar[5] further generalized the Vasuki Fuzzy Contraction Theorem by extending it to four mappings.

In the study of reciprocal continuity, Pant[11] has established a situation in which a collection of mappings has a fixed point which is a point of discontinuity for all mappings. The utility of reciprocally continuous selfmaps in diverse settings to establish fixed point theorems can be understood from the fact that while studying the common fixed point theorems which may admit discontinuity at the fixed point.

Two selfmaps f and g of a metric space (X, d) are called reciprocally continuous[11], whenever $\{x_n\}$ is a sequence in X satisfying $lim_n fgx_n = f\,t$ and $lim_n gfx_n = g\,t$ for some t in X such that $lim_n fx_n = f\,t$ and $lim_n gx_n = g\,t$. If f and g are both continuous then they are obviously reciprocally continuous but the converse is not true[11].

Two selfmaps f and g of a metric space (X, d) are called conditionally reciprocally continuous (CRC)[13], whenever $\{x_n\}$ is a sequence in X satisfying $lim_n fx_n = lim_n gx_n$ is nonempty, there exists a sequence $\{y_n\}$ satisfying $lim_n fy_n = lim_n gy_n = t$ for some t in X such that $lim_n fx_n = lim_n gx_n = t$.

Two selfmaps f and g of a metric space (X, d) are called g- compatible[16], if $lim_n d(ffx_n, gfx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some t in X and two selfmaps f and g of a metric space (X, d) are called f- compatible[16], if $lim_n d(fgx_n, ggx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some t in X.

Two selfmaps f, g of a metric space (X, d) are called R-weakly commuting[10] if there exists some real number R > 0 such that $d(fgx, gfx) \leq R(d(fx, gx))$ for all x in X. f and g are called pointwise R-weakly commuting if given x in X, there exists $R > 0$ such that $d(fgx, gfx) \leq R(d(fx, gx))$.

Two selfmaps f and g of a metric space (X, d) are called compatible [8] if $lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some t in X. It is clear from the above definition that f and g will be noncompatible if there exists at least one sequence $\{x_n\}$ such that $lim_n fx_n = lim_n gx_n = t$ for some t in X but $lim_n d(fgx_n, gfx_n)$ is either non-zero or non-existent. Compatibility implies pointwise R-weak commutativity since compatible maps commute at their coincidence points.

In the present paper, we obtain common fixed point theorems for conditionally reciprocally continuous selfmappings. We now give some relevant definitions.

Definition 1: Let X be any set. A fuzzy set in *X* is a function with domain X and values in [0, 1].

Definition 2 : A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is defined to be a *continuous t-norm* if ([0, 1], $*$) is an abelian topological monoid with unit 1 such that $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for all a, b, c, d $\in$ [0, 1]. Examples of t-norm are $a*b = ab$ and $a*b = min\{a, b\}$.

Definition 3: A 3-tuple (X, M, $*$) is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$,

(i) $M(x, y, 0) = 0$

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$

(iii) $M(x, y, t) = M(y, x, t) \neq 0$ for $t \neq 0$

(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$

(v) $M(x, y, .) : [0, \infty) \to [0, 1]$ is continuous.

In this paper, (X, M, $*$) will denote a fuzzy metric space in the sense of the above definition with the following condition

(vi) $\lim_{t \to \infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

M(x, y, t) can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with $d(x, y) = \infty$. The following example shows that every metric induces a fuzzy metric.

Example 1: Let (X, d) be a metric space. Define $a*b = ab$ or $a*b = \min\{a, b\}$ and for all $x, y \in X$, $t > 0$

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then (X, M, $*$) is a fuzzy metric space and the fuzzy metric M induced by the metric d is often referred to as the standard fuzzy metric.

Definition 4: If (X, M, $*$) is a fuzzy metric space, a sequence $\{x_n\}$ in X is said to converge to a point x in X (denoted by $\lim_{n \to \infty} x_n = x$) if $\lim_{n \to \infty} M(x_n, x, t) = 1$ for all $t > 0$.

Definition 5: If (X, M, $*$) is a fuzzy metric space, a sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ for all $p > 0$, $t > 0$. A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 6: A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is a Cauchy Sequence if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_m, x_n, t) > 1 - \varepsilon$ for all $m, n \geq n_0$, where N is the set of natural numbers.
G. Song [21] has proposed the following definition

Definition 7: A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is defined to be a Cauchy sequence if $M(x_{n+p}, x_n, t) \to 1$ (for all $t > 0$) as $n \to \infty$ uniformly on $p \in N$, N being the set of natural numbers.

Definition 8: A mapping f of a fuzzy metric space (X, M, *) is called continuous if $\lim_n fx_n = fz$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n x_n = z$.

Results

Theorem 1: *Let (X, M, *) be a complete fuzzy metric space and let f and g be conditionally reciprocally continuous selfmappings satisfying the conditions*

(i) $fX \subset gX$

(ii) $M(fx, fy, t) \geq \{M(gx, gy, t)\}$

(iii) $M(fx, ffx, t) > \{M(gx, ggx, t)\}$ *whenever* $gx \neq ggx$

If f and g are either compitable or g – compitable or f – compitable then f and g have a unique common fixed point.

Proof: Let x_0 be any point in *X*. Then $fX \subset gX$, define sequences $\{x_n\}$ and $\{y_n\}$ in *X* given by the rule $s_n = fx_n = gx_{n+1}$, $n = 0,1,2,\ldots$.
We claim that $\{s_n\}$ is a Cauchy sequence. Using (ii) we obtain
$M(fx_n, fx_{n+1}, t) \geq \{M(gx_n, gx_{n+1}, t\}$, $t > 0$
$M(s_n, s_{n+1}, t) \geq \{M(s_{n-1}, s_n, t\}$, $t > 0$

$$M(s_n, s_{n+1}, t) \geq M(s_{n-1}, s_n, t). \quad (1)$$

To prove that $\{s_n\}$ is a Cauchy sequence we prove that (1) is true for all $n \geq n_0$ and for every $m \in N$,

$$M(s_n, s_{n+m}, t) > 1-\lambda. \quad (2)$$

Here we use induction method.
$M(s_n, s_{n+1}, t) \geq M(s_{n-1}, s_n, t) \geq M(s_{n-2}, s_{n-1}, t) \geq \ldots\ldots \geq M(s_0, s_1, t) \to 1$ as $n \to \infty$ i.e. for $0 \leq \lambda < 1$, $t > 0$, we can choose $n_0 \in N$, such that $M(s_n, s_{n+1}, t) > 1-\lambda$. Thus (2) is true for m = 1. Suppose (2) is true for m then we shall show that it is also true for m + 1. By the definitions of fuzzy metric space, we have

$$(s_n, s_{n+m+1}, t) \geq \{M(s_n, s_{n+m}, t), M(s_{n+m}, s_{n+m+1}, t)\} > 1\text{-}\lambda .$$

Hence (2) is true for $m + 1$. Thus $\{s_n\}$ is a Cauchy sequence. By completeness of (X, M, *), $\{ s_n \}$ converges to some point t in X. Moreover, $lim_n fx_n = lim_n gx_n = t$. Since conditionally reciprocally continuous selfmappings of f and g implies that $lim_n fx_n = lim_n gx_n = t$ for some t in X there exists a sequence $\{y_n\}$ satisfying $lim_n fy_n = lim_n gy_n = u$ such that. $lim_n fgy_n = f\,u$ and $lim_n gfy_n = gu$. Since $fX \subset gX$, for each y_n there exists a z_n in X such that $lim_n fy_n = lim_n gz_n$. Thus $lim_n fy_n \rightarrow u$, $lim_n gy_n \rightarrow u$. Therefore, we have

(3) $$\lim_n gy_n \rightarrow u, \lim_n fz_n \rightarrow u \lim_n fy_n = \lim_n gz_n \rightarrow u.$$

Since f and g are compatible, then $\lim_n M(fgy_n, gfy_n, t) = 1$, that is $fu = gu$. Again , since compatibility implies commutativity at coincidence points, we get $ffu = fgu = gfu = ggu$. If $fu \neq ffu$, Using (iii), we get

$$M(fu, ffu, t) > max\{M(gu, ggu, t)]\} = M (fu, ffu, t)$$

a contradiction. Hence, $fu = ffu$ and $fu = ffu = fgu = gfu = ggu$. Hence fu is a common fixed point of f and g. The case when fX is a complete subspace of X is similar to the above case since $fX \subset gX$. Since f and g are g - compatible, then $\lim_n M(ffy_n, gfy_n, t) = 1$, that is $ffy_n \rightarrow gu$ using (ii), we get

$$M(fu, ffy_n, t) \geq max\{M(gu, gfy_n, t)\} = M(gu, gfy_n, t).$$

On letting $n \rightarrow \infty$ we get $fu = gu$. Now g – compatibility implies commutativity at coincident points, we obtain fgu = gfu and fu $\neq ffu = fgu = gfu = ggu$. Using (iii), we get

$$M(fu, ffu, t) > max\{M(gu, ggu, t)]\} = M (fu, ffu, t)$$

a contradiction. Hence, $fu = ffu$ and $fu = ffu = fgu = gfu = ggu$. Hence fu is a common fixed point of f and g. The case when fX is a complete subspace of X is similar to the above case since $fX \subset gX$.
Since f and g are f - compatible, then $\lim_n M(fgy_n, ggy_n, t) = 1$, by (3) $ggy_n = gfy_n \rightarrow gu$ we get $fgy_n \rightarrow gu$ Using (ii), we get

$$M(fu, fgy_n, t) \geq max\{M(gu, ggy_n, t\} = M (gu, ggy_n, t).$$

On letting $n \to \infty$ we get $fu = gu$, since $h < 1$. Now f – compatibility implies commutativity at coincident points, we obtain $fgu = gfu$ *and* $fu \neq ffu = fgu = gfu = ggu$. Using (iii), we get

$$M(fu, ffu, t) > \max\{M(gu, ggu, t)]\} = M(fu, ffu, t)$$

a contradiction. Hence, fu = ffu and fu = ffu = fgu =gfu = ggu. Hence fu is a common fixed point of f and g. Hence the theorem.

Example 2: Let X = [2, 20] equipped with the fuzzy metric on (X,M,*). Define a*b = ab or a*b = min{a, b}, a, b in [0, 1] and for all x, y ∈ X, t > 0,

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

Let $f, g: X \to X$ be defined by

$$f2 = 2, \ \text{if } x = 2 \text{ or } > 5, \qquad fx = 6 \text{ if } 2 < x \leq 5,$$
$$g2 = 2, \ gx = x + 4 \text{ if } 2 < x \leq 5, \quad gx = (4x + 10)/15 \text{ if } x > 5.$$

Then f and g satisfy all the conditions of Theorem 1 and have a unique common fixed point $x = 2$. In this example $fX = \{2\} \cup \{6\}$ and $gX = [2, 6] \cup \{7\}$. It may be seen that $fX \subset gX$. It can be verified also that f and g are conditionally reciprocally continuous, let $\{x_n\}$ be a sequence given by $x_n = 2$. Then $f x_n \to 2$, $gx_n \to 2$. Also $fg x_n \to 2 = f2 = 2$, $gfx_n \to 2 = g2 = 2$. Hence f and g are conditionally reciprocally continuous. In this case, if we have $x_n = 2$ the mappings f and g are compatible since the compatibility implies commutativity at coincidence points. If we consider by $x_n = 5+1/n : n>1$, then $f x_n = 2$, $gx_n \to 2$. Also $fgx_n = 6$, $gfx_n = 2$. Hence f and g are non-compatible. Similarly considering again $x_n = 5+1/n : n >1$, then $f x_n \to 2$, $gx_n \to 2$. Also $fgx_n = f(2+1/n) = 6 \neq f2$, $gfx_n = g2 = 2$. Therefore $gfx_n = g2$ but $fgx_n \neq f2$. Hence f and g are not reciprocally continuous. The mappings f and g are g – compatible and f – compatible also.

Theorem 2: *Let (X, M, *) be a complete fuzzy metric space and let f and g be conditionally reciprocally continuous selfmappings satisfying the conditions*

(iv) $fX \subset gX$

(v) $M(fx, fy, t) \geq \{M(gx, gy, th)\}$, $0 \leq h < 1$, $t > 0$.

If f and g satisfy the property (E.A) and the range of either f or g is a complete subspace of X, then f and g have a unique common fixed point.

Proof: Since f and g satisfy the property (E.A), there exists a sequence $\{x_n\}$ such that $fx_n \to t$ and $gx_n \to t$ for some t in X. Since $fX \subset gX$, there exists some point u in such that $t = gu$ where $t = \lim_n gx_n$.

Since conditionally reciprocally continuous selfmappings of f and g implies that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = \lim_n gy_n = u$ such that. $\lim_n fgy_n = f\,u$ and $\lim_n gfy_n = gu$. Since $fX \subset gX$, for each y_n there exists a z_n in X such that $\lim_n fy_n = \lim_n gz_n$. Thus $\lim_n fy_n \to u$, $\lim_n gy_n \to u$. Therefore, we have

(4) $$\lim_n gy_n \to u,\ \lim_n fz_n \to u,\ \lim_n fy_n = \lim_n gz_n \to u.$$

Since $t \in fX$ and $fX \subset gX$, if u in X such that $t = gu$ where $t = \lim_n gx_n$. If $fu \neq gu$, the inequality

$$M(fx_n, fu, t) \geq \max\{M(gx_n, gu, th\}.$$

On letting $n \to \infty$ yields, $M(gu, fu, t) \geq \max\{M(gu, gu, th\}$. Hence $fu = gu$. Again $M(ffu, fgu, t) \geq \max\{M(gfu, ggu, th)\}$, that is, $ffu = fgu$ and $ffu = fgu = gfu = ggu$. If $fu \neq ffu$, using (v), we get $M(fu, ffu, t) > \max\{M(gu, gfu, th)\} = M(fu, ffu, th)$
a contradiction. Hence, $fu = ffu$ and $fu = ffu = fgu = gfu = ggu$. Hence fu is a common fixed point of f and g. The case when fX is a complete subspace of X is similar to the above case since $fX \subset gX$. Hence we have the theorem.

Example 3: Let X = [2, 20] equipped with the fuzzy metric on (X, M, *). Define $a*b = ab$ or $a*b = \min\{a, b\}$, a, b in [0, 1] and

for all $x, y \in X$, $t > 0$, $M(x, y, t) = \dfrac{t}{t + |x - y|}$.

Let $f, g: X \to X$ be defined by

$f2 = 2$, if $x = 2$ or > 5, $fx = 6$ if $2 < x \leq 5$,

$g2 = 2$, $gx = x + 4$ if $2 < x \leq 5$, $gx = (4x + 10)/15$ if $x > 5$.

Then f and g satisfy all the conditions of Theorem 1 and have a unique common fixed point $x = 2$. In this example $fX = \{2\} \cup \{6\}$ and $gX = [2, 6] \cup \{7\}$. It may be seen that $fX \subset gX$. It can be verified that f and g are satisfy the property (E.A).

Remark: Property (E.A) of Aamri and Moutwakil[1] is more general then the notion of noncompatibility. It is however, worth to mention here that if we take noncompatibility aspect instead of the property (E.A) we can show, in addition, that the mappings are discontinuous at the common fixed point. Aforesaid results illustrate our assertion in the fuzzy metric fixed point theory. This is, however also true for the study of fixed points in metric space.

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