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On New Countability and Separationaxioms in Fuzzy Topological Spaces

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Abstract: In this paper, the concepts of α-countability (first and second) and α-separability for fuzzy topological spaces are introduced and studied. The invariance of first α-countability and α-separability under F-continuous surjections are some of the results proved.

Keywords: α-countability andα-separability, fts.

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1. Introduction

C. K. Wong[1] introduced the concept of the first hypothesis of countability and separability for fuzzy topological spaces and proved some properties. The definitions of these spaces involve the concept of a 'fuzzy point'-the fuzzification of the singleton set in the ordinary set theory-which is not a proper generalization of the concept of the singleton set. More specifically, the definition of 'fuzzy point' does not reduce to that of a point in ordinary set theory, even if all the fuzzy sets are restricted to take the value 0 and 1 only. S. S. Benchalli[3] introduced the alternative definitions of countability and separability for fuzzy topological spaces and proved some of their properties.

In this paper, the concept of α-countability and α-separability for fuzzy topological spaces are introduced and studied. The invariance of first α-

countability and α-separability under F-continuous surjections are among the results proved.

2. Main Results

Definition 2.1.[1] *Let (X,τ) be a fts and p be a fuzzy point. A subfamily B_p of T is called a local base of p if $p \in B$ for every member B of B_p and for every member A of T such that $p \in A$ there exists a member B of B_p such that $p \in B \subset A$.*

Definition 2.2.[2] *Anfts (X,τ) is said to be C_1 if every fuzzy point in X has a countable local base.*

Definition 2.3.[1] Anfts (X, τ) is said to be separable if there exists a sequence of fuzzy points $\{p_i\}$, i =1,2,3, . . . such that for every member A of T and $A \neq 0$ there exists a p_i such that $p_i \in A$.

Definition 2.4.[3] *Let (X,τ) be a fts and $x \in X$. A subfamily B_x of T is called a local base at x if and only if $B(x) > 0$ for each $B \in B_x$ and every $A \in T$ with $A(x) > 0$ there exits a member $B_0 \in B_x$ such that $B_0 < A$.*

Definition 2.5.[3] Anfts (X, τ) is said to be C_1 if and only if every $x \in X$ has a countable local base.

Definition 2.6.[3] *Anfts (X,τ) is said to be separable if and only if there exists a sequence of points $\{x_i\}$, i = 1,2,3, ... such that for every member A of T and $A \neq 0$ there exits a x_i such that $A(x_i) > 0$.*

Definition 2.7. *Let $\alpha \in [0,1]$. Let (X,τ) be fts and $x \in X$. A subfamily B_x of T is called a α-local base at x iff for each $B \in B_x$ with $B_x > 0$ and every $A \in T$ with $A(x) > \alpha$, there exists a member $B_0 \in B_x$ such that $B_0 \leq A$.*

Definition 2.8. *Let $\alpha \in [0,1]$. Anfts (X,τ) is said to be α-C_1 if and only if every $x \in X$ has a countable α-local base.*

Definition 2.9. *Let $\alpha \in [0,1]$. Anfts (X,τ) is said to be α-separable if and only if there exists a countable sequence of points $\{x_i\}$, i = 1,2,3, ... such that for every member A of T and $A \neq 0$ there exists a x_i such that $A(x_i) > \alpha$.*

Theorem 2.1. *Everyα-local base family in anfts (X,τ) is a local base.*

Proof: Let (X, τ) be fts for each $x \in X$, B_x be a α-local base at x in T. Since B_x is a α-local base at x, for each $B \in B_x$ with $B(x) > \alpha \geq 0$ and for each $A \in T$ with $A(x) > \alpha \geq 0$ there exists a member $B_0 \in B_x$ such that $B_0 \leq A$.

Thus, for each $x \in X$ and each $B \in B_x$ with $B(x) > 0$ and for each $A \in T$ with $A(x) > 0$, there exists $B_0 \in B_x$ such that $B_0 \leq A$. Hence B_x is a local base at x in fts T.

Theorem 2.2. *Everyα-C_1 fts is a C_1 fts.*

Proof: Follows from the two concepts.

Theorem 2.3. *Everyα-separable fts is a separable fts.*

Proof: Follows from the two concepts.

Theorem 2.4. *If (X, τ) is C_1 fts then for each $x \in X$ there exists a countable α-local base of x, say ϑ= $\{A_i\}$, i=1,2, . . .such that $A_1 \geq A_2 \geq A_3 \geq$. ... .*

Proof: Let (X, τ) be α-C_1 fts. Then for each $x \in X$ there exists a countable α-local base, say, $= \{B_i\}$, i =1,2,3, . . . of x. Now we define, $A_1 = B_1$, $A_2 = B_1 \wedge B_2$, $A_3 = B_1 \wedge B_2 \wedge B_3$, $A_n = \wedge_{i=1}^{n} B_i$. Clearly $A_1 \geq A_2 \geq A_3 \geq$ . ...

Let ϑ= $\{A_i\}$, i=1,2,3, . . . Then ϑ is a α-local base at x. Since B is a α-local base at x, for each $B_i \in B$, $B_i(x) > \alpha$. Therefore $A_i(x) > \alpha$ for each i. Let $G \in T$ with $G(x) > \alpha$. Since B is a α-local base at x, there exists a $B_i \in B$ such that $B_i \leq G$ and $B_i(x) > \alpha$ for each i=1,2,3,..,i_0. Therefore $\wedge_{i=1}^{n} B_i > \alpha$. That is $A_{i0}(x) > \alpha$ and $A_{i0} \leq B_{i0}$. But $B_{i0} \leq G$. Therefore $A_{i0} \leq G$.Thus, at $x \in X$ and $G(x) > \alpha$ of T, there exists $A_{i0} \in \vartheta$such that $A_{i0} \leq G$ and $A_{i0} > \alpha$.

Hence ϑ is countable α-local base at x.

Definition 2.10.[4,5] *Let (X, τ) and (Y, S) be two fts's and let $f: (X, \tau) \to (Y, S)$ be a function. Then f is said to be continuous (Fuzzy continuous) if $f^{-1}(B) \in \tau$ for each $B \in S$.*

Definition 2.11. *A function $f: X \to Y$ is said to be F-open (respectively F-closed) if and only if for each open (respectively closed) fuzzy set A in X, $F(A)$ is open (respectively closed) fuzzy set in Y.*

Theorem 2.5. *Let $f: X \to Y$ be an F-continuous F-open surjection. If (X, τ) is α-C_1 fts, then (Y, S) is also α-C_1.*

Proof: Let $y \in Y$, then $x \in X$ exists such that $f(x) = y$. Since (X, τ) is a α-C_1 fts, so x has a countable α-local base for τ, say B. Then the family $\vartheta_y = \{f(A): A \in B_x\}$ forms a countable α-local base of y in S; for each $A \in B_x$, $f(A) \in S$. Therefore ϑ_y is a subfamily of S and countable. Let $f(A) \in \vartheta_y$, then $\big(f(A)\big)(y) = \vee_{z \in f^{-1}(y)} A(z) > \alpha$. Further, let $G \in S$ with

$G(y) > \alpha$. Then $f^{-1}(G) \in \tau$ and $\left(f^{-1}(G)\right)(x) = G\left(f(x)\right) = G(y) > \alpha$. Therefore $f^{-1}(G) \in \tau$ and $\left(f^{-1}(G)\right)(x) > \alpha$. Since B_x is a α-local base at x, there exists $A_0 \in B_x$ such that $A_0 \leq f^{-1}(G)$ and $A_0 > \alpha$. Therefore $f(A_0) \leq f\left(f^{-1}(G)\right) = G$ and $\left(f(A_0)\right)(y) = \vee_{z \in f^{-1}(y)} A(z) > \alpha$. Therefore $f(A_0) > \alpha$. Thus $G \in S$ with $G(y) > \alpha$, there exists $f(A_0)$ in ϑ_y such that $f(A_0) \leq G$ and $\left(f(A_0)\right)(y) > \alpha$. Therefore ϑ_y is a countable α-local base at y in S. Hence (Y, S) is a α-C_1.

Theorem 2.6. *Let $f: X \to Y$ be an F continuous surjection. If (X, τ) is a α-separable fts, then (Y, S) is also α-separable fts.*

Proof: Let (X, τ) be a α-separable fts. Then there exists a sequence of points $\{x_i\}$, i=1,2,3,..in X such that for every member A of τ with $A \neq 0$ there exists an x_i such that $A(x_i) > \alpha$. Consider $\{f(x_i): i = 1,2,3, \ldots\}$ is a sequence of points in Y. Let $B \in S$ and $B \neq 0$. Then $f^{-1}(B) \in \tau$ and $f^{-1}(B) \neq 0$, for $B \neq 0$ implies that there exists a $y \in Y$ such that $B(y) > 0$. Since f is on to, there exists an x in X such that $f(x) = y$ and $\left(f^{-1}(B)\right)(x) = B\left(f(x)\right) = B(y) > 0$. Therefore $f^{-1}(B) \neq 0$. Since (X, τ) is α-separable, there exists x_i such that $\left(f^{-1}(B)\right)(x_i) > \alpha$, implies that $B\left(f(x_i)\right) > \alpha$. Thus for $B \in S$ and $B \neq 0$, there exists a point $f(x_i)$ in $\{f(x_i): i = 1,2,3, \ldots\}$ such that $B\left(f(x_i)\right) > \alpha$. Hence (Y, S) is α-separable.

Definition 2.12.[1] *Let τ be a topology. A subfamily B of τ is a base for τ if and only if each member of τ can be expressed as the union of some members of B.*

Definition 2.13. *Let $\alpha \in [0,1)$ (respectively $\alpha^* \in (0,1]$). Let (X, τ) be a fts. A subfamily B defined by $B = \{B: B \in \tau, B > \alpha\}$ ($B^* = \{B: B \in \tau, B \geq \alpha\}$ of τ is said to be α-base (respectively α^*-base) if every member A of τ and $A \neq 0$ is expressed as the union of members of B (respectively B^*).*

Definition 2.14. *A fts (X, τ) is said to be α-C_{11} (respectively α^*-C_{11}) if there exists a countable α-base (respectively α^*-base) for τ.*

Theorem 2.7. *Every α-C_{11} fts is α-C_1.*

Proof: Let X be α-C_{11} fts. Let $x \in X$. To prove that x has a countable α-localbase. Since X is α-C_{11}, by definition τhas a countable α-base, say $B = \{B: B \in \tau, B > \alpha\}$. Let $B_x \subset B$ be defined by $B_x = \{B: B \in \tau, B(x) > \alpha\}$. Clearly, B_x is countable. Let $A \in \tau$ with $A(x) > \alpha$. Now since $A \in \tau$ and B is a α-base for τ, A can be expressed as a union of some members of B. Therefore, $A = \vee_{B_i \in B} B_i$, but $A(x) > \alpha$. Therefore $V_{B_i \in B} B_i(x) > \alpha$ implies that there exists $B_{i0} \in B$, such that $B_{i0} > \alpha$. Therefore $B_{i0} \in B_x$ and $B_{i0} \leq$

A.Thus for each $x \in X$ with $A(x) > \alpha$, there exists a B_{i0} in B_x with $B_{i0} > \alpha$ such that $B_i < A$. Therefore, B_x is a α-local base for τ and therefore, every $x \in X$ has a countable α-local base. Hence (X, τ) is α-C_1.

Definition 2.15.[5] *Let (X, τ) be a fts and let A be a subset of X. Then the family $\tau_A = \{G_A : G \in \tau\}$ is a fuzzy topology on A, where G_A is the restriction of G to A. The fuzzy topology τ_A is called the relative fuzzy topology on A induced by the fuzzy topology τ on X. Also (A, τ_A) is called the fuzzy subspace of (X, τ).*

Theorem 2.8. *Every open crisp subspace of a α-separable fts is α-separable.*

Proof: Let X be a α-separable space and Y be an open crisp subspace of X. Since X is α-separable, there exists a countable sequence of points say $S = \{x_i : i = 1,2,3, \ldots\}$ such that for each $A \in \tau$ with $A \neq 0$ there exists x_i, such that $(A(x_i) > \alpha$.Now, let $S_1 = \{x_n \in S : n \in N$ and $x_n \in Y\}$ which is a countable sequence of points in Y.
Let U be any open fuzzy set in Y. Since Y is open, U is also open in X. As X is α-separable, there exists $x_{i0} \in S$ such that $U(x_{i0}) > \alpha$. But $\leq Y$, that is $Y \geq U(x_{i0}) > \alpha$ implies that $Y(x_{i0}) > \alpha$ which implies that $Y(x_{i0}) = 1$. Therefore $x_{i0} \in Y$. Therefore $x_{i0} \in S_1 = \{x_n \in S : n \in N$ and $x_n \in Y\}$.Thus for each open fuzzy set U in Y, there exists a point x_{i0} in S_1 which is the countable sequence of points in Y such that $U(x_{i0}) > \alpha$. Hence Y is α-separable. Therefore open crisp subspace of α-separable fts is α-separable.

Theorem 2.9. *Every crisp subspace of α-C_{11} fts is α-C_{11}.*

Proof: Let X be a α-C_{11} fts, and Y be a crisp subspace of X. Since X is α-C_{11},there exists a countable α-base for τ, say $B = \{B_i : i \in I;\ B_i > \alpha\}$. Then $B_Y = \{B_i \wedge Y : i \in I\}$ is a countable α-base for crisp space Y. If U is an open fuzzy set in Y with $U \neq 0$ then $U = Y \wedge G$, where $G \in \tau$. Now, $G \in \tau$ and B is a α-base for τ, implies that $G = \vee B_{in}$, $B_{in} \in B$. Therefore $U = (\vee B_{in}) \wedge Y$ for $B_{in} \in B$, that is $U = \vee(B_{in} \wedge Y)$ for some $B_{in} \in B$ and $(B_{in} \wedge Y) \in B_Y$ for each n. Clearly, $B_{in} \wedge Y > \alpha$ for each i. Thus, for each open fuzzy set U with $U \neq 0$ in Y can be expressed as aunion of members of B_Y. Hence B_Y is α-base for Y. Therefore, Y is a α-C_{11} fts. Hence every crisp subspace of α-C_{11} fts is α-C_{11}.

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References

1. C. K. Wong, Fuzzy Points and Local Properties of Fuzzy Topology, *J. Math. Anal. Appl*., **46** (1974), 316-328.
2. C. L. Chang, Fuzzy Topological Spaces, *J. Math. Anal. Appl*., **24** (1968), 182-190.
3. S. S. Benchalli, *Studies in Point Set Topology, Contribution to the Theory of Fuzzy Topological Spaces*, PhD. Thesis, Karnatak University, Dharwad (1984).
4. D. M. Ali and A. K. Srivastava, On Fuzzy Connectedness, *Fuzzy Sets and Systems* **28** (1998), 203-208.
5. R. H. Warren, Neighborhood Basis and Continuity in Fuzzy Topological Spaces, *Rocky Mount. J. Math*., **8(3)** (1978), 459-470.