ISSN 0974 - 9373

Vol. 25 No. 4 (2021) **Journal of International Academy of Physical Sciences** pp. 475- 494

Generalized *QR (GQR)*- Lightlike Submanifolds of Indefinite Quaternion Kähler Manifolds

Gauree Shanker and Ankit Yadav

Department of Mathematics and Statistics
Central University of Punjab, Bathinda, Punjab-151 401, India.

(Received October 24, 2021)

Abstract: In this article, we introduce a new class of submanifolds namely GQR- lightlike submanifold of indefinite quaternion Kähler manifolds. It is observed that there is no inclusive relation between QR lightlike submanifolds and SQR lightlike submanifolds. We introduce a generalized submanifold, namely GQR-lightlike submanifold of indefinite quaternion Kähler manifold which contains QR lightlike submanifolds and SQR lightlike submanifolds. Further, we study various distributions of GQR- lightlike submanifolds and establish some important results.

Keywords: Lightlike submanifolds; indefinite quaternion Kähler manifolds; QR-lightlike submanifolds, and SQR-lightlike submain-folds.

2010 AMS Classification Number: Primary 53C15, Secondary 53C40, 53C50.

1. Introduction

A quaternionic Kähler manifold is an oriented $4n$-dimensional Riemannian manifold whose restricted holonomy group is contained in the subgroup[1] $Sp(n)\,Sp(1)$ of $SO(4n)$. As the restricted holonomy group of the manifold to be a part of Burger's work[2], the study on these manifolds drew attention to researchers. The indefinite quaternion Kähler manifolds were studied in[3,4]. Further, the lightlike hypersurfaces, quaternion lightlike submanifolds, Screen real lightlike submanifolds, QR-lightlike submanifolds, screen QR lightlike submanifolds, and screen CR-lightlike submanifolds of indefinite quaternion Kähler manifolds were studied in[1]. We observed that QR lightlike submanifold does not contain quaternion lightlike submanifold, screen-real lightlike submanifold but lightlike real hypersurfaces are its special cases. Also, the screen QR-lightlike submanifold does not contain screen-real lightlike submanifolds and screen real hypersurfaces but quaternion lightlike submanifolds are its special cases. Due to this, we conclude that there is no inclusive relation between QR-lightlike submanifold and screen-QR lightlike submanifolds. Therefore, we introduce a new class of submanifolds of indefinite quaternion Kähler manifold namely generalized QR (GQR) lightlike submanifold as an umbrella space that contains QR-lightlike submanifolds and Screen-QR lightlike submanifolds as its special cases. This paper is organized as follows:

In section 2, we recall all basic terminologies of the quaternion Kähler manifold which are used throughout the paper. In section 3, we give the definition of GQR-lightlike submanifold, an example of GQR-lightlike submanifold and some necessary conditions on the dimension of GQR lightlike submanifold. Further, we establish some important equations on GQR-lightlike submanifolds and study various distributions of these submanifolds.

2. Preliminaries

Suppose $\tilde{N}$ be an n-dimensional manifold with a 3-dimensional vector bundle $\mathbb{Q}$ consisting three tensors J_1, J_2 and J_3 of the type $(1,1)$ over $\tilde{N}$ Let a local basis $\{J_1, J_2, J_3\}$ of $\mathbb{Q}$ in any coordinate neighbourhood U of $\tilde{N}$ such that

$$J_1^2 = -I, \quad J_2^2 = -I, \quad J_3^2 = -I, \tag{2.1}$$

and

$$J_2 J_3 = -J_3 J_2 = J_1, \; J_3 J_1 = -J_1 J_3 = J_2, \; J_1 J_2 = -J_2 J_1 = J_3. \tag{2.2}$$

The local basis $\{J_1, J_2, J_3\}$ is called a canonical local basis of the bundle $\mathbb{Q}$ in U If U is another coordinate neighborhood such that $U \cap U' \neq \{0\}$ then we have the following expression

$$J'_a = \sum_{b=1}^{3} S_{ab} J_b, \qquad a = 1,2,3, \tag{2.3}$$

where (S_{ab}) is an element of the proper orthogonal subgroup $SO(3)$. Suppose $\tilde{g}$ is an indefinite metric on $\tilde{N}$ such that

(2.4) $$\tilde{g}(J_aU, J_aV) = \tilde{g}(U,V), \ \forall U,V \in T_x\tilde{N}, x \in \tilde{N},$$

then $(\tilde{N}, \tilde{g}, \mathbb{Q})$ is known as indefinite almost quaternion manifold[1,3]. Suppose $\{E_i\}_{i=1,...n_1}$ and $\{F_j\}_{i=1,...n_2}$ are timelike and spacelike vector fields, then it is possible to construct a basis B for $\tilde{N}$ such that

$$B = \{E_1, \ J_1E_1, J_2E_1, J_3E_1, ..., E_{n_1}, J_1E_{n_1}, J_2E_{n_1}, J_3E_{n_1},$$
$$F_1, J_1F_1, J_2F_1, J_3F_1, ..., F_{n_2}, J_1F_{n_2}, J_2F_{n_2}, J_3F_{n_2}\}.$$

If, for any $U \in T\tilde{N}$, the metric connection ∇ of $\tilde{N}$ satisfies the following equations

$$\nabla_U J_1 = r(U)J_2 - q(U)J_3,$$

(2.5) $$\nabla_U J_2 = -r(U)J_1 + p(U)J_3,$$

$$\nabla_U J_3 = q(U)J_1 - p(U)J_2,$$

then $(\tilde{N}, \tilde{g}, \mathbb{Q})$ is known as quaternion Kähler manifold, where p, q and r are local 1-forms. The equation (2.5) can be re-written as

(2.6) $$\nabla_U J_a = \sum_{b=1}^{3} Q_{ab}(U)J_b \quad a = 1,2,3,$$

where Q_{ab} are local 1-forms locally defined on $\tilde{N}$ such that $Q_{ab} = -Q_{ba}$. Also, the local 2-forms η_1, η_2, η_3 with respect to the above manifold are defined as

(2.7) $$\eta_i(U,V) = \tilde{g}(U, J_iY), \ i = 1,2,3,$$

for any vector fields U and V on $\tilde{N}$ [1]. Clearly,

(2.8) $$\eta = \eta_1 \wedge \eta_1 + \eta_2 \wedge \eta_2 + \eta_3 \wedge \eta_3$$

is a 4-form defined on $\tilde{N}$.
The Gauss-Weingarten formulae are[3]

(2.9) $$\bar{\nabla}_U V = \nabla_U V + h^l(U,V) + h^s(U,V),$$

(2.10) $$\bar{\nabla}_U N = -A_N U + \nabla_U^l(N) + D^s(U,N)$$

and

(2.11) $$\bar{\nabla}_V W = -A_W V + \nabla_V^s(W) + D^l(V,W),$$

where $W \in \Gamma\left(S\left(TN^{\perp}\right)\right)$ and $N \in \Gamma\left(ltr\left(TN\right)\right)$.
Let S' be a projection map of TN to $S(TN)$, then we have

(2.12) $$\nabla_U S'V = \nabla_U^* S'V + h^*(U,S'V)$$

and

(2.13) $$\nabla_U \xi = A_\xi^* U + \nabla_U^{*t}(\xi),$$

where $\left\{\nabla_U^* S'V, A_\xi^* U\right\} \in S(TN)$ and $\left\{h^*(U,S'V), \nabla_U^{*t}(\xi)\right\} \in Rad(TN)$. We refer[5] to understand the construction of all the above equations.

Theorem[1] 2.1: *An indefinite almost quaternion manifold* $(\tilde{N}, \tilde{g}, \mathbb{Q})$ *is an indefinite quaternion Kähler manifold if and only if* $\nabla \eta = 0$.

3. Generalized QR-lightlike submanifold

Definition 3.1: *Let* $(N, g, S(TN))$ *be a lightlike submanifold of an indefinite quaternion Kähler manifold* $(\tilde{N}, \tilde{g}, \mathbb{Q})$, *then N is a generalized QR(GQR)-lightlike submanifold of* $\tilde{N}$ *if the following conditions are satisfied:*

(i) There exist two sub-bundles D_1 *and* D_2 *of* $Rad(TN)$ *such that*

$$Rad(TN) = D_1 \oplus D_2, \quad J_a(D_1) = D_1, \quad J_a(D_2) \subset S(TN).$$

(ii) There exist two sub-bundles D_0 *and* D' *of* $S(TN)$ *such that*

$$S(TN)=\{J_a(D_2\oplus D')\}\perp D_0,$$

where $a=1,2,3,$ D_0 *is invariant,* $J_a(\mathcal{L}_1\perp\mathcal{L}_2)=D'$ *and* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are vector sub-bundles of* $ltr(TN)$ *and* $S(TN^{\perp})$ *respectively.*

The tangent bundle TN can be written as

(3.1) $$TN=D\oplus D',\quad D=D_1\oplus_{orth.}D_2\oplus_{orth.}J_a(D_2)\oplus_{orth.}D_0.$$

Remark:

(A) N is a proper GQR-lightlike submanifold if $D_1\neq\{0\}$, $D_2\neq\{0\}$, $D_0\neq\{0\}$ and $\mathcal{L}_2\neq\{0\}$.

(B) N is a proper QR-lightlike submanifold if $D_1\neq\{0\}$, $D_2=\{0\}$, $D_0\neq\{0\}$ and $\mathcal{L}_2\neq\{0\}$.

(C) N is a proper Screen QR-lightlike submanifold if $D_1=\{0\}$, $D_2\neq\{0\}$, $D_0\neq\{0\}$ and $\mathcal{L}_2\neq\{0\}$.

From remarks (B) and (C), it is clear that QR-lightlike submanifolds and screen QR lightlike submanifolds are special cases of GQR lightlike submanifolds.The followings are the conditions on the dimensions of various distributions of N with respect to a proper GQR lightlike submanifold

(a) Condition (i) implies that $dim(Rad(TN))\geq 5.$

(b) Conditions (i) and (ii) imply that $dim(D)=4s\geq 12$ and $dim(D')=3k\geq 6.$

(c) $dim(N)\geq 12$ and $dim(\tilde{N})=4n\geq 16$

(d) Any proper 12-dimensional GQR-lightlike submanifold is 5-lightlike.

(e) If $\tilde{N}$ quaternion Kähler manifold and N is GQR-lightlike submanifold then index$(\tilde{N})\geq 8.$

Example 3.1: Let $\mathbb{R}^{16}_8$ be an indefinite Quaternion Kähler manifold with coordinate $(x_1,y_1,...,x_8,y_8)$ and signature $(-,-,-,-,-,-,-,-,+,+,+,+,+,+,+,+)$. Let us define a submanifold such that

$$x_1=u_1,\ y_1=u_2,\ x_2=u_3,$$

$$y_2 = u_4, x_3 = u_1\sin\theta - u_2\cos\theta\,,\ y_3 = \cos\theta u_1 + u_2\sin\theta$$

(3.2) $$x_4 = u_4 cos\theta + u_3 sin\theta\,,\ y_4 = u_4 sin\theta - u_3 cos\theta$$

$$x_5 = u_5 - u_6,\ y_5 = u_5 + u_6,\ x_6 = u_7 + u_8,\ y_6 = -u_7 + u_8,$$

$$x_7 = u_9,\ y_7 = u_{10},\ \ x_8 = u_{11},\ y_8 = const.$$

Then the tangent bundle TN is spanned by the following vectors

$$Z_1 = \partial x_1 + \sin\theta\,\partial x_3 + \cos\theta\,\partial y_3,\ \ Z_2 = \partial y_1 - \cos\theta\,\partial x_3 + \sin\theta\,\partial y_3,$$

$$Z_3 = \partial x_2 + \sin\theta\,\partial x_4 - \cos\theta\,\partial y_4,\ \ \ Z_4 = \partial y_2 + \cos\theta\,\partial x_4 + \sin\theta\,\partial y_4,$$

$$Z_5 = \partial x_5 + \partial y_5,\ Z_6 = -\partial x_5 + \partial y_5,\ Z_7 = \partial x_6 - \partial y_6,$$

$$Z_8 = \partial x_6 + \partial y_6,\ \ Z_9 = \partial x_7,\ Z_{10} = \partial y_7,\ Z_{11} = \partial x_8$$

Hence N is 5-lightlike submanifold with $Rad(TN) = \{Z_1, Z_5, Z_6, Z_7, Z_8\}$. Also, $D_1 = \{Z_5, Z_6, Z_7, Z_8\}$, $D_2 = \{Z_1\}$ and $J(D_2) = \{Z_2, Z_3, Z_4\}$. It can be easily verified that the above vectors satisfy the conditions of a $12 - GQR$-lightlike submanifold of $\mathbb{R}^{16}_8$.

Suppose P be the projection map from TN to the invariant distribution D and $\{e_1, ...e_{r'}, f_1, ..., f_{k'}\}$ be a local orthonormal frame on the vector sub-bundle $\mathcal{L}_1 \perp \mathcal{L}_2$.

Applying J_a on above orthonormal frames we get an orthonormal frame

$$\mathcal{F} = \{E_{11}, ..., E_{1r'}, E_{21}, ...E_{2r'}, E_{31}, ..., E_{3r'}, F_{11}, ..., F_{1k'}, F_{21}, ..., F_{2k'}, F_{31}, ..., F_{3k'}\} \text{ for } D',$$

where $J_a(V_i) = E_{ai}$, $J_a(f_j) = F_{aj}$, $i = 1,..,r'$ $j = 1, ..., k'$ and $a = 1, 2, 3.$

Now, any vector field $U \in TN$ can be written as

(3.3) $$U = PU + \sum_{b=1}^{3}\left\{\left\{\sum_{i=1}^{r}\theta_{bi}(Y)E_{bi}\right\} + \left\{\sum_{k=1}^{k_1}\theta'_{bj}(Y)F_{bj}\right\}\right\},$$

where

(3.4) $$\theta_{bi}(U) = \tilde{g}\left(U, J_{b\xi_i}\right)$$

(3.5) $$\theta'_{bj}(U) = \tilde{g}(U, F_{bj}),$$

are 1-forms defined on N Applying J_a on equation (3.3), we get the following expression:

(3.6) $$J_a U = J_a PU + \sum_{b=1}^{3}\{\left\{\sum_{i=1}^{r}\theta_{bi}(Y)E_{ci} - \theta_{ci}(Y)E_{bi} - \theta_{ai}(Y)e_i\right\} + \left\{\sum_{i=1}^{r}\theta'_{bj}(Y)f_{cj} - \theta'_{cj}(Y)f_{bj} - \theta'_{aj}(Y)f_j\right\}$$

It is known that a transversal bundle can be decomposed in the following sense,

$$tr(TN) = \mathcal{L}_1 \perp \mathcal{L}_2 \oplus_{orth.} \mathcal{L}_1^{\perp} \perp \mathcal{L}_2^{\perp}$$

therefore, for any vector $V \in tr(TN,)$ we can decompose V in two components as follows

(3.7) $$J_a(V) = B_a V + C_a V,$$

where $B_a V \in \Gamma(D')$ and $C_a V \in \mathcal{L}_1^{\perp} \perp \mathcal{L}_2^{\perp}$.
Now, we derive some important equations for GQR-lightlike submanifold. Since N is GQR lightlike submanifold of a quaternion Kähler manifold $\tilde{N}$, for any $U,\ V \in \Gamma(TN)$, we obtain

(3.8) $$\bar{\nabla}_U J_a V = (\bar{\nabla}_U J_a)V + J_a(\bar{\nabla}_U V).$$

In equation (3.8), the terms $\bar{\nabla}_U J_a V$, $(\bar{\nabla}_U J_a)V$, and $J_a(\bar{\nabla}_U V)$ can be expanded, respectively, in the following manner

(3.9) $$(\bar{\nabla}_U J_a V = \nabla_U J_a PV + h^l(U, J_a PV) + h^s(U, J_a PV) + \sum_{i=1}^{r}\{\ \theta_{bi}(V)\nabla_U E_{ci} + \theta_{bi}(V)h^l(U, E_{ci}) + \theta_{bi}(V)h^s(U, E_{ci}) + U(\theta_{bi}(V))E_{ci}\} - \{(U\theta_{ci}(V))E_{bi} + \theta_{ci}(V)\nabla_U E_{bi} + \theta_{ci}(V)h^l(U, E_{bi}) + \theta_{ci}(V)h^s(U, E_{bi})\} - \{U\theta_{ai}(V)e_i$$

$$-\theta_{ai}(V)A_{e_i}U+\theta_{ai}(V)\nabla^l_U e_i+\theta_{ai}(V)D^s(U,e_i)\}\}+\sum_{j=1}^{k}\{\ U\left(\theta'_{bj}(V)\right)f_{cj}$$

$$+\theta'_{bj}(V)\nabla_U F_{cj}+\theta'_{bj}(V)h^l\left(U,F_{cj}\right)+\theta'_{bj}(V)h^s\left(U,F_{cj}\right)\}-\{U\left(\theta'_{cj}(V)\right)F_{bj}$$

$$+\theta'_{cj}(V)\nabla_U F_{bj}+\theta'_{cj}(V)h^l\left(U,F_{bj}\right)+\theta'_{cj}(V)h^s\left(U,F_{bj}\right)\}-\{U\left(\theta'_{aj}(V)\right)f_j$$

$$-\theta'_{aj}(V)A_{f_j}U+\theta'_{cj}(V)\nabla^s_U f_j+\theta'_{cj}(V)D^l\left(U,f_j\right)\Big\},$$

(3.10) $$\left(\bar{\nabla}_U J_a\right)V=Q_{ab}(U)\left(PJ_bV\right)+Q_{ac}(U)\left(PJ_cV\right)$$

$$+Q_{ab}\sum_{i=1}^{r}\left\{-\theta_{ai}(U)E_{ci}-\theta_{bi}(U)e_i+\theta_{ci}(U)E_{ai}\right\}$$

$$+Q_{ab}\sum_{j=1}^{k}\left\{-\theta'_{ai}(U)F_{cj}-\theta'_{bj}(U)f_j+\theta'_{cj}(U)F_{aj}\right\}$$

$$+Q_{ac}\sum_{i=1}^{r}\left\{\theta_{ai}(U)E_{bi}-\theta_{bi}(U)E_{ai}-\theta_{ci}(U)e_i\right\}$$

$$+Q_{ac}\sum_{j=1}^{k}\left\{\theta'_{aj}(U)F_{bj}-\theta'_{bj}(U)F_{aj}+\theta'_{cj}(U)f_j\right\}$$

and

(3.11) $$J_a\left(\bar{\nabla}_U V\right)=\sum_{i=1}^{r}\left\{\theta_{bi}\left(\nabla_U V\right)E_{ci}-\theta_{ci}\left(\nabla_U VE_{bi}-\theta_{ai}\left(\nabla_U V\right)e_i\right)\right\}$$

$$+J_aP\left(\bar{\nabla}_U V\right)+\sum_{j=1}^{k}\left\{\theta'_{bj}\left(\nabla_U V\right)F_{cj}-\theta'_{cj}\left(\nabla_U V\right)F_{bj}-\theta'_{aj}\left(\nabla_U V\right)f_j\right\}$$

$$+B_a\left(h^l(X,Y)\right)+C_a\left(h^l(X,Y)\right)+B_a\left(h^s(X,Y)\right)+C_a\left(h^s(X,Y)\right)$$

Substituting the values from the equations (3.9), (3.10) and (3.11) in (3.8) and equating tangential and transversal components, we obtain the following equations, respectively.

(3.12) $$\nabla_U J_a PY+\sum_{i=1}^{r}\left\{\theta_{bi}(V)\nabla_U E_{ci}+U\left(\theta_{bi}(V)\right)E_{ci}-U\left(\theta_{ci}E_{bi}\right)\right.$$

$$-\theta_{ci}(V)\nabla_U E_{bi}+\theta_{ai}(V)A_{ei}U\Big\}+\sum_{j=1}^{k}\Big\{U\left(\theta_{bj}^{'}\right)F_{cj}+\theta_{bj}^{'}\nabla_U F_{cj}$$

$$-U\left(\theta_{cj}(V)\right)F_{bj}-\theta_{bj}^{'}(V)\nabla_U F_{bj}+\theta_{cj}^{'}(V)A_{f_j}U\Big\}=Q_{ab}(U)PJ_bY$$

$$+Q_{ac}(U)PJ_cV+J_a\nabla_U V+\sum_{i=1}^{r}\{-Q_{ab}\theta_{ai}E_{ci}+Q_{ab}\theta_{ci}E_{ai}+Q_{ac}\theta_{ai}E_{bi}$$

$$-\theta_{bi}E_{ai}+\theta_{bi}(\nabla_U V)E_{ci}-\theta_{ci}(\nabla_U V)E_{bi}\}+\sum_{j=1}^{k}\{Q_{ab}(U)\theta_{cj}^{'}F_{aj}$$

$$-Q_{ab}(U)\theta_{aj}^{'}F_{cj}+Q_{ac}\theta_{aj}^{'}F_{bj}-\theta_{bj}^{'}F_{aj}+\theta_{bj}^{'}(\nabla_U V)F_{cj}-\theta_{cj}^{'}(\nabla_U V)F_{bj}\Big\}$$

(3.13)
$$h^l(U,J_aPV)+h^s(U,J_aPV)+\sum_{i=1}^{r}\Big\{\theta_{bi}(V)h^l(U\ E_{ci})$$

$$+\theta_{bi}(V)h^s(U,E_{ci})+\theta_{ci}(V)h^l(U,E_{bi})+\theta_{ci}(V)h^s(U)(X,E_{bi})$$

$$-U(\theta_{ai})e_i-\theta_{ai}(V)\nabla_U^l e_i-\theta_{ai}(V)D^s(U,e_i)\}+\sum_{j=1}^{k}\Big\{\theta_{bj}^{'}(V)h^l\left(U\ F_{cj}\right)$$

$$+\theta_{bj}^{'}(V)h^s\left(U,F_{cj}\right)+\theta_{cj}^{'}(V)h^l\left(U,F_{bj}\right)+\theta_{cj}^{'}(V)h^s\left(U,F_{bj}\right)-U\left(\theta_{aj}\right)f_j$$

$$-\theta_{cj}^{'}(V)\nabla_U^s f_j-\theta_{cj}^{'}(V)D^l\left(U,f_j\right)\}=\sum_{i=1}^{r}\{-Q_{ab}\theta_{bi}e_i-Q_{ac}\theta_{ci}e_i$$

$$-\theta_{ai}(\nabla_U V)e_i+\sum_{j=1}^{k}\Big\{-Q_{ab}\theta_{bj}^{'}f_j-Q_{ac}\theta_{cj}f_j-\theta_{aj}^{'}(\nabla_U V)f_j\Big\}$$

$$+C_ah^l(U,V)+C_ah^s(U,V)$$

Equating $ltr(TN)$ components in (3.13), we obtain

(3.14)
$$h^l(U,J_aPV)+\sum_{i=1}^{r}\Big\{\theta_{bi}(V)h^l(U,E_{ci})+\theta_{ci}(V)h^l(U,E_{bi})_i$$

$$-U\left(\theta_{ai}(V)\right)e_i-\theta_{ai}(V)\nabla_{U^l}f\Big\}+\sum_{j=1}^{k}\Big\{\theta_{bj}^{'}(V)h^l\left(U,F_{cj}\right)$$

$$+\theta_{cj}^{'}(V)h^l\left(U,F_{bj}\right)-\theta_{cj}(V)D^l\left(U,w_j\right)\Big\}=\sum_{i=1}^{r}\{-\theta_{ab}e_i$$

$$-Q_{ac}\theta_{ci}(V)e_i-\theta_{ai}(\nabla_U V)e_i\Big\}+C_ah^l(U,V)$$

Equating $S(TN)^{\perp}$ components in (3.13), we obtain

(3.15)
$$h^s(U,J_aPV)+\sum_{i=1}^{r}\{\theta_{bi}(V)h^s(U,E_{ci})+\theta_{ci}(V)h^s(U,E_{bi})$$
$$-U(\theta_{ai}(V))e_i-\theta_{ai}(V)D^s(U,e_i)\}+\sum_{j=1}^{k}\{\theta'_{bj}(V)h^s(U,F_{cj})$$
$$+\theta'_{cj}(V)h^s(U,F_{bj})-U(\theta'_{aj}(V))f_j-\theta'_{cj}(V)\nabla^s_U f_j\}$$
$$=\sum_{i=1}^{r}\{-Q_{ac}\theta'_{cj}(V)f_j-Q_{ab}\theta'_{bj}(V)f_j-\theta'_{aj}(\nabla_U V)f_j\}+C_a h^s(U,V)$$

Theorem 3.2: *Let N be a GQR-lightlike submanifold of a quaternion Kähler manifold $\tilde{N}$, then*
(i) the distribution D is integrable if and only if, $h(U,J_aV)=h(V,J_aU)$.
(ii) the distribution D defines a totally geodesic foliations if and only if, $h^l(U,J_aV)=0$ and $h^s(U,J_aV)\notin\mathcal{L}_2$.

Proof: The distribution D of GQR-lightlike submanifold $\tilde{N}$ is integrable if and only if, for any $U,\ V\in\Gamma(D)$, $[U,V]\in D.$
Claim: $[U,V]\notin D'.$ Suppose $U,V\in\Gamma(D)$, then from (3.13), we obtain

(3.16)
$$h(U,J_aPY)-C_ah(U,V)=-w_{ai}(\nabla_UV)v_i-w'_{aj}(\nabla_UV)w_j.$$

Interchanging U and V in (3.16), we get

(3.17)
$$h(V,J_aPU)-C_ah(V,U)=-w_{ai}(\nabla_VU)v_i-w'_{aj}(\nabla_VU)w_j.$$

Subtracting (3.17) from (3.16), we obtain

(3.18)
$$h(U,J_aPY)-h(V,J_aPU)=-w_{ai}([U,V])v_i-w'_{aj}([U,V])w_j.$$

If $h(U,J_aPY)=h(V,J_aPU)$, (3.18) becomes

(3.19)
$$-w_{ai}([U,V])v_i-w'_{aj}([U,V])w_j=0.$$

Since v_i and w_j are linearly independent vectors, (3.19) reduces to

(3.20) $$w_{ai}([U,V])=0, \& \; w'_{aj}([U,V])=0.$$

Using (2.7) in (3.20), we obtain

(3.21) $$g([U,V],J_a\xi_i)=0, \& \; g([U,V],F_{bj})=0.$$

From (3.21), we get $[U,V]\notin J_a(\mathcal{L}_1)$ and $[U,V]\notin J_a(\mathcal{L}_2)$, this implies $[U,V]\notin D'$. Clearly, $[U,V]\notin D'$ implies that $[U,V]\in D$. Therefore, D is integrable if and only if $h(U,J_aV)=h(V,J_aU)$.

(ii) **Claim:** D defines totally geodesic foliations if and only if $g(\nabla_U V,J_a\xi_i)=0,$ and $g(\nabla_U V,F_{aj})=0.$ Suppose $U,\ V\in\Gamma(D)$ and $\xi\in Rad(TN)$, then

$$g(\nabla_U V,J_a\xi_i)=g(\bar{\nabla}_U V-h(U,V),J_a\xi_i)=g(\bar{\nabla}_U V,J_a\xi_i).$$

Above equation reduces to

(3.22) $$g(\bar{\nabla}_U V,J_a\xi_i)=g((\bar{\nabla}_U J_a)V-\bar{\nabla}_U J_aV,\xi_i)=-g(h^l(U,J_aV),\xi).$$

On the other hand, we have

(3.23) $$g(\nabla_U V,F_{bj})=g(\bar{\nabla}_U V-h(U,V),F_{bj})=-g(h^s(U,J_aV),w_j).$$

From (3.22) and (3.23), we conclude that D defines totally geodesic foliations if and only if $h^l(U,J_aV)=\ 0$ and $h^s(U,J_aV)\notin\mathcal{L}_2$.

Theorem 3.3: *Let N be a GQR-lightlike submanifold of a quaternion Kähler manifold $\tilde{N}$, then the distribution D' is integrable if, for any $G,G'\in D',N\in ltr(TN)$, $n_s\in\{e_1,...,e_{r_1},f_1,...f_{k_1}\}$, the following equations are satisfied*

(i) $g(\nabla^*_G G',J_cX)=0.$

(ii) *for* $N \in \mathcal{L}_1$, *then* $\left(h^*(G,G'),J_cN\right)=0,\ g\left(f_l,D^s(J_aN,G)\right)=0,$ $g\left(e_j,A_{J_aNG}\right)=0$ *and* *for* $N \in \mathcal{L}_1^{\perp}$ then$\left(\nabla^*_G G',J_cN\right)=0,\ g\left(n_s,\nabla^*_G J_aN\right)=0,$ $g\left(f_l,h^s(J_aN,F_{al'})\right)=0,\ g\left(e_j,h^*(J_aN,E_{ai})\right)=0$

(iii) $g\left(h^*(G,G'),e_k\right)=0,\ g\left(\nabla^*_G G',E_{ai}\right)=0$ *and* $g\left(A_{n_s}G,e_k\right)=0.$

Proof: We know that D' is integrable if and only if, for any G,G' belongs to $\Gamma(D')$, $[G,G']\in\Gamma(D')$. To prove $[G,G']\in D'$ is equivalent to prove $[G,G']\notin D$, therefore, we will prove the following claims.

Claim 1: For any $G,G'\in D'$, if $g\left(\nabla^*_G G',J_cX\right)=0$ then $[G,G']\notin D_0$. Suppose $G,G'\in D'$ and $X\in D_0$, then using (2.6), (3.7) and Gauss-Weingarten equations, we obtain

$$g\left([G,G'],X\right)=g\left(\bar{\nabla}_G G'-\bar{\nabla}_G G',X\right). \tag{3.24}$$

If $G=E_{ai},G'=E_{aj}$, then (3.24) reduces to

$$g\left(\left[E_{ai},E_{aj}\right],X\right)=g\left(\bar{\nabla}_{E_{ai}}E_{aj}-\bar{\nabla}_{E_{aj}}E_{ai},X\right).$$

After some calculation calculations, we get

$$g\left(\left[E_{ai},E_{aj}\right],X\right)=g\left(A_{e_j}E_{ai},J_aX\right)-g\left(A_{e_i}E_{aj},J_aX\right)=0. \tag{3.25}$$

If $G=F_{al}$ and $G'=F_{al'}$ then (3.24) reduces to

$$g\left(\left[F_{al},F_{al'}\right],X\right)=g\left(A_{f_{l'}}F_{al},J_aX\right)-g\left(A_{f_l}F_{al'},J_aX\right)=0. \tag{3.26}$$

If $G=E_{ai}$ and $G'=F_{al}$, then (3.24) reduces to

$$g\left(\left[E_{ai},F_{al}\right],X\right)=g\left(A_{f_l}E_{ai},J_aX\right)-g\left(A_{e_i}F_{al},J_aX\right)=0. \tag{3.27}$$

If $G=E_{ai}$ and $G'=E_{bj}$, using (3.11), (2.6) and (3.8), (3.24) reduo

(3.28) $$g\left(\left[E_{ai},E_{bj}\right],X\right)=g\left(\overline{\nabla}_{E_{ai}}E_{bj}-\overline{\nabla}_{E_{bj}}E_{ai},X\right).$$

Since $J_a=J_bJ_c=-J_cJ_b$ and $J_b=J_cJ_a=-J_aJ_c$, to above equation reduces to

(3.29) $$g\left(\left[E_{ai},E_{bj}\right],X\right)=-g\left(\nabla^*_{E_{ai}}E_{aj},J_cX\right)-g\left(\nabla^*_{E_{bj}}E_{bi},J_cX\right).$$

Similarly, If $G=F_{al}$, $G'=F_{bl'}$, then

(3.30) $$g\left(\left[F_{al},F_{bl'}\right],X\right)=-g\left(\nabla^*_{F_{al}}F_{al'},J_cX\right)-g\left(\nabla^*_{F_{bl'}}F_{bl},J_cX\right)$$

and if $G=$, $G'=F_{bl}$, then

(3.31) $$g\left(\left[E_{ai},F_{bl}\right],X\right)=-g\left(\nabla^*_{E_{ai}}F_{al},J_cX\right)-g\left(\nabla^*_{F_{bl}}E_{ai},J_cX\right).$$

If $g\left(\nabla^*_G G',J_cX\right)=0,$ using (3.25),(3.26), (3.27), (3.29), (3.30) and (3.31), we obtain $\left[G,G'\right]\notin D_0$.

Claim 2: $\left[G,G'\right]\notin Rad\left(TN\right)$ If, for any $N\in\mathcal{L}_1$,

(3.32) $$g\left(h^*\left(G,G'\right),J_cN\right)=0,\ g\left(f_l,D^s\left(J_aN,G\right)\right)=0, g\left(e_j,A_{J_aN}G\right)=0,$$

and, for any $N\in\mathcal{L}_1^\perp$,

(3.33) $$g\left(\nabla^*_G G',J_cN\right)=0,\ g\left(n_s,\nabla^*_G J_aN\right)=0,$$
$$g(f_l,h^s\left(J_aN,F_{al'}\right)=0,\ g\left(e_j,h^*\left(J_aN,E_{ai}\right)\right)=0.$$

Suppose $G=E_{ai}$, $G'=E_{aj}\in D'$ and $N\in ltr\left(TN\right)$, from Gauss-Weingarten equations, (2.4), (2.6) and (3.7)

(3.34) $$g\left(\left[E_{ai},E_{aj}\right],N\right)=g\left(\overline{\nabla}_{E_{ai}}E_{aj}-\overline{\nabla}_{E_{aj}}E_{ai},N\right).$$

For $N\in\mathcal{L}_1\subset ltr\left(TN\right)$, (3.34) reduces to

$$g\left(\left[E_{ai},E_{aj}\right],N\right)=-g\left(e_j,A_{J_aN}E_{ai}\right)+g\left(e_i,A_{J_aN}E_{aj}\right).$$

For $N\in\mathcal{L}_1^{\perp}\subset ltr(TN)$, (3.34) reduces to

(3.35) $$g\left(\left[E_{ai},E_{aj}\right],N\right)=g\left(e_j,h^*\left(J_aN,E_{ai}\right)\right)-g\left(e_i,h^*\left(J_aN,E_{aj}\right)\right).$$

Similarly, If $G=F_{al}$, $G'=F_{al'}$ and $N\in ltr(TN)$, we have

$$g\left(\left[F_{al},F_{al'}\right],N\right)=g\left(\overline{\nabla}_{F_{al}}F_{al'}-\overline{\nabla}_{F_{al'}}F_{al},N\right).$$

For $N\in\mathcal{L}_1^{\perp}\subset ltr(TN)$, the above equation is equivalent to

(3.36) $$g\left(\left[F_{al},F_{al'}\right],N\right)=g\left(f_l,D^s\left(J_aN,F_{al'}\right)\right)-g\left(f_{l'},D^s\left(J_aN,F_{al}\right)\right).$$

For $N\in\mathcal{L}_1\subset ltr(TN)$, above equation equivalent to

(3.37) $$g\left(\left[F_{al},F_{al'}\right],N\right)=g\left(f_l,h^s\left(J_aN,F_{al'}\right)\right)-g\left(f_{l'},h^s\left(J_aN,F_{al}\right)\right).$$

Suppose $G=E_{ai}$, $G'=E_{bj}$, then (3.34) reduces to

(3.38) $$g\left(\left[E_{ai},E_{bj}\right],N\right)=g\left(\overline{\nabla}_{E_{ai}}E_{bj},N\right)-g\left(\overline{\nabla}_{E_{bj}}E_{ai},N\right),$$

if $N\in\mathcal{L}_1^{\perp}$, then above equation is equivalent to

(3.39) $$g\left(\left[E_{ai},E_{bj}\right],N\right)=-g\left(h^*\left(E_{ai},E_{aj}\right),J_cN\right)-g\left(h^*\left(E_{bj},E_{bi}\right),J_cN\right).$$

If $N\in\mathcal{L}_1$ then above equation is equivalent to

(3.40) $$g\left(\left[E_{ai},E_{bj}\right],N\right)=-g\left(\nabla^*_{E_{ai}}E_{aj},J_cN\right)-g\left(\nabla^*_{E_{bj}}E_{bi},J_cN\right).$$

Suppose $G=F_{al}$, $G'=F_{bl'}$, then (3.34) reduces to

(3.41) $$g\left(\left[F_{al},F_{bl'}\right],N\right)=-g\left(\overline{\nabla}_{F_{al}}F_{al'},J_cN\right)-g\left(\overline{\nabla}_{F_{bl'}}F_{bl},J_cN\right).$$

If $N\in\mathcal{L}_1^{\perp}$ then above equation is equivalent to

(3.42) $$g\left(\left[F_{al},F_{bl'}\right],N\right)=-g\left(h^*\left(F_{al},F_{al'}\right),J_cN\right)-g\left(h^*\left(F_{bl},F_{bl'}\right),J_cN\right).$$

If $N\in\mathcal{L}_1$, then above equation is equivalent to

(3.43) $$g\left(\left[F_{al},F_{bl'}\right],N\right)=-g\left(\nabla^*_{F_{al}}F_{al'},J_cN\right)-g\left(\nabla^*_{F_{bl'}}F_{bl},J_cN\right).$$

Suppose $G=E_{ai}$, $G'=F_{al}$, then (3.34) reduces to

(3.44) $$g\left(\left[E_{ai},F_{al}\right],N\right)=g\left(f_l,\overline{\nabla}_{E_{ai}}J_aN\right)-g\left(e_i,\overline{\nabla}_{F_{al}}J_aN\right).$$

If $N\in\mathcal{L}_1^{\perp}$, then above equation is equivalent to

(3.45) $$g\left(\left[E_{ai},F_{al}\right],N\right)=g\left(f_l,D^s\left(J_aN,E_{ai}\right)\right)+g\left(e_i,A_{J_aN}F_{al}\right).$$

If $N\in\mathcal{L}_1$ then above equation is equivalent to

(3.46) $$g\left(\left[E_{ai},F_{al}\right],N\right)=-g\left(f_l,\nabla^*_{E_{ai}}J_aN\right)+g\left(e_i,\nabla^*_{F_{al}}J_aN\right).$$

Suppose $G=E_{ai}$, $G'=F_{bl}$, then (3.34) rduces to

(3.47) $$g\left(\left[E_{ai},F_{bl}\right],N\right)=-g\left(\nabla_{E_{ai}}F_{al},J_cN\right)-g\left(\nabla_{F_{bl}}E_{bi},J_cN\right).$$

If $N\in\mathcal{L}_1^{\perp}$ then above equation is equivalent to

(3.48) $$g\left(\left[E_{ai},F_{bl}\right],N\right)=-g\left(h^*\left(E_{ai},F_{al}\right),J_cN\right)-g\left(h^*\left(F_{bl},E_{bi}\right),J_cN\right),$$

if $N\in\mathcal{L}_1$ then above equation is equivalent to

(3.49) $$g\left(\left[E_{ai},F_{bl}\right],N\right)=-g\left(J_cN,\nabla^*_{E_{ai}}F_{al}\right)-g\left(J_cN,\nabla^*_{F_{bl}}E_{bi}\right).$$

From above sub-cases on the choice of G and G', we obtain that if $N \in \mathcal{L}_1$, then

(3.50) $$\left(h^*\left(G,G'\right),J_cN\right)=0,\ g\left(f_l,D^s\left(J_aN,G\right)\right)=0,g\left(e_j,A_{J_aNG}\right)=0,$$

and if $N \in \mathcal{L}_1^\perp$, then

(3.51) $$\left(\nabla^*_G G',J_cN\right)=0,\ g\left(n_s,\nabla^*_G J_aN\right)=0,$$
$$g(f_l,h^s\left(J_aN,F_{al'}\right)=0,\ g\left(e_j,h^*\left(J_aN,E_{ai}\right)\right)=0.$$

In, either case $\left[G,G'\right]\notin Rad\left(TN\right)$.

Claim 3: $\left[G,G'\right]\notin J_a\left(D_2\right)$, if $g\left(h^*\left(G,G'\right),e_k\right)=0,g\left(\nabla^*_G G',E_{ai}\right)=0$ and $g\left(A_{n(s)}G,e_k\right)=0$. Suppose $G=E_{ai}$, $G'=E_{aj}$, then

(3.52) $$g\left(\left[E_{ai},E_{aj}\right],E_{bk}\right)=g\left(h^*\left(E_{ai},E_{cj}\right),e_k\right)-g\left(h^*\left(E_{aj},E_{ci}\right),e_k\right)$$

and

(3.53) $$g\left(\left[E_{ai},E_{aj}\right],E_{ak}\right)=-g\left(A_{e_j}E_{ai},e_k\right)+g\left(A_{e_i}E_{aj},e_k\right).$$

Suppose $G=F_{al}$, $G'=F_{al'}$, then

(3.54) $$g\left(\left[F_{al},F_{al'}\right],E_{bk}\right)=g\left(h^*\left(F_{al},F_{al'}\right),e_k\right)-g\left(D^l\left(F_{al},f_l\right),e_k\right)$$

and

(3.55) $$g\left(\left[F_{al},F_{al'}\right],F_{ak}\right)=-g\left(A_{f_{al'}}F_{al},e_k\right)+g\left(A_{f_{al}}F_{al'},e_k\right).$$

Suppose $G=E_{ai}$, $G'=E_{bj}$, then

(3.56) $$g\left(\left[E_{ai},E_{bj}\right],E_{ck}\right)=g\left(h^*\left(E_{ai},E_{aj}\right),e_k\right)+g\left(h^*\left(E_{bi},E_{bj}\right),e_k\right)$$

and

(3.57) $$g\left(\left[E_{ai},E_{bj}\right],E_{bk}\right)=g\left(\nabla^{*}_{E_{ai}}E_{aj},E_{ak},\right)-g\left(\nabla^{*}_{E_{bj}}E_{bi},E_{ak}\right).$$

Suppose $G=F_{al}$, $G'=F_{bl'}$, then

(3.58) $$g\left(\left[F_{al},F_{bl'}\right],E_{ck}\right)=g\left(h^{*}\left(F_{al},F_{al'}\right),e_{k}\right)+g\left(h^{*}\left(F_{bl},F_{bl'}\right),e_{k}\right)$$

and

(3.59) $$g\left(\left[F_{al},F_{bl'}\right],E_{ak}\right)=g\left(\nabla^{*}_{F_{al}}F_{al'},E_{bk}\right)+g\left(\nabla^{*}_{F_{bl'}}F_{bl},E_{bk}\right).$$

Suppose $G=E_{ai}$, $G'=F_{bl}$, then

(3.60) $$g\left(\left[E_{ai},F_{bl}\right],E_{ck}\right)=g\left(h^{*}\left(E_{ai},F_{al}\right),e_{k}\right)+g\left(h^{*}\left(F_{bl},E_{bi}\right),e_{k}\right)$$

and

(3.61) $$g\left(\left[E_{ai},F_{bl}\right],E_{ak}\right)=-g\left(\nabla^{*}_{E_{ai}}F_{al},,E_{bk}\right)-g\left(\nabla^{*}_{F_{bl}}E_{bi},E_{bk}\right).$$

From above sub-cases on the choice of G and G', we obtain that, if $g\left(h^{*}\left(G,G'\right),e_{k}\right)=0$, $g\left(\nabla^{*}_{G}G',E_{ai}\right)=0$ and $g\left(A_{n(s)}G,e_{k}\right)=0$, then $\left[G,G'\right]\notin J_{a}\left(D_{2}\right)$. Therefore, $G\notin D=D_{1}\oplus_{orth.}D_{2}\oplus_{orth.}J_{a}\left(D_{2}\right)\oplus_{orth.}D_{0}$ if above claims 1, 2 and 3 are true.

Theorem 3.4: *Let N be a GQR-lightlike submanifold of a quaternion Kähler manifold N', then the distribution D' defines a totally geodesic foliation if and only if, for any $V\in\left\{e_{1},...e_{r_{1}},f_{1},...,f_{k_{1}}\right\}$ and $G\in B$, $A_{V}G\notin D.$*

Proof: By definition, D' defines a totally geodesic foliation if and only if, for any $G, G'\in D'$, $\nabla_{G}G\in$ D$'$. Suppose $G=E_{ai}$ and $G'=E_{aj}$, then

$$\nabla_{E_{ai}}E_{aj}=\left(\nabla_{E_{ai}}J_{a}\right)e_{j}+J_{a}\nabla_{E_{ai}}e_{j},$$

which is equivalent to

(3.62) $$\nabla_{E_{ai}} E_{aj} = Q_{ab}\left(E_{ai}\right)E_{bj} + Q_{ac}\left(E_{ai}\right)E_{cj} + J_a\left(-A_{e_j}E_{ai} + \nabla^s_{E_{ai}} e_j + D^l\left(E_{ai}, e_j\right)\right).$$

Further , eqn (3.62) can be expanded in the following manner

(3.63) $$\begin{aligned}\nabla_{E_{ai}} E_{aj} &= Q_{ab}\left(E_{ai}\right)E_{bj} + Q_{ac}\left(E_{ai}\right)E_{cj} - J_a S\left(A_{e_j}E_{ai}\right)\\ &-\sum_{b=1}^{3}\left\{\sum_{i=1}^{r_1}\theta_{bi}\left(A_{e_j}E_{ai}\right)E_{ci} - \theta_{ci}\left(A_{e_j}E_{ai}\right)E_{bi} - \theta_{ai}\left(A_{e_j}E_{ai}\right)e_j\right\}\\ &+\sum_{l=1}^{k'}\left\{\theta'_{bl}\left(A_{e_j}E_{ai}\right)F_{cl} - \theta_{cl}\left(A_{e_j}E_{ai}\right)F_{bl} - \theta_{al}\left(A_{e_j}E_{ai}\right)f_l\right\}\\ &+B_a\left(\nabla^s_{E_{ai}} e_j + D^l\left(E_{ai}, e_j\right)\right).\end{aligned}$$

Considering tangential components and neglecting other terms from the R.H.S of equation (3.63), we obtain

(3.64) $$\begin{aligned}\nabla_{E_{ai}} E_{aj} &= Q_{ab}\left(E_{ai}\right)E_{bj} + Q_{ac}\left(E_{ai}\right)E_{cj} - J_a S\left(A_{e_j}E_{ai}\right)\\ &-\sum_{b=1}^{3}\left\{\sum_{i=1}^{r_1}\theta_{bi}\left(A_{e_j}E_{ai}\right)E_{ci} - \theta_{ci}\left(A_{e_j}E_{ai}\right)E_{bi}\right\} + \sum_{l=1}^{k'}\theta'_{bl}\left(A_{e_j}E_{ai}\right)F_{cl}.\end{aligned}$$

From (3.64), $\nabla_{E_{ai}} E_{aj} \in D'$ if and only if $A_{e_j}E_{ai} \notin D$. Similarly, if $G = F_{al}$, $G' = F_{al'}$, then $\nabla_{F_{al}} F_{al'} \in D'$ if and only if $A_{f_{l'}}F_{al} \notin D$. Now, let $G = E_{ai}$ and $G' = E_{bj}$, then

(3.65) $$\nabla_{E_{ai}} E_{bj} = J_b \nabla_{E_{ai}} e_j + \left(\nabla_{E_{ai}} J_b\right)e_j .$$

Which is equivalent to

(3.66) $$\nabla_{E_{ai}} E_{bj} = J_b \nabla_{E_{ai}} e_j + Q_{bc}\left(E_{ai}\right)E_{cj} + Q_{ba}\left(E_{ai}\right)E_{aj} .$$

Further, eqn (3.66) can be expanded in the following manner

(3.67) $$\nabla_{E_{ai}} E_{bj} = J_b(-A_{e_j}E_{ai} + \nabla^s_{E_{ai}} e_j + D^l\left(e_j, E_{ai}\right) + Q_{bc}\left(E_{ai}\right)E_{cj} + Q_{ba}\left(E_{ai}\right)E_{aj}.$$

Now, using (3.2) and (3.5) in (3.67), we obtain

$$(3.68)\qquad \nabla_{E_{ai}} E_{bj} = -J_b S A_{e_j} E_{ai} - \sum_{b=1}^{3}\Bigg[\sum_{i=1}^{r_1}\Big\{\theta_{ai}\left(A_{e_j}E_{bi}\right)E_{bi} - \theta_{bi}E_{ai} - \theta_{ci}e_i\Big\}$$
$$+\sum_{i=1}^{k'}\Big\{\theta_{cl}\left(A_{e_j}E_{bi}\right)F_{al} - \theta'_{al}\left(A_{e_j}E_{bi}\right)F_{cl} - \theta'_{b}\left(A_{e_j}E_{bi}\right)f_j\Big\}\Bigg]$$
$$+B_a(\nabla^s_{(E_a i)}e_j + D^l(e_j, E_a i)) + Q_b c(E_a i)E_c j + Q_b a(E_a i)E_a j.$$

Now, equating tangential components from equation (3.68), we obtain

$$(3.69)\qquad \nabla_{E_{ai}} E_{bj} = -J_b S A_{e_j} E_{ai} - \sum_{b=1}^{3}\Bigg[\Bigg\{\sum_{i=1}^{r_1}\Big\{\theta_{ai}\left(A_{e_j}E_{bi}\right)E_{bi} - \theta_{bi}E_{ai}\Big\}$$
$$+\theta_{cl}\left(A_{e_j}E_{bi}\right)F_{al} - \theta'_{al}\left(A_{e_j}E_{bi}\right)F_{cl}\Bigg\}\Bigg] + B_a\left(\nabla^s_{E_{ai}}e_j + D^l\left(e_j, E_{ai}\right)\right)$$
$$+Q_{bc}\left(E_{ai}\right)E_{cj} + Q_{ba}\left(E_{ai}\right)E_{aj}.$$

Therefore, from (3.69), $\nabla_{E_{ai}} E_{bj} \in D'$ if and only if $A_{e_j}E_{ai}$ does not have components in D. Similarly, if $G = F_{al}$, $G' = F_{bl'}$, then $\nabla_{F_{al}} F_{al'} \in D'$ if and only if $A_{f_{l'}}F_{al}$ does not have components in D, and if $G = E_{ai}$, $G' = F_{al}$ or F_{bl}, then $\nabla_{E_{ai}} F_{al}$ or $\nabla_{E_{ai}} F_{bl} \in D'$ if and only if $A_{f_l}E_{ai}$ does not have components in D. From above all conditions, we conclude that $\nabla_G G' \in D'$ if and only if, for any $V \in \left\{e_1,\ldots e_{r_1}, f_1,\ldots, f_{k_1}\right\}$ and $G \in B$, $A_V G$ does not belong to D.

4. Conclusion

The generalization of any theory is always fascinating due to its nature to bring various cases under one roof, particularly, when there are no intrinsic relations between these cases. In this article, we introduce a new class of submanifolds, namely, Generalized-QR or GQR lightlike submanifolds which is a generalized class of QR and SQR lightlike submanifolds of quaternion Kähler manifolds. We establish this class by giving a definition and one example which contains QR lightlike and SQR lightlike submanifolds as its sub-cases. Further, we study the geometry of distributions and derive various important results from them. These results are helpful for further study of totally umbilical, totally geodesic, and mixed geodesic GQR-lightlike submanifolds.

Acknowledgments: The second author is thankful to CSIR for facilitating SRF fellowship through letter no. (09/1051/(0022)/2018-EMR-1).

References

1. K. L. Duggal and B. Sahin, *Differential Geometry of Lightlike Submanifolds*, Birkhäuser Verlag AG, Berlin, 2010.
2. M. Berger M., Sur les groupes d'holonomie homog'enes devarietees a connexion affine et. de varieteest riemanniennes, *Bull. Soc. Math. Fr.*, **83** (1995), 279-330.
3. J. D. Perez, Indefinite quaternion space forms II, *Ann. Stinnt. Univ. Al. I. Cuza Iasi. Sect. I Mat.*, **3** (1986), 63-68.
4. J. D. Perez and F. G. Santos, Indefinite quaternion space forms, *Ann. Mat. Pura. Appl.*, **132** (1982), 383-398.
5. K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic, 1996.
6. A. I. Cătălin, Totally Umbilical Lightlike submanifolds, *Bull. Math. Sc. Math. Roumanie*, **37** (1996), 151-172
7. A. Duggal and D. H. Jin, Geometry of null curves, *Math. J. Toyama Univ.*, **22** (1999), 95-120.