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WC* Partner Curves in E_1^3

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Abstract: In this paper, we investigate the existence/non-existence of WC* partner curves in $\mathbb{E}^3$ and $\mathbb{E}_1^3$. We obtain that there do not exist WC* partner curves in $\mathbb{E}^3$ and $\mathbb{E}_1^3$.

Keywords: Alternative frame, darboux vector, slant helices, euclidean space, minkowski space.

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1. Introduction

The fundamental theory of curves, their characterization and corresponding relations between the curves are very interesting and important topics in Differential geometry. Two curves that have some special properties at some corresponding points are called curve pairs. One of the most interesting curve pairs is the Bertrand curve which was discovered by J. Bertrand in 1850. It has a special property that the principal normal vector of α coincides with the principal normal vector of α^* at the corresponding points of curves. The characterization for these types of curves is a Bertrand curve if and only if $ak(s)+b\tau(s)=1$ holds, where $k(s)$ and $\tau(s)$ represents the curvature and torsion of the curve, respectively and a, b are real constants[1]. In[2], Matsuda and Yorozu gave a definition of the generalized Bertrand curve in $\mathbb{E}^4$. In[3], Balgetir et al. defined a non-null Bertrand curve in 3-dimensional Lorentzian space. In[4], Ucum and Ilarslan studied a timelike Bertrand curve in $\mathbb{E}_1^3$. In[5], Ilarslan and Aslan studied a spacelike Bertrand curve in $\mathbb{E}_1^3$. In[6], Ucum et al. defined a generalized Bertrand curve with a timelike (1, 3)-normal plane in

Minkowski space-time. Recently in[7], Uzunoglu et al. introduced an alternative frame (N, C, W) in $\mathbb{E}^3$ where N is a unit normal vector, W is Darboux vector, and $C = W \times N$. In[8], Yilmaz and Has defined WC^* partner curves in $\mathbb{E}^3$ as the vector field W of α coincides with the vector field C^* of α^* at the corresponding points of the curves. In view of this, we investigate the existence/non-existence of WC^* partner curves in $\mathbb{E}^3$ and $\mathbb{E}_1^3$.

2. Preliminaries

Let $\alpha = \alpha(s)$ be a regular unit speed curve in the $\mathbb{E}^3$ where s measures its arc length. Then, the Frenet formula for the Frenet frame (T, N, B) of the curve α is given as[1]:

(2.1) $$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where $k(s)$ and $\tau(s)$ are curvature and torsion of α, respectively. When the Frenet frame moves along a curve α in $\mathbb{E}^3$, there exists an axis of instantaneous frame's rotation and the direction of such an axis is given by the Darboux vector. From (2.1), the unit Darboux vector W of α is defined as:

(2.2) $$W = \frac{\tau T + kB}{\sqrt{\tau^2 + k^2}}.$$

Uzunoglu et al. introduced the alternative frame (N, C, W) and its derivative formula in $\mathbb{E}^3$ are given as[7]:

(2.3) $$\begin{pmatrix} N \\ C \\ W \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{-k}{\sqrt{\tau^2 + k^2}} & 0 & \dfrac{\tau}{\sqrt{\tau^2 + k^2}} \\ \dfrac{\tau}{\sqrt{\tau^2 + k^2}} & 0 & \dfrac{k}{\sqrt{\tau^2 + k^2}} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

(2.4) $$\begin{pmatrix} N' \\ C' \\ W' \end{pmatrix} = \begin{pmatrix} 0 & f & 0 \\ -f & 0 & g \\ 0 & -g & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix},$$

where

(2.5) $$f = k\sqrt{1+H^2}, \; g = \frac{H'}{1+H^2},$$

(2.6) $$C = -\tilde{k}T + \tilde{\tau}B, \qquad W = \tilde{\tau}T + \tilde{k}B,$$

(2.7) $$T = -\tilde{k}C + \tilde{\tau}W, \; B = \tilde{\tau}C + \tilde{k}W,$$

and $$\tilde{k} = \frac{k}{f}, \tilde{\tau} = \frac{\tau}{f}.$$

On the other hand, the Lorentz-Minkowski $\mathbb{E}_1^3$ is a space with metric,

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system. With respect to this metric, an arbitrary vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is said to be spacelike if $\langle \alpha, \alpha \rangle > 0$, timelike if $\langle \alpha, \alpha \rangle < 0$ and lightlike if $\langle \alpha, \alpha \rangle = 0$. Similarly, if $\alpha = \alpha(s)$ denotes the position vector of an arbitrary non-null curve in $\mathbb{E}_1^3$, then it is called timelike and spacelike if all of its velocity vector $\alpha'(s)$ are timelike and spacelike, respectively. The norm of the vector α is given by $\| \alpha' \| = \sqrt{| \langle \alpha', \alpha' \rangle |}$. A non-null curve $\alpha(s)$ is parameterized by arc length s if $\langle \alpha'(s), \alpha'(s) \rangle = \pm 1$.

Let $\alpha(s)$ be a regular unit speed curve in $\mathbb{E}_1^3$ where s measures its arc length. Then, the Frenet-Serret formula of the curve α is given in[9] as follows:

(2.8) $$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_1 k & 0 \\ -\epsilon_0 k & 0 & \epsilon_2 \tau \\ 0 & -\epsilon_1 \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where $\langle T,T\rangle=\epsilon_0$, $\langle N,N\rangle=\epsilon_1$, $\langle B,B\rangle=\epsilon_2$, and $\epsilon_0,\epsilon_1,\epsilon_2\in\{-1,1\}$, and $T\times N=B$, $N\times B=\epsilon_0\epsilon_2 T$, $B\times T=\epsilon_1\epsilon_2 N$, and $k(s)$, $\tau(s)$ are curvature and torsion of α.

The unit Darboux vector of the unit speed non-null curve α is given in[9] as follows:

(2.9) $$W=\frac{(\epsilon_2\tau T+\epsilon_2 kB)}{\sqrt{|\epsilon_0\tau^2+\epsilon_2 k^2|}}.$$

3. WC* Partner Curves in $\mathbb{E}^3$

Definition[8] 3.1: *Let $\alpha(s)$ and $\alpha^*(s^*)$ be two regular space curves in $\mathbb{E}^3$ parameterized by its arc length s and s^*, having Frenet frames (T, N, B) and (T^*,N^*,B^*) with curvatures k and k^*, and torsions τ and τ^*, respectively. Also, let (N, C, W) and (N^*,C^*,W^*) be the alternative moving frames of curves with alternative curvatures f, g, and f^*,g^*, respectively. Curves α and α^* are called WC^* partner curves if the vector fields W and C^* coincide, i.e., $W=C^*$ holds at the corresponding points of curves. The parametric representation of $\alpha^*(s^*)$ is defined as*

(3.1) $$\alpha^*(s^*)=\alpha(s)+\lambda(s)W(s),$$

where $\lambda=\lambda(s)$ denotes the distance function between corresponding points of curves α and α^*.

If θ is the angle between vector fields W^* and N, then from[8]

(3.2) $$N^*=cos\theta C+sin\theta N,\quad W^*=sin\theta C-cos\theta N,$$

(3.3) $$N=-cos\theta W^*+sin\theta N^*,\quad C=sin\theta W^*+cos\theta N^*.$$

Theorem 3.1: *There do not exist any WC* partner curves in $\mathbb{E}^3$.*

Proof: Differentiating (3.1) with respect to s, we have

(3.4) $$T^*\frac{ds^*}{ds}=T+\lambda' W-\lambda gC.$$

Using (2.6) and (2.7), we get

$$(-\tilde{k}^* C^* + \tilde{\tau}^* W^*)\frac{ds^*}{ds} = -\tilde{k}C + \tilde{\tau}W + \lambda' W - \lambda g C\,,$$

wherein using the second relation of (3.3) and $W = C^*$, we find

$$\left(-\tilde{k}^* C^* + \tilde{\tau}^* W^*\right)\frac{ds^*}{ds} = -\tilde{k}\left(\sin\theta W^* + \cos\theta N^*\right) + \tilde{\tau}W + \lambda' W - \lambda g\left(\sin\theta W^* + \cos\theta N^*\right). \tag{3.5}$$

Comparing coefficients of C^*, N^*, and W^* in (3.5), we obtain

$$\lambda' = -\tilde{\tau} - \tilde{k}^* \frac{ds^*}{ds}, \tag{3.6}$$

$$\lambda = -\frac{\tilde{k}}{g}\,, \tag{3.7}$$

and

$$\lambda = \frac{\left(\tilde{\tau}^* \dfrac{ds^*}{ds} + \tilde{k}\sin\theta\right)}{-g\sin\theta}\,, \tag{3.8}$$

respectively.

Now, taking the inner product of (3.4) with itself, we get

$$\left(\frac{ds^*}{ds}\right)^2 = 1 + \lambda'^2 + \lambda^2 g^2 + 2\lambda'\tilde{\tau} + 2\lambda g \tilde{k}\,. \tag{3.9}$$

Using (3.8) and (3.6) in (3.9), we obtain

$$\left(\frac{ds^*}{ds}\right)^2 = 1 + \tilde{\tau}^2 + \tilde{k}^{*2}\left(\frac{ds^*}{ds}\right)^2 + \frac{1}{(\sin\theta)^2}\left(\tilde{\tau}^* \frac{ds^*}{ds} + \tilde{k}\sin\theta\right)^2 \tag{3.10}$$

$$-2\left(\tilde{\tau}+\tilde{k}^{*}\frac{ds^{*}}{ds}\right)\tilde{\tau}-\frac{2\tilde{k}}{sin\theta}\left(\tilde{\tau}^{*}\frac{ds^{*}}{ds}+\tilde{k}\,sin\theta\right)+2\tilde{\tau}\tilde{k}^{*}\frac{ds^{*}}{ds}.$$

Further, solving (3.10), we find

$$\tilde{\tau}^{*2}cos^{2}\theta\left(\frac{ds^{*}}{ds}\right)^{2}=0, \tag{3.11}$$

whereby we get $\tilde{\tau}^{*}=0$ as $cos\theta\neq 0$ and $\frac{ds^{*}}{ds}\neq 0$. Since $\tilde{\tau}^{*}=\frac{\tau^{*}}{f^{*}}$, which gives $\tau^{*}=0$. Hence, the curve α^{*} has to be a plane curve.

On the other hand, differentiating the first relation of (3.2) with respect to s, we get

$$f^{*}C^{*}\frac{ds^{*}}{ds}=cos\theta\left(\frac{d\theta}{ds}-f\right)N+sin\theta\left(-\frac{d\theta}{ds}+f\right)C+g\,cos\theta W. \tag{3.12}$$

Since $W=C^{*}$, therefore from (3.12), we have

$$f=\frac{d\theta}{ds}. \tag{3.13}$$

Using (3.13) in (3.12), we find

$$f^{*}C^{*}\frac{ds^{*}}{ds}=g\,cos\theta W. \tag{3.14}$$

Similarly, differentiating the second relation of (3.2) with respect to s and using (3.13), we obtain

$$-g^{*}C^{*}\frac{ds^{*}}{ds}=g\,sin\theta W. \tag{3.15}$$

Using (3.14) and (3.15), we find

$$\frac{g^{*}}{f^{*}}=-tan\theta,$$

which gives

$$\frac{k^{*2}}{(k^{*2}+\tau^{*2})^{3/2}}\left(\frac{\tau^{*}}{k^{*}}\right)' = -tan\theta \,. \tag{3.16}$$

Putting $\tau^{*}=0$ in (3.16), we get $-tan\theta = 0$, which implies $\theta = 0$. Using this in second relation of (3.2), we have

$$W^{*} = -N \,. \tag{3.17}$$

Differentiating (3.17) with respect to s and using (2.4), we get

$$-g^{*}C^{*}\frac{ds^{*}}{ds} = -fC \,. \tag{3.18}$$

Since $\tau^{*}=0$, we get $g^{*}=0$ and using this in (3.18), we get

$$f = 0 \,. \tag{3.19}$$

Using (3.19) in (2.5), we have $0 = k^{2}+\tau^{2}$, which gives

$$k = 0, \text{ and } \tau = 0 \,. \tag{3.20}$$

Using (3.20) in (2.2), we get the Darboux vector $W = 0$, a contradiction, which completes the proof.

Remark 3.1: In[8], the authors studied WC* partner curves in $\mathbb{E}^{3}$. They obtained the following expressions on page 7 in lines 8, 13, and 14:

$$T^{*}\frac{ds^{*}}{ds} = T + \lambda' W + \lambda W', \tag{3.21}$$

$$\tilde{k}^{*}\frac{ds^{*}}{ds} = -(\tilde{\tau} + \lambda'), \tag{3.22}$$

$$\tilde{\tau}^{*}\frac{ds^{*}}{ds} = -(\tilde{k} + \lambda g)sin\theta \,. \tag{3.23}$$

Unfortunately, they did not solve (3.21), (3.22) and (3.23), further. Infact taking the inner product of (3.21) with itself, we get

$$\left(\frac{ds^*}{ds}\right)^2 = 1 + \lambda'^2 + \lambda^2 g^2 + 2\lambda'\tilde{\tau} + 2\lambda g\tilde{k}\,. \tag{3.24}$$

Using (3.22) and (3.23) in (3.24), we obtain

$$\tau^* = 0\,. \tag{3.25}$$

Using (3.25) and following the proof of Theorem 3.2 of the present paper, we get $W = 0$, which is a contradiction.

4. WC* Partner Curves in $\mathbb{E}_1^3$

Similar to an alternative frame given in[8] in $\mathbb{E}^3$, we obtain an alternative frame (N, C, W) for a curve $\alpha(s)$ in $\mathbb{E}_1^3$ as follows:

$$\begin{pmatrix} N \\ C \\ W \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{-\epsilon_0 k}{\sqrt{|\epsilon_0\tau^2 + \epsilon_2 k^2|}} & 0 & \dfrac{\epsilon_2 \tau}{\sqrt{|\epsilon_0\tau^2 + \epsilon_2 k^2|}} \\ \dfrac{\epsilon_2 \tau}{\sqrt{|\epsilon_0\tau^2 + \epsilon_2 k^2|}} & 0 & \dfrac{\epsilon_2 k}{|\sqrt{\epsilon_0\tau^2 + \epsilon_2 k^2}|} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{4.1}$$

Theorem 4.1: *Let $\alpha = \alpha(s)$ be a unit speed regular non-null curve in $\mathbb{E}_1^3$. Then, the derivative formula for an alternative frame (N, C, W) is given by:*

$$\begin{pmatrix} N' \\ C' \\ W' \end{pmatrix} = \begin{pmatrix} 0 & f & 0 \\ -\epsilon_0\epsilon_1\epsilon_2 f & 0 & \epsilon_2 g \\ 0 & -\epsilon_0 g & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix}, \tag{4.2}$$

where

(4.3) $$f = k\sqrt{|\epsilon_0 H^2 + \epsilon_2|},\ \ g = \frac{H'}{|(\epsilon_0 H^2 + \epsilon_2)|},\ \ H = \frac{\tau}{k}.$$

Proof: Differentiating (2.9) with respect to arc-length parameter s and using (2.8), we get

$$W' = \frac{(k\tau' - k'\tau)(kT - \epsilon_0\epsilon_2\tau B)}{(\epsilon_0\tau^2 + \epsilon_2 k^2)^{3/2}},$$

which gives

(4.4) $$W' = -\epsilon_0 gC.$$

Using (2.9) in $C = W \times N$, we obtain

(4.5) $$C = \frac{(\epsilon_2\tau B - \epsilon_0 kT)}{\sqrt{|\epsilon_0\tau^2 + \epsilon_2 k^2|}}.$$

Differentiating (4.5) with respect to arc-length parameter s and using (2.8), we have

$$C' = \frac{\epsilon_2 k^2}{(\epsilon_0\tau^2 + \epsilon_2 k^2)}\frac{(\epsilon_2\tau T + \epsilon_2 kB)}{\sqrt{|\epsilon_0\tau^2 + \epsilon_2 k^2|}}\left(\frac{\tau}{k}\right)' - \epsilon_0\epsilon_1\epsilon_2\sqrt{|\epsilon_0\tau^2 + \epsilon_2 k^2|}N,$$

which gives

(4.6) $$C' = \epsilon_2 gW - \epsilon_0\epsilon_1\epsilon_2 fN.$$

Using (2.8) in (4.5), we find

(4.7) $$N' = fC$$

Combining (4.4), (4.6) and (4.7), we complete the proof of Theorem 3.1.

From (4.1) and (4.3), we obtain

(4.8) $$C = -\epsilon_0\tilde{k}T + \epsilon_2\tilde{\tau}B,\quad W = \epsilon_2\tilde{\tau}T + \epsilon_2\tilde{k}B,$$

(4.9) $$T=\epsilon_0\epsilon_2(\tilde{\tau}W-\tilde{k}C),\quad B=\epsilon_0\epsilon_2\tilde{\tau}C+\tilde{k}W\,,$$

where $\tilde{k}=\dfrac{k}{f}$, $\tilde{\tau}=\dfrac{\tau}{f}$.

From the definition of WC^* partner curves, the parametric representation of $\alpha^*(s^*)$ is as follows

(4.10) $$\alpha^*(s^*)=\alpha(s)+\lambda(s)W(s)\,.$$

From (2.9) and (4.5), we have

(4.11) $$\langle W,W\rangle=\epsilon=\pm1, \langle C,C\rangle=-\epsilon\epsilon_1, \langle N,N\rangle=\epsilon_1\,.$$

Now, we consider following cases of WC* partner curves in $\mathbb{E}_1^3$.

Case 1. If α be a spacelike or timelike curve with a spacelike principal normal vector in $\mathbb{E}_1^3$. If W be a spacelike then W^* is a spacelike or timelike vector, and

(4.12) $$C=sinh\theta W^*+cosh\theta N^*,\qquad N=sinh\theta N^*+cosh\theta W^*\,,$$

or

(4.13) $$C=sinh\theta N^*+cosh\theta W^*,\qquad N=sinh\theta W^*+cosh\theta N^*\,,$$

for some function $\theta=\theta(s)$.

Case 2. If α be a spacelike or timelike curve with a spacelike principal normal vector in $\mathbb{E}_1^3$. If W be a timelike then W^* and N^* are spacelike vectors, and

(4.14) $$N=sin\theta N^*-cos\theta W^*,\qquad C=cos\theta N^*+sin\theta W^*\,,$$

for some function $\theta=\theta(s)$.

Case 3. If α be a spacelike curve with timelike principal normal vector N in $\mathbb{E}_1^3$ then N^* is a timelike or spacelike vector, and

(4.15) $$C = sinh\theta N^* + cosh\theta W^*, \qquad N = sinh\theta W^* + cosh\theta N^*,$$

or

(4.16) $$C = sinh\theta W^* + cosh\theta N^*, \qquad N = sinh\theta N^* + cosh\theta W^*,$$

for some function $\theta = \theta(s)$.

Theorem 4.2: *There do not exist any WC* partner curves in* $\mathbb{E}_1^3$.

Proof: We give proof of this Theorem under the assumption of Case 1 of the WC^* partner curve. The proof of the other cases is similar to this proof.

Differentiating (4.10) with respect to s and using (4.2), we obtain

(4.17) $$T^* \frac{ds^*}{ds} = T + \lambda' W + \lambda g C .$$

Using (4.9) in (4.17), we get

$$-(-\tilde{k}^* C^* + \tilde{\tau}^* W^*)\frac{ds^*}{ds} = -(\tilde{\tau} W - \tilde{k} C) + \lambda' W + \lambda g C ,$$

wherein using (4.13) and $W = C^*$, we have

(4.18) $$-(-\tilde{k}^* C^* + \tilde{\tau}^* W^*)\frac{ds^*}{ds} = (\lambda' - \tilde{\tau})C^* + (\tilde{k} + \lambda g) sinh\theta N^* + (\tilde{k} + \lambda g) cosh\theta W^*$$

Comparing coefficients of C^*, N^*, and W^* in (4.18), we obtain

(4.19) $$\lambda' = \tilde{\tau} + \tilde{k}^* \frac{ds^*}{ds},$$

(4.20) $$\lambda = -\frac{\tilde{k}}{g},$$

and

$$(4.21) \qquad \lambda = -\frac{\left(\tilde{\tau}^{*}\frac{ds^{*}}{ds} + \tilde{k}cosh\theta\right)}{g\,cosh\theta},$$

respectively.

Now, taking the inner product of (4.17) with itself we get,

$$(4.22) \qquad \left(\frac{ds^{*}}{ds}\right)^{2} = -1 + \lambda'^{2} - \lambda^{2}g^{2} - 2\lambda'\tilde{\tau} - 2\lambda g\tilde{k}.$$

Using (4.19) and (4.21) in (4.22), we obtain

$$(4.23) \qquad \left(\frac{ds^{*}}{ds}\right)^{2} = -1 + \tilde{\tau}^{2} + \tilde{k}^{*2}\left(\frac{ds^{*}}{ds}\right)^{2} - \frac{1}{(cosh\theta)^{2}}\left(\tilde{\tau}^{*}\frac{ds^{*}}{ds} + \tilde{k}cosh\theta\right)^{2}$$
$$-2\tilde{\tau}\left(\tilde{\tau} + \tilde{k}^{*}\frac{ds^{*}}{ds}\right) + \frac{2\tilde{k}}{cosh\theta}\left(\tilde{\tau}^{*}\frac{ds^{*}}{ds} + \tilde{k}cosh\theta\right) + 2\tilde{\tau}\tilde{k}^{*}\frac{ds^{*}}{ds}.$$

Further, solving (4.23), we obtain

$$(4.24) \qquad \tilde{\tau}^{*2}sinh^{2}\theta\left(\frac{ds^{*}}{ds}\right)^{2} = 0,$$

where by we get $\tilde{\tau}^{*} = 0$ as $sinh\theta \neq 0$ and $\frac{ds^{*}}{ds} \neq 0$. Since $\tilde{\tau}^{*} = \frac{\tau^{*}}{f^{*}}$, which gives $\tau^{*} = 0$. Hence, the curve α^{*} has to be a plane curve.

On the other hand, solving (4.13), we obtain

$$(4.25) \qquad N^{*} = \cos h(\theta)N + \sinh(\theta)C, \qquad W^{*} = -sinh\theta N + cosh\theta C$$

Differentiating the first relation of (4.25) with respect to s and using (4.2), we find

$$(4.26) \qquad f^{*}C^{*}\frac{ds^{*}}{ds} = sinh\theta\left(\frac{d\theta}{ds} - f\right)N + cosh\theta\left(f - \frac{d\theta}{ds}\right)C - g\,sinh\theta W,$$

wherein using $W = C^*$, we have

(4.27) $$f = \frac{d\theta}{ds}.$$

Using (4.27) in (4.26), we obtain

(4.28) $$f^* C^* \frac{ds^*}{ds} = -g\, sinh\theta W.$$

Similarly, differentiating the second relation of (4.25) with respect to s and using (4.2) and (4.27), we get

(4.29) $$-\epsilon_0^* g^* C^* \frac{ds^*}{ds} = g\, cosh\theta W.$$

Using (4.28) and (4.29), we obtain

(4.30) $$\frac{\epsilon_0^* g^*}{f^*} = coth\theta,$$

which gives

(4.31) $$\frac{k^{*2}}{|\epsilon_0^* \tau^{*2} + \epsilon_2^* k^{*2}|^{3/2}} \left(\frac{\tau^*}{k^*}\right)' = coth\theta.$$

Putting $\tau^* = 0$ in (4.30), we get $coth\theta = 0$, which is a contradiction. This completes the proof of the theorem.

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