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Common Fixed Point of Absorbing Mappings in Weak Partial Metric Spaces using Implicit Relation

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Abstract: The aim of this paper is to prove some common fixed point theorems of absorbing mappings in weak partial metric spaces using implicit relation.

Keywords: Partial metric space, Weak partial metric space, Common fixed Point.

1. Introduction

In 1994, Matthews[1], introduced the concept of partial metric space and proved the Banach Contraction Principle in complete partial metric space. After that many authors, Valero[2], Oltra et al.[3] and Altun et al.[4,5] gave some generalizations of the result of Matthews. One of the most interesting properties of a partial metric is that $p(x, x)$ may not be zero for $x \in X$. In (1999), Heckmann[6] introduced the concept of weak partial metric space

(WPMS), which is a generalized version of Matthews partial metric space by omitting the small self-distance axiom.

Using the concept of weak partial metric space many authors as Altun et al.[7], Durmaz et al.[8] obtained some interesting results. In (2004), Ranadive et al.[9] introduced the concept of absorbing maps in metric space and proved the common fixed point theorems. They observed that the new notion of absorbing map is neither a subclass of compatible maps nor a subclass of non-compatible maps. In (2008), Mishra et al.[10] proved some common fixed point theorems in fuzzy metric spaces by using the notion of absorbing maps. In (2001), Popa[11], introduced the notion of implicit relation and obtained many interesting results. In this paper first we introduce the notion of absorbing maps in weak partial metric spaces and then we prove some common fixed point results in weak partial metric spaces by using the notion of absorbing mapping.

2. Preliminaries

In this section, we recall some definitions and some properties of theirs. For more details, see [Matthews[1], Valero[2], Oltra et al.[3], Altun et al.[4,5], S. Romaguera[12])

Definition 2.1: *(Matthews[1]). A partial metric on a non-empty set on* X *is a function* $p: X \times X \to \mathbb{R}^+$ *such that for all* $x, y, z \in X$ *:*

(P_1) $x = y$ *if and only if* $p(x, x) = p(y, y) = p(x, y)$,

(P_2) $p(x, x) \le p(x, y)$,

(P_3) $p(x, y) = p(y, x)$,

(P_4) $p(x, z) \le p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) *is called a partial metric space.*

It is clear that, if $p(x, y) = 0$, then, from (P_1) and (P_2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a PMS is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$.

Each partial metric p on X generates a T_0 – topology τ_p on X which has as base the family of open p – balls $\{B_p(x, \varepsilon): x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) \le p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then $(x, y) = p(x, y) - \min\{p(x, x),$

$p(y, y)\}$ is a ordinary metric on X .

Definition 2.2: *(Matthews[1]). Let* (X, p) *be a partial metric space and* $\{x_n\}$ *be a sequence in* X *. Then*

(i) $\{x_n\}$ *convergence to a point* $x \in X$ *if and only if*

$p(x, x) = \lim_{n \to \infty} p(x_n, x)$.

(ii) A sequence $\{x_n\}$ *is called a Cauchy sequence if* $\lim_{n, m \to \infty} p(x_n, x_m)$ *exists and is finite.*

Definition 2.3: *(I. Altun and G. Durmaz[7]). A partial metric space* (X, p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges with respect to* τ_p *to a point* $x \in X$ *such that*

$p(x, x) = \lim_{n, m \to \infty} p(x_n, x_m)$.

Remark 2.1: Let $\{x_n\}$ be a sequence in a partial metric (X, p) and $x \in X$. Then $\lim_{n \to \infty} d_\omega(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n \to \infty} p(x_n, x_m)$.

Following is the definition of weak partial metric space.

Definition 2.4: *(Heckmann[6]) A weak partial metric on a nonempty set* X *is a function* $p: X \times X \to \mathbb{R}^+$ *such that for all* $x, y, z \in X$ *:*

(ωP_1) $x = y$ *if and only if* $p(x, x) = p(y, y) = p(x, y)$,

(ωP_2) $p(x, y) = p(y, x)$,

(ωP_3) $p(x, z) \le p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) *is called a weak partial metric space.*

Obviously, every partial metric space is a weak partial metric space, but the converse is not true. A basic example of a WPMS but not a PMS is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \left(\frac{x+y}{2}\right)$ for all $x, y \in \mathbb{R}^+$. We need the following lemmas to give some fixed point results on aWPMS.

Lemma 2.1: *(I. Altun and G. Durmaz[7]). Let* (X, p) *be a weak partial metric space. Then*

(a) $\{x_n\}$ *is a Cauchy sequence in* (X, p) *if and only if* $\{x_n\}$ *is a Cauchy sequence in metric space* (X, d_ω).

(b) (X, p) *is complete if and only if* (X, d_ω) *is complete.*

Lemma 2.2: *Let* (X, p) *be a weak partial metric space and* $\{x_n\}$ *is a sequence in* X. *If* $\lim_{n\to\infty} x_n = x$ *and* $p(x, x) = 0$, *then* $\lim_{n\to\infty} p(x_n, y) = p(x, y)$, *for all* $y \in X$.

Proof: By (ωP_3),

$$p(x, y) \le p(x, x_n) + p(x_n, y).$$

Hence

$$p(x, y) - p(x, x_n) \le p(x_n, y) \le p(x_n, x) + p(x, y).$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} p(x_n, y) = p(x, y).$$

Definition 2.5: *(J. Ali and M. Imdad[13]) Two pairs* (A, S) *and* (B, T) *of self-maps of a weak partial metric space* (X, p) *are said to satisfy the common property (E. A.) if there exists two sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *such that,*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t, \text{ for some } t \in X.$$

Inspired by Ranadive et al.[9] we also define here the same notion of absorbing mappings in the weak partial metric space as follows:

Definition 2.6: *Let* A *and* S *be two self-maps of a metric space* (X, p). *The map* A *is said to be* S*-absorbing if there exists* $\Re > 0$ *such that,*

$$p(Sx, SAx) \le \Re\, p(Sx, Ax), \text{ for all } x \in X.$$

Similarly the map S *is said to be* A*-absorbing if there exists* $\Re > 0$ *such that,*

$$p(Ax, ASx) \le \Re\, p(Ax, Sx), \text{ for all } x \in X.$$

Similarly, we can defined point wise absorbing map, mapping A *is said to be point-wise* S*-absorbing if for given* $x \in X$ *there exists* $\Re > 0$ *such that,*

$$p(Sx, SAx) \leq \Re\, p(Sx, Ax),$$

and S *is said to be point-wise* A*-absorbing if for given* $x \in X$ *there exists* $\Re > 0$ *such that,*

$$p(Ax, ASx) \leq \Re\, p(Ax, Sx)$$

It is clear that the notion of absorbing maps is different from other generalizations of commutativity which force the mappings to commute at coincidence points (For detail see Mishra et al.[10], Gopal et al.[14], Mishra et al.[15]). Recently Ali & Imdad[13] gave implicit function as follows:

Let ψ_6 be the family of lower semi-continuous functions $F(t_1, t_2, t_3, t_4, t_5, t_6): \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) $F(u, 0, u, 0, 0, u)$ for all $u > 0$,

(F_2) $F(u, 0, 0, u, u, 0)$ for all $u > 0$,

(F_3) $F(u, u, 0, 0, u, u)$ for all $u > 0$.

Now we give some examples for above implicit function:

Example 2.1: Define $F(t_1, t_2, t_3, t_4, t_5, t_6): \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_3 - b\left(\frac{t_5 + t_6}{1 + t_2 + t_4}\right),$$

where $a, b \geq 0$ and $a + b < 1$.

Example 2.2: Define $F(t_1, t_2, t_3, t_4, t_5, t_6): \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\left\{t_2 t_3, \frac{t_3 + t_5}{2}, \frac{t_4 + t_6}{2}\right\},$$

where $k \in [0, 1]$.

Example 2.3: Define $F(t_1, t_2, t_3, t_4, t_5, t_6): \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k_1 \max\{t_2, t_3, t_4\} - k_2 \max\{t_5, t_6\},$$

where $k_1, k_2 \geq 0$ and $k_1 + k_2 < 1$.

Example 2.4: Define $F(t_1, t_2, t_3, t_4, t_5, t_6):\mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \alpha\left(\frac{t_3t_5 + t_4t_6}{t_2t_3 + 1}\right),$$

where $\alpha \in [0, 1]$.

Example 2.5: Define $F(t_1, t_2, t_3, t_4, t_5, t_6):\mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\left\{t_2t_4, \alpha(t_3t_5)^{\frac{1}{2}}, \frac{t_4 + t_6}{2}\right\},$$

where $\alpha > 0$.

Example 2.6: Define $F(t_1, t_2, t_3, t_4, t_5, t_6):\mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha(t_2t_3) - \beta(t_3t_5 + t_4t_6),$$

where $\alpha, \beta \in (0, 1)$ and $\alpha + \beta < 1$.

Example 2.7: Define $F(t_1, t_2, t_3, t_4, t_5, t_6):\mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = \begin{cases} t_1^2 - \alpha \max\{t_3t_5, t_4t_6\} \\ -\beta\left(\dfrac{t_2^3 + t_5^3}{t_2 + t_5}\right) & \text{if } t_2 + t_5 \neq 0 \\ t_1 & \textit{if } t_2 + t_5 = 0. \end{cases}$$

where $\alpha + \beta < 1$ and $\beta \in (0, 1)$.

3. Main Results

Now we are ready to state and prove our main result as follows:

Theorem 3.1: *Let A, B, S and T be four self maps on a weak partial metric space (X, p) satisfying:*

(1) $S(X)$ and $T(X)$ are closed subsets of X,

(2) the pairs (A, S) *and* (B, T) *satisfy the common property (E. A.),*

(3) for all $x, y \in X$ *and* $F \in \psi_6$,

$$F\big(p(Ax, By),\ p(Sx, Ty),\ p(Ax, Sx),\ p(By, Ty),\ p(Sx, By),\ p(Ty, Ax)\big) \le 0$$

Then the pairs (A, S) *and* (B, T) *have coincidence point. Moreover if* A *is pointwise* S*- absorbing and* B *is pointwise* T*-absorbing then* A, B, S *and* T *have unique common fixed point.*

Proof: Suppose that the pairs (A, S) and (B, T) satisfy the common property (E. A.). Then by condition (2) there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

for some $t \in X$. Since $S(X)$ is closed subset of X, $\lim_{n\to\infty} Sy_n = t \in S(X)$, therefore there exists a point $u \in X$ such that $Su = t$. Now we claim that $Au = Su$, for if not then by condition (3), we have,

$$F\big(p(Au, By_n),\ p(Su, Ty_n),\ p(Au, Su),\ p(By_n, Ty_n),$$
$$p(Su, By_n),\ p(Ty_n, Au)\big) \le 0.$$

Letting $n \to \infty$, we get,

$$F\big(p(Au, t),\ p(Su, t),\ p(Au, t),\ p(t, t),\ p(t, t),\ p(t, Au)\big) \le 0,$$

i.e. $$F\big(p(Au, t), 0,\ p(Au, t), 0, 0,\ p(t, Au)\big) \le 0.$$

So by (F_1) we get a contradiction. Hence $Au = Su$. Thus u is a coincidence point of A and S. Now if $T(X)$ is closed subset of X, then $\lim_{n\to\infty} Ty_n = t \in T(X)$. Therefore there exists a point w in X such that $Tw = t$. We claim that $Bw = Tw$, for if not then by (3), we have,

$$F\big(p(Ax_n, Bw),\ p(Sx_n, Tw),\ p(Ax_n, Sx_n),\ p(Bw, Tw),$$
$$p(Sx_n, Bw),\ p(Tw, Ax_n)\big) \le 0$$

Letting $n \to \infty$, we get,

$$F\left(p(t, Bw), p(t, Tw), p(t, t), p(Bw, Tw), p(t, Bw), p(Tw, t)\right) \le 0,$$

i.e. $$F\left(p(Tw, Bw), 0, 0, p(Bw, Tw), p(Bw, Tw), 0\right) \le 0,$$

a contradiction to (F_2). Hence $Bw = Tw$. Thus w is a coincidence point of the pair (B, T). Now since A is point wise S-absorbing, there exists $\Re > 0$ such that,

$$p(Su, SAu) \le \Re\, p(Su, Au) \text{ i.e. } Su = SAu \Rightarrow t = St.$$

Now we assert $At = t$, for if not then by condition (3), we get,

$$F\left(p(At, By_n), p(St, Ty_n), p(At, St), p(By_n, Ty_n), p(St, By_n), p(Ty_n, At)\right) \le 0.$$

Letting $n \to \infty$,

$$F\left(p(At, t), p(t, t), p(At, t), p(t, t), p(t, t), p(t, At)\right) \le 0,$$

i.e. $$F\left(p(At, t), 0, p(At, t), 0, 0, p(t, At)\right) \le 0,$$

a contradiction to (F_1). Hence $At = St = t$. Also, B is point wise T absorbing, there exists $\Re > 0$ such that,

$$p(Tw, TBw) \le \Re\, p(Tw, Bw) \quad \text{i.e.} \quad Tw = TBw \Rightarrow t = Tt.$$

Now suppose $t \ne Bt$ then by condition (3), we get,

$$F\left(p(Au, Bt), p(Su, Tt), p(Au, Su), p(Bt, Tt), p(Su, Bt), p(Tt, Au)\right) \le 0,$$

or $$F\left(p(t, Bt), p(t, t), p(t, t), p(Bt, t), p(t, Bt), p(t, t)\right) \le 0,$$

i.e. $$F\left(p(t, Bt), 0, 0, p(Bt, t), p(t, Bt), 0\right) \le 0,$$

a contradiction to (F_2). Thus $Bt = Tt = t$. Therefore we have $At = Bt = St = Tt = t$ and hence t is a common fixed point of A, B, S and T.

For uniqueness of common fixed point, let as assume that v be another common fixed point of A, B, S and T. Then putting $x=v$, $y=t$ in (3), we get,

$$F\left(p(Au, Bt),\ p(Sv, Tt),\ p(Av, Sv),\ p(Bt, Tt),\ p(Sv, Bt),\ p(Tt, Av)\right)\le 0,$$

or $$F\left(p(t, Bt),\ p(t, t),\ p(t, t),\ p(Bt, t),\ p(t, Bt),\ p(t, t)\right)\le 0,$$

i.e. $$F\left(p(t, Bt),\ 0,\ 0,\ p(Bt, t),\ p(t, Bt),\ 0\right)\le 0,$$

a contradiction to (F_2). Thus $v=t$. Therefore we have $At=Bt=St=Tt=t$ and hence t is the unique common fixed point of A, B, S and T. This completes the proof.

Theorem 3.2: *Let A, B, S and T be four self maps on a weak partial metric space (X, p) satisfying:*

(1) $\overline{A(X)}\subset T(X)$ and $\overline{B(X)}\subset S(X)$,

(2) the pairs (A, S), (B, T) satisfy the common property (E. A.),

(3) for all $x, y\in X$ and $F\in\psi_6$,

$$F\left(p(Ax, By),\ p(Sx, Ty),\ p(Ax, Sx),\ p(By, Ty),\ p(Sx, By),\ p(Ty, Ax)\right)\le 0.$$

Then the pairs (A, S) and (B, T) have coincidence point. Moreover if A is pointwise S-absorbing and B is pointwise T-absorbing then A, B, S and T have unique common fixed point.

Proof: The proof of this theorem will remain same as the proof of theorem 3.1.

Theorem 3.3: *Let A, B, S and T be four self-maps on a weak partial metric space (X, p) satisfying:*

(1) $A(X)\subset T(X)$ (or $B(X)\subset S(X)$),

(2) $S(X)$ (or $T(X)$) is closed subsets of X,

(3) the pairs (A, S) and (B, T) enjoy the property (E. A.),

(4) for all $x, y\in X$ and $F\in\psi_6$,

$$F\left(p(Ax, By),\ p(Sx, Ty),\ p(Ax, Sx),\ p(By, Ty),\ p(Sx, By),\ p(Ty, Ax)\right)\le 0.$$

Then the pairs (A, S) *and* (B, T) *have coincidence point. Moreover, if* A *is pointwise* S*-absorbing and* B *is pointwise* T*-absorbing then* A, B, S *and* T *have unique common fixed point.*

Proof: As the pairs (A, S) and (B, T) satisfy the common property (E.A.). Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Sz,$$

for some $z \in X$. Now we assert that $Az = Sz$, for if not then putting $x = z$, $y = y_n$
by condition (4), we have,

$$F\left(p(Az, By_n), p(Sz, Ty_n), p(Az, Sz), p(By_n, Ty_n), p(Sz, By_n), p(Ty_n, Az)\right) \le 0$$

Letting $n \to \infty$, we get,

$$F\left(p(Az, Sz), p(Sz, Sz), p(Az, Sz), p(Sz, Sz), p(Sz, Sz), p(Sz, Az)\right) \le 0,$$

i.e. $$F\left(p(Az, Sz), 0, p(Az, Sz), 0, 0, p(Sz, Az)\right) \le 0,$$

which is contradiction. Hence $Az = Sz$. Thus z is a coincidence point of A and S. Now if $A(X) \subset T(X)$, then there exists $t \in X$ such that $Az = Tt$. Further we assert that $Bt = Tt$. If not then putting $x = z$, $y = t$ in (4), we have,

$$F\left(p(Az, Bt), p(Sz, Tt), p(Az, Sz), p(Bt, Tt), p(Sz, Bt), p(Tt, Az)\right) \le 0,$$

or $$F\left(p(Tt, Bt), p(Tt, Tt), p(Sz, Sz), p(Bt, Tt), p(Tt, Bt), p(Tt, Tt)\right) \le 0,$$

i.e. $$F\left(p(Tt, Bt), 0, 0, p(Bt, Tt), p(Tt, Bt), 0\right) \le 0,$$

a contradiction. Hence $Bt = Tt$. Thus t is a coincidence point of the pair (B, T). Thus we have $u = Bt = Tt = Az = Sz$. Now, since A is pointwise S-absorbing, there exists $\Re > 0$ such that,

$$p(Sz, SAz) \le \Re\, p(Sz, Az) \text{ i.e. } Sz = SAz \Rightarrow u = Su$$

Now we assert $Au = u$, If not, then putting $x = u$, $y = y_n$ in condition (4), we get,

$$F\left(p\left(Au, By_n\right), p\left(Su, Ty_n\right), p\left(Au, Su\right), p\left(By_n, Ty_n\right), p\left(Su, By_n\right), p\left(Ty_n, Au\right)\right) \le 0$$

Letting $n \to \infty$,

$$F\left(p\left(Au, Sz\right), p\left(Su, Sz\right), p\left(Au, Su\right), p\left(Sz, Sz\right), p\left(Su, Sz\right), p\left(Sz, Au\right)\right) \le 0,$$

i.e. $F\left(p\left(Au, u\right), p\left(u, u\right), p\left(Au, u\right), 0, p\left(u, u\right), p\left(u, Au\right)\right) \le 0$,

i.e. $F\left(p\left(Au, u\right), 0, p\left(Au, u\right), 0, 0, p\left(u, Au\right)\right) \le 0$,

a contradiction. Hence $Au = u = Su$, which shows that u is common fixed point of the pair $\left(A, T\right)$. Similarly, B is pointwise T -absorbing, there exists $\Re > 0$ such that

$$p\left(Tt, TBt\right) \le \Re\, p\left(Tt, Bt\right) \quad \text{i.e. } Tt = BTt \Rightarrow u = Tu .$$

Now suppose $u \ne Bu$, then putting $x = z$, $y = u$ in condition (4), we get,

$$F\left(p\left(Az, Bu\right), p\left(Sz, Tu\right), p\left(Az, Sz\right), p\left(Bu, Tu\right), p\left(Sz, Bu\right), p\left(Tu, Az\right)\right) \le 0,$$

or $F\left(p\left(u, Bu\right), p\left(u, u\right), p\left(u, u\right), p\left(Bu, u\right), p\left(u, Bu\right), p\left(u, u\right)\right) \le 0$,

i.e. $F\left(p\left(u, Bu\right), 0, 0, p\left(Bu, u\right), p\left(u, Bu\right), 0\right) \le 0$,

a contradiction. Thus $Bu = Tu = u$ Therefore, we have $Au = Bu = Su = Tu$ $= u$ and hence u is a common fixed point of A, B, S and T . For uniqueness of common fixed point, let as assume that w be another common fixed point of A, B, S and T . Then putting $x = w$, $y = u$ in (4), we get

$$F\left(p\left(Aw, Bu\right), p\left(Sw, Tu\right), p\left(Aw, Sw\right), p\left(Bu, Tu\right), p\left(Sw, Bu\right), p\left(Tu, Aw\right)\right) \le 0,$$

or $F\left(p\left(w, u\right), p\left(w, u\right), p\left(w, w\right), p\left(u, u\right), p\left(w, u\right), p\left(u, w\right)\right) \le 0$, i.e. $F\left(p(w, u), p(w, u), 0, 0, p(w, u), p(u, w)\right) \leq 0$,

a contradiction. Thus $w = u$. Therefore, we have $Au = Bu = Su = Tu = u$ and hence u is the unique common fixed point of A, B, S and T. This completes the proof.

Now we give an example which illustrates our Theorems 3.1, 3.2 and 3.3.

Example 3.1: Let $X = [2, 20]$ be a metric space with weak partial metric p. Define A, B, S and T self - mappings of (X, p) as:

$$Ax = 2 \text{ if } 2 \leq x \leq 5, \quad Ax = 3 \text{ if } x > 5;$$

$$Bx = 2 \text{ if } 2 \leq x < 5, \quad Bx = 3 \text{ if } x \in [5, 20];$$

$$S_2 = 2, \; Sx = 3, \; 2 < x \leq 5, \; Sx = \frac{x+1}{3} \text{ if } x > 5 \text{ and}$$

$$T_2 = 2, \; T_3 = 3, \; Tx = 12 + x \text{ if } 3 < x < 8, \; Tx = x - 5 \text{ if } x \geq 8.$$

Clearly, both the pairs (A, S) and (B, T) satisfy the common property (E. A.) as there exists two sequences $x_n = \left\{8 + \frac{1}{n}\right\}$, $y_n = \left\{8 + \frac{1}{n}\right\} \in X$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n \lim_{n \to \infty} Ty_n = 3.$$

Also $A(x) = \{2, 3\} \subset [2, 20] = T(x)$ and $B(x) = \{2, 3\} = [2, 7] \cup \{3\} = S(x)$
Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - b\left(\frac{t_3 + t_4}{2}\right) - c\left(\frac{t_{35} + t_6}{2}\right),$$

where $a, b, c \geq 0$ and $a + b + c < 1$.

By routine calculation we can verify that contraction condition of theorem is satisfied for $a = 0$, $b = 1/4$, $c = 1/2$. If $x, y \in [2, 5)$ then $p(Ax, By) = 0$ and also if $x, y \in (5, 20]$ then $p(Ax, By) = 0$ and verification is of contractive condition is trivial. If $x = 5$ and $y \in [5, 8)$, then

$$ap(Sx, Ty)+b\left(\frac{p(Ax, Sx)+p(By, Ty)}{2}\right)+c\left(\frac{p(Sx, By)+p(Ty, Ax)}{2}\right)$$

$$=0.|3-12-y|+\frac{1}{4}\left(\frac{|3-2|+|12+y-3|}{2}\right)+\frac{1}{2}\left(\frac{|3-3|+|12+y-2|}{2}\right)$$

$$=0+\frac{1}{4}\left(\frac{1+|9+y|}{2}\right)+\frac{1}{2}\left(\frac{|10+y|}{2}\right)\geq 1=p(Ax, By).$$

Similarly, one can verify the other cases. Thus all the conditions of above theorems are satisfied and 2 is the unique common fixed point of A, B, S and T. Here one may notice that all the mappings in this example are even discontinuous at their unique common fixed point 2.

4. Application

Let (X, p) be a weak partial metric space and X is a function $f:(X, p)\to(X, p)$ such that $\mathrm{Fix}(f)=\{x\in X:\ x=fx\}$.

Theorem 4.1: Let A, B, S and T be four self-maps on a weak partial metric space (X, p). Since theorem 3.1 holds for all $x, y\in X$ and $F\in\psi_6$, then

$$[\mathrm{Fix}(A)\cap\mathrm{Fix}(S)]\cap\mathrm{Fix}(T)=[\mathrm{Fix}(B)\cap\mathrm{Fix}(S)]\cap\mathrm{Fix}(T).$$

Proof: Let $x\in[\mathrm{Fix}(A)\cap\mathrm{Fix}(S)]\cap\mathrm{Fix}(T)$. Then by theorem 3.1, we have

$$F\left(p(Ax,Bx),\ p(Sx, Tx),\ p(Ax, Sx),\ p(Tx, Bx),\ p(Sx, Bx),\ p(Tx, Ax)\right)\leq 0,$$

$$F\left(p(x, Bx), 0, 0,\ p(x, Bx),\ p(x, Bx), 0\right)\leq 0,$$

If $p(x, Bx)>0$, then a contradiction of (F_2). Hence, $p(x, Bx)=0$, which implies $x=Bx$. Therefore

$$[\mathrm{Fix}(A)\cap\mathrm{Fix}(S)]\cap\mathrm{Fix}(T)\subset[\mathrm{Fix}(B)\cap\mathrm{Fix}(S)]\cap\mathrm{Fix}(T).$$

Similarly, by theorem 3.1 and (F_1), we obtain,

$$\left[\mathrm{Fix}(B)\cap\mathrm{Fix}(S)\right]\cap\mathrm{Fix}(T)\subset\left[\mathrm{Fix}(A)\cap\mathrm{Fix}(S)\right]\cap\mathrm{Fix}(T).$$

Hence,

$$\left[\mathrm{Fix}(A)\cap\mathrm{Fix}(S)\right]\cap\mathrm{Fix}(T)=\left[\mathrm{Fix}(B)\cap\mathrm{Fix}(S)\right]\cap\mathrm{Fix}(T).$$

This completes the proof.

5. Conclusion

In this attempt we prove common fixed point theorems by using the new notion called absorbing mappings for satisfying implicit relation and we have given one interesting example in support of our theorem. In the last we present the application of our result.

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References

1. S. G. Matthews, Partial Metric Topology, *Proc. 8th Summer Conference General Topology Appl., Ann. New Work Acad. Sci.*, **728** (1996), 183-197.
2. O. Valero, On Banach Fixed Point Theorems for Partial Metric Spaces, *Appl. General Topology,* **6** (2005), 229-240.
3. S. Oltra and O. Valero, Banach's Fixed Point Theorem for Partial Metric Spaces, *Rend. Istid. Math. Univ. Trieste*, **36** (2004), 17-26.
4. I. Altun and H. Simsek, Some Fixed Point Theorems on Dualistic Partial Metric Spaces, *Jour. Adv. Math. Stud.,* **1** (2008), 1-8.
5. I. Altun, F. Sola and H. Simsek, Generalized Contractions on Partial Metric Spaces, *Topology and its Applications*, **157** (2010), 2778-2785.
6. R. Heckmann, Approximation of Metric Spaces by Partial Metric Spaces, *Appl. Categ. Struct.*, 7 (1999), 71-88.
7. I. Altun and G. Durmaz, Weak Partial Metric Spaces and Some Fixed Point Results, *Appl. Gen. Topo.,* **13(2)** (2012), 179-191.
8. G. Durmaz, O. Acar and I. Altun, Some Fixed Point Results on Weak Partial Metric Spaces, *Filomat*, **27(2)** (2013), 317-326.
9. D. Gopal, A. S. Ranadive and U. Mishra, On Some Open Problems of Common Fixed Point Theorems for a Pair of Non-Compatible Self-Maps, *Proc. Math. Soc., BHU*, **20** (2004), 135-141.

10. U. Mishra, A. S. Ranadive and D. Gopal, Fixed Point Theorems Via Absorbing Maps, *Thai Jour. Math.*, **6(1)** (2008), 49-60.

11. V. Popa, Some Fixed Point Theorems for Weakly Compatible Mappings, *Rad. Mat.*, **10** (2001), 245-252.

12. S. Romaguera, A Kirk Type Characterization of Completeness for Partial Metric Spaces, *Fixed Point Theory and Applications,* **2010** (2010), 1-6.

13. J. Ali and M. Imdad, An Implicit Function Implies Several Contraction Conditions, *Sarajevo Jour. Math.,* **4(17)** (2008), 269-285.

14. D. Gopal, R. P. Pant and A. S. Ranadive, Common Fixed Point of Absorbing Maps, *Bull. Marathwada Math. Soc.,* **9(1)** (2008), 43-48.

15. U. Mishra, A. S. Ranadive, A. K. Dubey and W. H. Lim, Common Fixed Point of Absorbing Mapping Satisfying Implicit Relation, *Non. Func. Ana. Appl.,* **25(1)** (2020), 117-126.