ISSN 0974 - 9373

Vol. 25 No. 3 (2021) **Journal of International Academy of Physical Sciences** pp. 393- 412

Mathematical Modeling of the Effect of Pollutant-Affected Resource Biomass on Biological Species

Alok Malviya

Department of Mathematics, V.S.S.D. (P.G.) College, Kanpur
E-mail: alok92nov@gmail.com

Sheodan Singh Bhadoriya

Commissioner of Income tax, Kanpur
E-mail: dr.sheodansingh@gmail.com

Maninder Singh Arora

Department of Mathematics, P.P.N. (P.G.) College, Kanpur
E-mail: maninderarora120@gmail.com

(Received July 07, 2021)

Abstract: Pollutants and toxicants are released into the environment as a result of numerous industrial and other human activities. These dangerous discharges affect the resource biomass which eventually affects the biological species. In this paper, we have proposed and analyzed a non-linear mathematical model to study the effect of pollutant-affected resource biomass on biological species. Five dependent variables have been considered in the modelling process. The model has been analyzed analytically through equilibrium and stability analysis. It has been found that the environmental tax imposed on the polluters proves as an important measure to maintain the resource dependent biological species at a desirable level. Numerical simulation is also done to illustrate the analytical findings.

Keywords: Resource biomass, Stability analysis, Numerical Simulation, Biological Species, Mathematical Model.

2010 AMS Subject Classification: 92D40.

1. Introduction

It is well known that due to the rapid growth of industrialization, various kinds of toxicants(pollutants) are discharged into both aquatic and terrestrial environment affecting resources as well as living beings in such a habitat. The ceaseless consumption of all natural resources without recharging or purifying them has created a hazardous situation. For instance, various kinds of industrial discharges and chemical spills have affected the water, air and land which in turn have affected a large number of biological species. In this paper, it has been considered that the biological species under consideration are either partially or wholly dependent on resource biomass. Therefore, the former gets affected due to the pollutants indirectly. The examples of these kinds of problems may be found in the ecosystem in which the air pollution affects the forests and then the forest dependent biological species. Another example is the water pollution affecting the planktons or the prey fishes on which the predator fishes or other aquatic creatures are dependent.

Hallam and Clark[1], Hallam *et al.*[2,3], De Luna and Hallam[4], Shukla et al.[5], Huaping and Zhein[6], Shukla and Dubey[7,8] have done valuable work in this field. Various mathematical models have been suggested and analyzed relating to the effects of the pollutants generated from various industrial activities and other activities of the human beings. In particular, Hallam *et al.*[3], studied the effects of the pollutants on the population. De Luna and Hallam[4] have discussed the impact of a toxicant on population and have shown that a population can only survive if it has a consistent growth potential. Huaping and Zhein[6] proposed a mathematical model to study the effects of polluting agents and toxicants in a two species competitive system. Shukla and Dubey[8] have investigated the detrimental effects of industrialization and population on forestry resources. Shukla *et al.*[9] have further studied the existence and survival of two competing species in a polluted environment using mathematical models. Shukla *et al.*[10] also modeled the effects of primary and secondary toxicants on renewable resources. Dubey and Hussain[11] have devised a model for studying the survival of a resource-dependent species in a polluted environment. Rai and Malviya[12] have proposed mathematical model for the survival of resource-based industries when its resource is being depleted by toxicants. Shukla *et al.*[13], Agarwal *et al.*[14] and Malviya[15,16] have proposed mathematical models for the survival of biological species which are directly affected by pollutants discharged from external sources as well as formed by its precursor. In this paper, we have proposed an ecological model for studying the conservation of resource-based biological species by imposing environmental tax on the polluters.

2. Ecological Model

We have considered biological species which are dependent on resource biomass, both growing logistically in their habitat. The intrinsic growth rate and carrying capacity of resource biomass are dependent on the concentration of the pollutant emitted from various external sources. We have assumed that the biological species and resource biomass under consideration are dynamic in nature i. e. regenerating themselves. It has been assumed that the cumulative rate of discharge of pollutants from the external sources is constant when there is no environmental tax imposed on the emitters. It is also assumed that pollutant concentration in the environment is reduced as a result of absorption, deposition, assimilation and uptake by resource biomass, the quantity being proportional to the biomass density of the resource as well as the toxicant concentration in the environment. It is further assumed that a fraction of the total amount of the pollutants become part of the uptake phase in the resource biomass. Some of the pollutants in the uptake phase are decomposed and the same reenter in the environment. It is also assumed that the environmental tax is imposed only when the concentration of the pollutants crosses the harmful limits.

Based on above mentioned assumptions, we propose the following ecological model to study the dynamics of the biological species depending on resource biomass affected by the pollutants and controlled by the environmental tax-

$$
\begin{aligned}
\frac{dN}{dt} &= rN - \frac{r_0 N^2}{K} + \beta_1 NB \\
\frac{dB}{dt} &= s(U)B - \frac{s_0 B^2}{L(T)} - \beta_2 NB - k\alpha BT \\
\frac{dT}{dt} &= Q - \rho I - \delta T - \alpha BT + \pi vUB \\
\frac{dU}{dt} &= (1-k)\alpha BT - \varnothing U - vUB \\
\frac{dI}{dt} &= \theta(T - T_0) - \theta_0 I,
\end{aligned}
\tag{2.1}
$$

where,

$$
\frac{dr(U)}{dU} < 0 \ \ for\, U \geq 0
$$

$$\frac{d\,K(T)}{dT} < 0 \ for\, T \geq 0$$

and

$$N(0) = N_0 \geq 0; B(0) = B_0; T(0) = T_0 \geq 0; U(0) = U_0 \geq 0;\ r(0) = r_0 > 0;$$

$$s(0) = s_0; K > 0;\ L(0) = L_0; \alpha; \beta_1; \beta_2; Q; \delta; v; \rho; \varnothing; \theta; \theta_0; T_0 > 0, 0 \leq k; \pi \leq 1.$$

Here, $N(t)$ is the density of population of biological species under consideration which is dependent on the resource biomass of density $B(t)$. $T(t)$ and $U(t)$ are the concentrations of the pollutant under consideration in the environment and in the uptake phase in the resource biomass respectively at any moment $t.Q$ and ρ are constants and I is the environment tax, introduced to control the emission of pollutants, collected in unit period of time from various industries. When there is no tax this emission of the pollutants has been considered as Q, which is constant at particular point of time but with the introduction of tax, this is limited by the factor $Q - \rho I$. The biological species under consideration are growing and β_1 is the growth rate coefficient of the same. It has been assumed that the resource biomass is growing intrinsically with the rate s, which is to some extent being affected by the concentration of the pollutants in the resource biomass, i.e. s is function of U which is the uptake concentration of the pollutant in resource biomass. Since the intrinsic growth of the resource biomass decreases with the presence of the pollutants,

(2.1a) $$\frac{dS(U)}{dU} < 0 \ for\, U \geq 0,$$

thus we take

$$s(U) = s_0 - \frac{a_1 U(t)}{1 + s_1 U(t)}.$$

It is assumed that the resource biomass is also depleting, β_2 is the depletion rate coefficient of the same. The toxicant is also depleting in the environment and in the resource biomass due to various internal physiological phenomenon, we are assuming the depletion rate coefficients of the pollutant in the environment as δ and in the resource biomass as ϕ.

The resource biomass also consumes some amount of pollutant. Therefore α is considered as depletion rate of the pollutants in the environment due to uptake of the pollutants by the resource biomass under consideration. Let the dying rate of resource biomass due to pollutants be ν and a fraction π of this be reentering in the environment. Let αBT be the rate with which concentration of the pollutant in the environment is decreasing and let us assume that k is fraction of this αBT with which the resource biomass is being affected, then $k\alpha BT$ will be rate of decrease of B due to this phenomenon. Then concentration of the pollutant in the resource biomass will be getting affected by the factor $(1-k)\alpha BT$. νUB is the rate with which concentration of the pollutants in the resource biomass is decreasing and $\pi\nu UB$ is the rate with which the pollutants are reentering in the environment. Let the function $L(T)$ denote the maximum density of the resource biomass which can be supported by the environment. then with the increase of concentration of pollutants in the environment. this function decreases i.e.

(2.1b) $$\frac{dL(T)}{dT} < 0 \ for\ T \geq 0,$$

thus we take

$$L(T) = L_0 - \frac{b_1 T(t)}{1 + m_1 T(t)}.$$

In this model, the carrying capacity of the biological species under consideration has been considered as constant K and r is rate of increase of the biological species. It has been assumed that if at a particular time the level of concentration of the pollutants in the ecosystem (T) is less than the permissible limit T_0, then no tax would be imposed on the various industries. Now the method of imposing the tax may be devised on the basis of emission of pollutants/toxicants by a particular industry. θ and θ_0 are the constants, where $\theta_0 I$ is the factor which has been considered due to some practical difficulties on implementing the foolproof tax system.

3. Equilibrium Analysis

The above Ecological model is such that the variable space is restricted to the positive quadrants. This model has four non-negative real equilibria in $N-B-T-U-I$ space, namely,

$$E_0\left(0,0,T,0,I\right)=E_0\left[0,0,\frac{Q\theta_0+\rho\theta T_0}{\delta\theta_0+\rho\theta},0,\frac{Q\theta-\delta\theta T_0}{\rho\theta+\delta\theta_0}\right],$$

$$E_1\left(\hat{N},0,\hat{T},0,\hat{I}\right)=E_1\left[\frac{rK}{r_0},0,\frac{Q\theta_0+\rho\theta T_0}{\delta\theta_0+\rho\theta},0,\frac{Q\theta-\delta\theta T_0}{\rho\theta+\delta\theta_0}\right],\ E_2\left(0,\tilde{B},\tilde{T},\tilde{U},\tilde{I}\right)$$

and $E^*\left(N^*,B^*,T^*,U^*,I^*\right)$, (where $\delta\theta_0+\rho\theta\neq 0$ since all these parameters are positive)

Existence of equilibrium at: $E^*\left(N^*,B^*,T^*,U^*,I^*\right)$: This equilibrium point is the solution of following system of equations-

(3.1a) $$N=\frac{\left(r+\beta_1 B\right)K}{r_0}=f\left(B\right)\ \text{(say)}$$

(3.1b) $$B=\frac{s\left(U\right)L\left(T\right)-\beta_2 NL\left(T\right)-k\alpha TL\left(T\right)}{s_0},$$

Provides $\left(U\right)-\beta_2 N-k\alpha T>0$.

(3.1c) $$T=\frac{\left(Q\theta_0+T_0\rho\theta\right)\left(\varnothing+vB\right)}{f_1\left(B\right)}=g\left(B\right),\ \text{(say)}$$

(3.1d) $$U=\frac{\left(Q\theta_0+T_0\rho\theta\right)\left(1-k\right)\alpha B}{f_1\left(B\right)}=h\left(B\right),\ \text{(say)}$$

(3.1e) $$I=\frac{\theta}{\theta_0}[g(B)-T_0]=i(B),\ \text{(say)}$$

Provided $g\left(B\right)>0$. Now to show the existence of the general equilibrium point $E^*\left(N^*,B^*,T^*,U^*,I^*\right)$, let $F\left(B\right)$ be a function such that;

(3.1f) $$F\left(B\right)=s_0 B-s\left(h\left(B\right)\right)L\left(g\left(B\right)\right)+\beta_2 f\left(B\right)L\left(g\left(B\right)\right)+k\alpha g\left(B\right)L\left(g\left(B\right)\right)$$

From the equation (3.1a) to (3.1e) and putting $B=0$ and L_0 in (3.1f), we get

$$F(0)=-L\left(\frac{Q\theta_0+\rho\theta T_0}{\delta\theta_0+\rho\theta}\right)\left[s_0-\beta_2\frac{rK}{r_0}-k\alpha\left(\frac{Q\theta_0+\rho\theta T_0}{\delta\theta_0+\rho\theta}\right)\right]$$

$$F(0)=-L(T)\left[s_0-\beta_2 N-k\alpha T\right] \, at \; B=0$$

Since $L(T)$ being carrying capacity of resource biomass is always positive and $s_0-\beta_2 N-k\alpha T>0$, being intrinsic growth rate, hence $F(0)<0$. Again, since

$$F(L_0)=L_0[s_0-s\left(h\left(L_0\right)\right)+\beta_2 f\left(L_0\right)]>0$$

Hence on the basis of mean value theorem it can be concluded that there exists a root B^* of $F(B)=0$ in the interval $0<B^*<L_0$.

Now to prove the uniqueness of the B^*, the necessary and sufficient condition is that F'(B) is positive in the interval $(0, L_0)$.

$$F'(B)=s_0+\frac{dL}{dg}\frac{dg}{dB}\left[-s\left(h(B)\right)+\beta_2 f\left(B\right)+k\alpha g\left(B\right)\right]$$

$$+L(g(B))\left[-\frac{ds}{dh}\frac{dh}{dB}+\beta_2\left(\frac{K\beta_1}{r_0}\right)+k\alpha\frac{dg}{dB}\right]$$

Now since $F(B)=0$ at $B=B^*$, we have from (3.1f),

$$F'\left(B\right)=s_0+\frac{dL}{dg}\frac{dg}{dB}\left[-s_0\frac{B}{L\left(g\left(B\right)\right)}\right]-L\left(g\left(B\right)\right)\frac{ds}{dh}\frac{dh}{dg}$$

$$+k\alpha L\left(g\left(B\right)\right)\frac{dg}{dB}+\beta_2 K\frac{\beta_1}{r_0}L(g(B))$$

This should be greater than 0 for the root B to be unique. and when B^* is determined, N^*, T^*, U^* and I^* can be found by solving the system (3.1a) to (3.1e).

Thus, the existence of equilibrium at $E^*\left(N^*,B^*,T^*,U^*,I^*\right)$ is proved and the condition of uniqueness of the equilibrium point has been obtained.

Effect of the cumulative rate of introduction of the pollutant on the density of the resource biomass: Now let us examine the effect of Q on B-From equation (3.1b),

$$s_0 B = s\left(h\right)L\left(g\right)-\frac{\beta_2}{r_0}(rK+\beta_1 KB)L\left(g\right)-k\alpha g L\left(g\right)$$

Therefore,

$$\frac{dB}{dQ}s_0 = s\left(h\right)\frac{dL}{dg}\frac{dg}{dQ}-\frac{\beta_2}{r_0}\left(rK+\beta_1 KB\right)\frac{dL}{dg}\frac{dg}{dQ}-k\alpha g\frac{dL}{dg}\frac{dg}{dQ}$$

$$+L\left(g\right)\frac{ds}{dh}\frac{dh}{dQ}-\frac{\beta_2}{r_0}L\left(g\right)\beta_1 K\frac{dB}{dQ}-k\alpha L(g)\frac{dg}{dQ}$$

Using

$$\frac{dg}{dQ}=\frac{\partial g}{\partial B}\left(\frac{dB}{dQ}\right)+\frac{\partial g}{\partial Q} \quad \text{and} \quad \frac{dh}{dQ}=\frac{\partial h}{\partial B}\left(\frac{dB}{dQ}\right)+\frac{\partial h}{\partial Q}$$

Therefore, on substituting and solving we get:

$$\frac{dB}{dQ}\left[s_0 - s_0\frac{B}{L\left(g\right)}\frac{dL}{dg}\frac{\partial g}{\partial B}+L\left(g\right)\frac{\beta_1\beta_2 K}{r_0}+k\alpha L\frac{\partial g}{\partial B}-L\frac{ds}{dh}\frac{\partial h}{\partial B}\right]$$

$$=s_0\frac{B}{L\left(g\right)}\frac{dL}{dg}\frac{\partial g}{\partial Q}-k\alpha L\frac{\partial g}{\partial Q}+L(g)\frac{ds}{dh}\frac{\partial h}{\partial Q}$$

In the condition of uniqueness, we have shown that F'(B)>0 in the interval $0<B<L_0$ and further using equations (3.1c) and (3.1d). Thus, we have

$$\frac{\partial g}{\partial Q}=\frac{\theta_0\left(\varnothing+vB\right)}{f_1\left(B\right)}>0 \quad \text{and} \quad \frac{\partial h}{\partial Q}=\frac{\theta_0\left(1-k\right)\alpha B}{f_1\left(B\right)}>0$$

From equations (2.1a) and (2.1b),

$$\frac{ds\left(U\right)}{dU}<0 \ for\ U\geq 0 \quad \text{and} \quad \frac{dL\left(T\right)}{dT}<0 \ for\ T\geq 0$$

From these conditions and results it is clear that:

$$\frac{dB}{dQ}\left(\text{+ve function}\right)=(\text{-ve function})$$

Therefore,

$$\frac{dB}{dQ}<0$$

From this it is concluded that the resource biomass density decreases with the increase in the cumulative emission rate of the pollutants in the environment. Now from (3.1e),

$$I=\frac{\theta}{\theta_0}[g(B)-T_0]=i(B)$$

Differentiating the same with respect to I we get:

$$\frac{\theta}{\theta_0}\left[\frac{dg}{dB}\frac{dB}{dI}\right]=1.$$

and since from (3.1c) in the condition of uniqueness, it has been taken that $\frac{dg}{dB}>0$,we can conclude that $\frac{dB}{dI}>0$.

Again from (3.1a), we have

$$N=\frac{(r+\beta_1 B)K}{r_0}=f\left(B\right)$$

$$\frac{dN}{dI}=\frac{\beta_1 K}{r_0}.\frac{dB}{dI}$$

and since $\frac{dB}{dI}>0$,

we also have $\frac{dN}{dI}>0$.

From the above finding it is clear that with the increase in the environment tax, the resource biomass density as well as the population of the biological species increases. Thus, the tax has a positive impact on the ecosystem and the same may be saved from getting deteriorated further.

4. Stability of Equilibrium

We will discuss local as well as the nonlinear stability of the equilibrium on all the four equilibria. For local stability, the perturbed variational matrices are constructed. For nonlinear stability, the Lyapunov functions are found to prove the stability of equilibrium.

4(a). Local stability of equilibrium points: The variational matrices are computed at all the four equilibrium points E_0, E_1, E_2 and E^*

$$M_{E_0(0,0,T,0,I)} = \begin{bmatrix} r & 0 & 0 & 0 & 0 \\ 0 & s_0 - k\alpha\left(\dfrac{Q\theta_0 + \rho\theta T_0}{\delta\theta_0 + \rho\theta}\right) & 0 & 0 & 0 \\ 0 & -\alpha\left(\dfrac{Q\theta_0 + \rho\theta T_0}{\delta\theta_0 + \rho\theta}\right) & -\delta & 0 & -\rho \\ 0 & (1-k)\alpha\left(\dfrac{Q\theta_0 + \rho\theta T_0}{\delta\theta_0 + \rho\theta}\right) & 0 & -\varnothing & 0 \\ 0 & 0 & \theta & 0 & -\theta_0 \end{bmatrix}$$

$$M_{E_1\left(\hat{N},0,\hat{T},0,\hat{I}\right)} = \begin{bmatrix} -r & \dfrac{\beta_1 rK}{r_0} & 0 & 0 & 0 \\ 0 & s_0 - \dfrac{\beta_2 rK}{r_0} - k\alpha\left(\dfrac{Q\theta_0 + \rho\theta T_0}{\delta\theta_0 + \rho\theta}\right) & 0 & 0 & 0 \\ 0 & -\alpha\left(\dfrac{Q\theta_0 + \rho\theta T_0}{\delta\theta_0 + \rho\theta}\right) & -\delta & 0 & -\rho \\ 0 & (1-k)\alpha\left(\dfrac{Q\theta_0 + \rho\theta T_0}{\delta\theta_0 + \rho\theta}\right) & 0 & -\varnothing & 0 \\ 0 & 0 & \theta & 0 & -\theta_0 \end{bmatrix}$$

$$M_{E_2\left(0,\tilde{B},\tilde{T},\tilde{U},\tilde{I}\right)} = \begin{bmatrix} r + \beta_1\tilde{B} & 0 & 0 & 0 & 0 \\ -\beta_2\tilde{B} & s\left(\tilde{U}\right) - \dfrac{2s_0\tilde{B}}{L(\tilde{T})} - k\alpha\tilde{T} & \dfrac{s_0\tilde{B}^2 L'\left(T\right)}{L\left(\tilde{T}\right)^2} - k\alpha\tilde{B} & s'\left(\tilde{U}\right)\tilde{B} & 0 \\ 0 & -\alpha\tilde{T} + \pi v\tilde{U} & -\delta - \alpha\tilde{B} & \pi v\tilde{B} & -\rho \\ 0 & \left(1-k\right)\alpha\tilde{T} - v\tilde{U} & \left(1-k\right)\alpha\tilde{B} & -\varnothing - v\tilde{B} & 0 \\ 0 & 0 & \theta & 0 & -\theta_0 \end{bmatrix}$$

and

$$M_{E^*\left(N^*,B^*,T^*,U^*,I^*\right)} = \begin{bmatrix} \dfrac{r_0 N^*}{K} & \beta_1 N^* & 0 & 0 & 0 \\ -\beta_2 B^* & -\dfrac{s_0 B^*}{L\left(T^*\right)} & \dfrac{s_0 B^{*2} L'\left(T^*\right)}{L\left(T^*\right)^2} - k\alpha B^* & s'\left(U^*\right) B^* & 0 \\ 0 & -\alpha T^* + \pi v U^* & -\delta - \alpha B^* & \pi v B^* & -\rho \\ 0 & \left(1-k\right)\alpha T^* - vU^* & \left(1-k\right)\alpha B^* & -\varnothing - vB^* & 0 \\ 0 & 0 & \theta & 0 & -\theta_0 \end{bmatrix}$$

From the matrix $M_{E_0}\left(0,0,T,0,I\right)$, it is clear that $E_0\left(0,0,T_0,I\right)$ is a saddle point for which manifold in $N-B$ direction is unstable if $\left(s_0 - k\alpha T\right) > 0$. This saddle point has stable manifold in U-direction. From the matrix $M_{E_1\left(\hat{N},0,\hat{T},0,\hat{I}\right)}$, it is clear that $E_1\left(\hat{N},0,\hat{T},0,\hat{I}\right)$ is a saddle point for which manifold in $N-U$ direction is stable. This saddle point has unstable manifold in B-direction. From the matrix $M_{E_2\left(0,\tilde{B},\tilde{T},\tilde{U},\tilde{I}\right)}$, it is clear that $E_2\left(0,\tilde{B},\tilde{T},\tilde{U},\tilde{I}\right)$ is a saddle point for which manifold in N-direction is unstable since $r + \beta_1\tilde{B}$ is positive always. The stability behavior of $E^*\left(N^*,B^*,T^*,U^*,I^*\right)$ is not obvious from its variational matrix. However, in the following theorem, we have found sufficient conditions for E_3 to be locally asymptotically stable.

Theorem 4.1: *Let the following inequality hold*

$$\left[\frac{\beta_1}{\beta_2}\left(\frac{s_0 B^*}{L\left(T^*\right)^2} L'\left(T^*\right) - k\alpha\right) - C_3\left(\alpha T^* - \pi v U^*\right)\right]^2 < 2\frac{C_3 \beta_1 s_0 \delta}{\beta_2\left[L\left(T^*\right)\right]}$$

where,

$$C_3 = \frac{-\beta_1 s'(U^*)(1-k)\alpha}{\pi v \beta_2[(1-k)\alpha T^* - vU^*]}$$

(where $v \neq 0$ *since it is > 0), then* $E^*\left(N^*,B^*,T^*,U^*,I^*\right)$ *will be locally asymptotically stable.*

Proof: First, let us linearize the system (2.1) about $E^*\left(N^*,B^*,T^*,U^*,I^*\right)$ by substituting the values as under:

$$N = N^* + n, B = B^* + b, T = T^* + \tau, U = U^* + u, I = I^* + i$$

Where n, b, τ, u, i are small perturbations around E^* and on simplifying these equations, we get,

(4.1a) $$\frac{dn}{dt} = -\frac{r_0 N^*}{K} n + \beta_1 N^* b$$

(4.1b) $$\frac{db}{dt} = -\beta_2 B^* n - \frac{s_0 B^*}{L\left(T^*\right)} b + \left[\frac{s_0 B^{*2}}{\left[L\left(T^*\right)\right]^2}\frac{\partial L\left(T^*\right)}{\partial T} - k\alpha B^*\right]\tau + \left(\frac{\partial s\left(U^*\right)}{\partial U} B^*\right) u$$

(4.1c) $$\frac{d\tau}{dt} = -\left(\alpha T^* - \pi v U^*\right) b - \left(\delta + \alpha B^*\right)\tau + \pi v B^* u - \rho i$$

(4.1d) $$\frac{du}{dt} = \left[\left(1-k\right)\alpha T^* - vU^*\right] b + \left[\left(1-k\right)\alpha B^*\right]\tau - \left(\phi + vB^*\right) u$$

(4.1e) $$\frac{di}{dt} = \theta\tau - \theta_0 i$$

To understand the local stability at $E^*\left(N^*,B^*,T^*,U^*,I^*\right)$, consider the following positive definite function:

$$V = \frac{1}{2}\frac{C_1 n^2}{N^*} + \frac{1}{2}\frac{C_2 b^2}{B^*} + \frac{1}{2}C_3\tau^2 + \frac{1}{2}C_4 u^2 + \frac{1}{2}C_5 i^2,$$

where $C_j\left(j = 1 \text{ to } 5\right)$ are positive constants

On differentiating the same and choosing suitable values of $C_j\left(j = 1 \text{ to } 5\right)$ we can show that $\dot{V}$ is negative definite under the conditions stated in theorem (4.1), thus equilibrium E^* is locally asymptotically stable.

4(b). Region of attraction: Before proving the global stability, we need to state the following set (region of attraction) which attracts all solutions of the system (2.1) initiating in the interior of the positive orthant.

$$\Gamma = \{(N, B, T, U, I) : 0 \le N \le \frac{K(r+\beta_1 L_0)}{r_0}; 0 \le B \le L_0;$$

$$0 \le T + U \le \frac{Q}{\delta_1}; 0 \le T + U + I \le \frac{Q\left(1+\frac{\theta}{\delta_1}\right)}{\delta_2}\},$$

where, $\delta_1 = \min(\delta, \phi)$; $\delta_2 = \min(\delta, \rho + \theta_0, \phi)$.

4(c). Global stability of E^*:

Theorem 4.2: *If in addition to the assumption (2.1a) and (2.1b), let L_m, p and q be positive constants such that in the region*

$$\Gamma, L_m \le L(T) \le L_0, 0 \le -s'(U) \le q, 0 \le -L'(T) \le p,$$

equilibrium point $E^*(N^*, B^*, T^*, U^*, I^*)$ will be non-linear asymptotically stable in the region Γ, if the following inequalities hold-

$$\left[\frac{\beta_1}{\beta_2}\left(s_0 B^* \frac{p}{L_m^2} + k\alpha\right) + C_3\left(\alpha T^* - \pi \nu U^*\right)\right]^2 < \frac{\beta_1 C_3 s_0 \delta}{\beta_2 L_0},$$

$$\left[\frac{\beta_1 q}{\beta_2} + (1-k)\alpha T^* - \nu U^*\right]^2 < \frac{\beta_1 s_0 \phi}{\beta_2 L_0},$$

where,

$$C_3 = \frac{(1-k)\alpha}{\pi \nu}$$

Proof: To prove this assertion, we construct a Lyapunov Function and prove that this Lyapunov Function with respect to $E^*(N^*, B^*, T^*, U^*, I^*)$ is such that its domain contains the region Γ as defined above.

Consider the positive definite function around $E^{*}\left(N^{*},B^{*},T^{*},U^{*},I^{*}\right)$:

$$W\left(N,B,T,U,I\right)=C_1\left(N-N^{*}-N^{*}ln\frac{N}{N^{*}}\right)+C_2\left(B-B^{*}-B^{*}ln\frac{B}{B^{*}}\right)$$

$$+\frac{1}{2}C_3\left(T-T^{*}\right)^2+\frac{1}{2}C_4\left(U-U^{*}\right)^2+\frac{1}{2}C_5\left(I-I^{*}\right)^2$$

Differentiating with respect to t we get:

$$\dot{W}=C_1\left(\frac{N-N^{*}}{N^{*}}\right)\frac{dN}{dt}+C_2\left(\frac{B-B^{*}}{B}\right)\frac{dB}{dt}+C_3\,(T-T^{*})\frac{dT}{dt}$$

$$+C_4(U-U^{*})\frac{dU}{dt}+C_5(I-I^{*})\frac{dI}{dt}$$

Now from the system of equations (2.1), the values of derivatives of N, B, T, U and I can be substituted and applying the conditions for $\dot{W}$ to be negative definite we get the following inequalities-

(4.2a) $$\left[-\frac{\beta_1}{\beta_2}k\alpha+s_0B^{*}\xi\left(T\right)+C_3\left(\pi vU^{*}-\alpha T^{*}\right)\right]^2<4\frac{\beta_1 s_0}{2\beta_2 L(T)}C_3\left(\frac{\delta}{2}\right)$$

(4.2b) $$\left[\frac{\beta_1}{\beta_2}\eta(U)+C_4\left(1-k\right)\alpha T^{*}-vU^{*}\right]^2<4\frac{\beta_1 s_0}{2\beta_2 L(T)}C_4\left(\frac{\phi}{2}\right)$$

(4.2c) $$\left[C_3\pi vB+C_4\left(1-k\right)\alpha B\right]^2<4C_3C_4\left(\alpha B\right)\left(\frac{\phi}{2}+vB\right)$$

(4.2d) $$\left[-C_3\rho+C_5\theta\right]^2<4C_3C_5\theta_0\left(\frac{\delta}{2}\right)$$

where $C_5=\dfrac{C_3\rho}{\theta}\xi\left(T\right)=\begin{cases}\dfrac{\dfrac{1}{L\left(T\right)}-\dfrac{1}{L\left(T^{*}\right)}}{\left(T-T^{*}\right)}, \text{ for } T\neq T^{*}\\ -\dfrac{L'\left(T^{*}\right)}{\left(L\left(T^{*}\right)\right)^2}, \text{ for } T=T^{*}\end{cases}$

and

$$\eta(U)=\begin{cases}\dfrac{s(U)-s(U^*)}{(U-U^*)}, \text{ for } U\neq U^* \\ s'(U^*), \text{ for } U=U^*\end{cases}$$

Now choosing $C_1=1$ and $C_2=\dfrac{\beta_1}{\beta_2}$ and taking the following assumptions into account-

$$L_m \le L(T) \le L_0, 0 \le -s'(U) \le q, 0 \le -L'(T) \le p,$$

where L_m, p and q are the positive constants in the region Γ, we have in the region Γ,

$$|\xi(T)| \le \frac{p}{L_m^2} \text{ and } |\eta(U)| \le q,$$

this gives

(4.2e) $$\left[\frac{\beta_1}{\beta_2}\left(s_0 B^* \frac{p}{L_m^2}+k\alpha\right)+C_3\left(\alpha T^*-\pi v U^*\right)\right]^2 < \frac{\beta_1 C_3 s_0 \delta}{\beta_2 L(T)}$$

(4.2f) $$\left[\frac{\beta_1}{\beta_2} q + C_4\{(1-k)\alpha T^* - vU^*\}\right]^2 < \frac{\beta_1 C_4 s_0 \phi}{\beta_2 L(T)}$$

(4.2g) $$\begin{aligned}&\left[C_3\pi v B - C_4(1-k)\alpha B\right]^2 + 4C_3C_4\alpha\pi v(1-k)B^2 \\ &< 4C_3C_4(\alpha B)\left(\frac{\phi}{2}+vB\right)\end{aligned}$$

(4.2h) $$\left[C_3\rho - C_5\theta\right]^2 < 2C_3C_5\theta_0\delta$$

The inequality (4.2g) is automatically satisfied if we choose $C_4=1$ and $C_3=\dfrac{(1-k)\alpha}{\pi v}$. On choosing the value of $C_5=\dfrac{C_3\rho}{\theta}$, inequality (4.2h) also holds good. For $L(T)$, the maximum value is L_0, therefore using the same and substituting the value of C_3 and C_4 in (4.2e) and (4.2f), we get the inequalities stated in theorem (4.2) under which W becomes a Lyapunov function whose domain contains the region Γ and hence system becomes non-linearly asymptotically stable. From this theorem it is found that the equilibrium level of the resource biomass density decreases as the concentration of the pollutant in the environment and in the uptake phase of the biological species increase. If this phenomenon continues, the resource biomass may lead to the extinction which in turn may seriously affect the biological species, which are dependent on this resource biomass.

5. **Numerical Simulation**

To illustrate the analytical finding we have done numerical simulation using Maple. We have considered the following set of values of parameters-

$K = 4.38$; $\beta_1 = 0.2$; $\alpha = 0.01$; $\delta = 12$; $k = 0.5$; $\beta_2 = 0.1$;
Q=11 $\pi = 0.03$ $\nu = 0.03$; $\phi = 14$; $\rho = 5$; $T_0 = 0.5$; $\theta = 100$;
$\theta_0 = 0.005$; $s_0 = 16$; $\alpha_1 = 1$; $s_1 = 3.9$; $r = 2$; $L_0 = 5.62$; $b_1 = 1$
$m_1 = 1.03$; $r_0 = 3.21$.

With these values we get the following 4 equilibria analogous to our finding in equilibrium analysis-

E_0 {B = 0, I = 0.9998800144, N = 0, T = 0.5000499940, U = 0},

E_1 {B = 0, I = 0.9998800144, N = 2.728971963, T = 0.5000499940, U = 0},

E_2 {B = 5.288811035, I = 0.9945922010, N = 0, T = 0.5000497296, U = 0.0009339399770},

E^* {B = 5.152107910, I = 0.9947288561, N = 4.134967766, T = 0.5000497364, U = 0.0009100634755}.

We have plotted following graphs to illustrate the analytical findings-

In figure 1, we see that the four different solutions originating from four different initial conditions converge to the equilibrium point E^*. In figure 2 we have plotted population density of biological species $N(t)$ vs time t for 3 different values of s_0 and have found that with increase in s_0, the population density of biological species increases. In figure 3 it is seen that population density of biological species N(t) decreases with increase in value of Q. Similarly in figure 4, we have observed the effect of s_0 on the density of resource biomass.

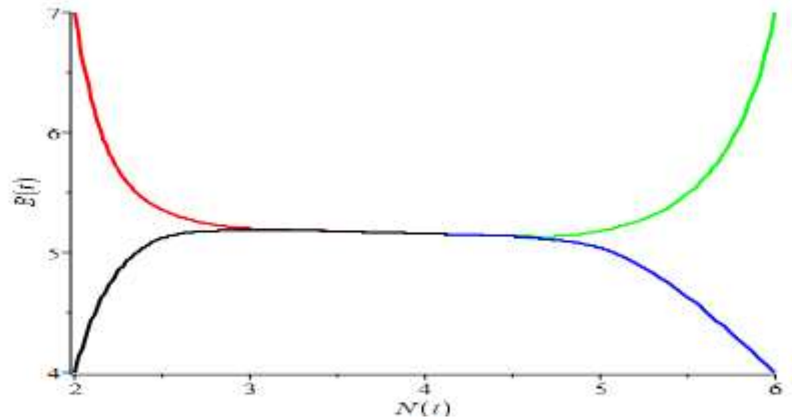

Figure 1. Population density of biological species N(t) with resource biomass B(t)

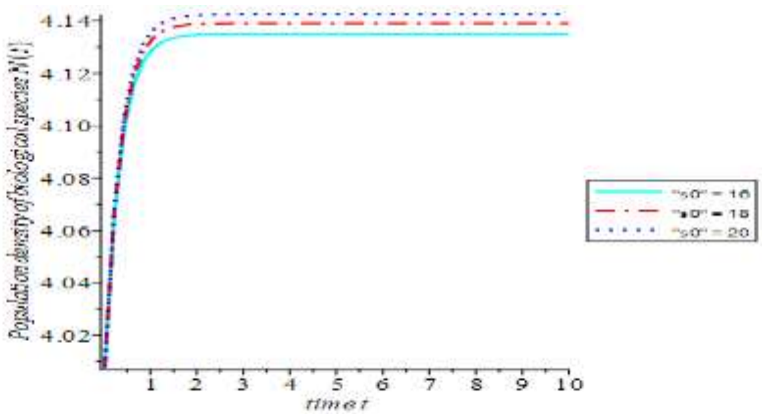

Figure 2. Population density of biological species N(t) with time t for Different values of s_0

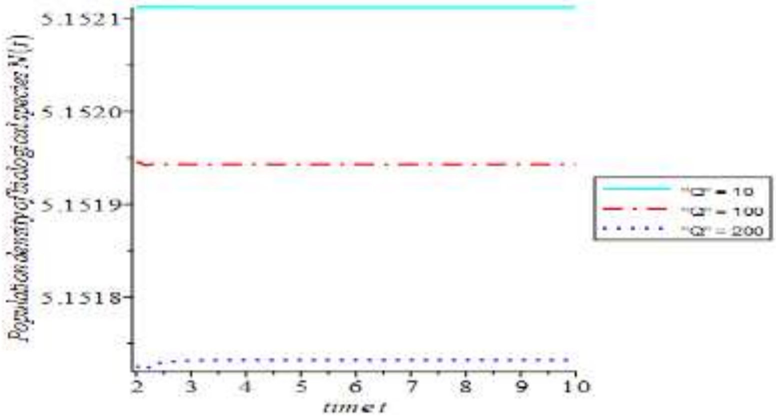

Figure 3. Population density of biological species N(t) with time t for different values of Q

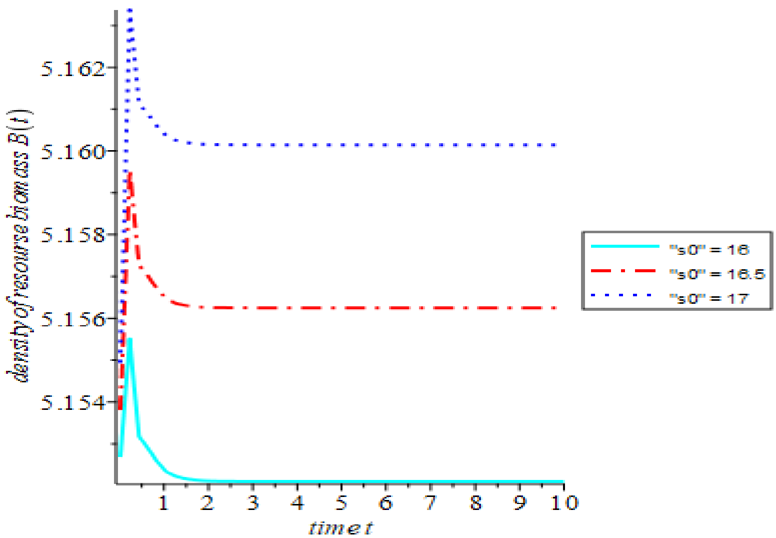

Figure 4. Density of resource biomass B(t) with time t for different values of s_0

6. Conclusion

We have considered an ecological model in which the density of the biological species under consideration, the density of the resource biomass, the concentration of the pollutant emitted from the external sources in the environment and in the uptake phase of the biological species have been considered as the functions of the time. In this paper the existence and stability of the equilibria has been analyzed. It has been shown that the density of the biological species decreases with the increase in the total emission rate of the pollutants when the cumulative rate of emission in the environment is considered to be getting reduced due to levy of taxes. Carrying capacity of the resource biomass decreases with the increased introduction of the pollutants in the environment. The analysis of the nonlinear stability shows that the system settles at much lower density of the resource biomass when the concentration of the pollutants in the environment and in the uptake phase of the biomass is high. But the environment taxes imposed on the polluters, control concentration of the pollutants in the environment and due to introduction of this, the equilibrium point shifts in such a way that the density of the resource biomass as well as biological species is much nearer to the density when ecosystem is pollution free.

Acknowledgement: We are very thankful to Prof. B. Rai (University of Allahabad) for his valuable suggestions regarding this manuscript.

References

1. T. G Hallam and C. E. Clark, Non-autonomous logistic equation as models of population in a deteriorating environment, *J. Theor. Biol.*, **93** (1982), 303-311.

2. T. G. Hallam, C. E. Clark and G. S. Jordan, Effects of toxicants on populations: a qualitative approach II. First order Kinetics, *J. Math. Biol.*, **18** (1983), 25-37.

3. T. G. Hallam, C. E. Clark and R. R. Lassiter, Effects of toxicants on populations: a qualitative approach I. Equilibrium environmental exposure, *Ecol. Modeling*, **18** (1983), 291-304.

4. J. T. De Luna and T. G. Hallam, Effect of toxicants on population: a qualitative approach IV. Resource-consumer-toxicant models, *Ecol. Modeling*, **35** (1987), 249-273.

5. J. B. Shukla, H. I. Freedman, V. N. Pal, O. P. Misra, M. Agarwal and A. Shukla, Degradation and subsequent regeneration of a forestry resource: A mathematical model, *Ecol. Modeling*, **44** (1989), 219-229.

6. L. Huaping and M. Zhein, The threshold of survival for system of two species in a polluted environment, *J Math. Biol.*, **30** (1991), 49-62.

7. J. B. Shukla and B. Dubey, Simultaneous effects of two toxicants on biological species: A mathematical model, *J. Biol. Systems*, **4** (1996), 109-130.

8. J. B. Shukla and B. Dubey, Modelling the depletion and conservation of forestry resource: effects of population and pollution, *J. Math. Biol.*, **35** (1997), 71-94.

9. J. B. Shukla, A. K. Agarwal, B. Dubey and P. Sinha, Existence and survival of two competing species in a polluted environment: A mathematical model, *Journal of Biological System*, **9**(2) (2001), 89-103.

10. J. B. Shukla, A. K. Agarwal, P. Sinha and B. Dubey, Modeling effects of primary and secondary toxicants on renewable resources, *Natural Resource Modeling*, **16**(1) (2003), 99-120.

11. B. Dubey and J. Hussain, Modelling the survival of species dependent on a resource in a polluted environment, *J. Math. Biol.*, **7** (2006), 187-210.

12. B. Rai and Alok Malviya, A control model in ecology for the survival of resources based industries when its resource is depleted by toxicants/pollutants and augmented precursors: A qualitative approach, *Journal of International Academy of Physical Sciences*, **12** (2008), 55-74.

13. J. B. Shukla, Shalini Sharma, B. Dubey and P. Sinha, Modelling the survival of resource dependent population: Effect of toxicants (pollutants) emitted from external sources as well as formed by its precursor, *J. Math. Biol.*, **10** (2009), 54-70.

14. M. Agrawal and Sapna Devi, A resource-dependent competition model: Effect of toxicants emitted from external sources as well as formed by precursor of competing species, *J. Math. Biol.*, **12** (2011), 751-766.

15. Alok Malviya, Control ecological model for the survival of biological species which are directly affected by pollutants emitted from external sources: A qualitative approach, *Journal of International Academy of Physical Sciences*, **19**(2) (2015), 81-101.

16. Alok Malviya, A control modelling effect of primary and secondary toxicants on the survival of extinction of resource based population in aquatic environment: A qualitative approach, *Journal of ultra scientist of physical sciences-Section B*, **29**(2) (2017), 35-39.