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On Projective *h*-Matsumoto Change

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Abstract: Recently, we have studied the h-Matsumoto change and obtained Cartan connection coefficient for the transformed space[1]. In this paper, we have derived the necessary and sufficient condition for the h-Matsumoto change to be projective and also obtained that there is no non-trivial projective h-Matsumoto change such that the h-vector b_i is gradient.

Keywords: Finsler space, Projective change, Matsumoto change, h-vector.
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1. Introduction

Projective changes of Finsler metric on a manifold in Finsler geometry is an interesting topic to study. Two Finsler metrics of the space F^n and $\overline{F}^n$ are said to be projectively related if any geodesic of F^n is also geodesic of $\overline{F}^n$ and vice versa. The projective relation is said to be trivial if the corresponding Spray coefficients are equal. In 1961, A. Rapscák has given a crucial result with the projective change[2]. He proved necessary and sufficient condition for a transformation to be projective transformation. The projective change between a Finsler space with (α,β)-metric[3] and the associated Riemannian metric have been studied by H. Park and Y. Lee[4], while S. Bácsó[5] has considered the projective change between Finsler spaces with (α,β)-metric. B. Tiwari *et al.*[6] discussed projective changes between two special Finsler space of (α,β)-metric. M. K. Gupta and P. N. Pandey[7] have derived the relation between projective change and Kropina change with an h-vector and also obtained the condition for which the Randers change with an h-vector to be projective[8].

In the previous paper[1], we have discussed the h-Matsumoto change given by

(1.1) $$\overline{L}(x,y) = \frac{L^2(x,y)}{L(x,y) - b_i(x,y)y^i},$$

where $b_i(x,y)$ is an h-vector[9] on (M^n, L). The aim of present paper is to derive the necessary and sufficient condition for the h-Matsumoto change (1.1) to be projective. The terminologies and notations are referred to Matsumoto[10].

2. Preliminaries

Let $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$ be two Finsler spaces on the same underlying manifold M^n, where $\overline{L}$ is defined as (1.1). The metric tensor g_{ij} for the transformed space $\overline{F}^n$ is obtained as follows[1]

(2.1) $$\overline{g}_{ij} = pg_{ij} + p_1 l_i l_j + p_2(m_i l_j + m_j l_i) + p_3 m_i m_j,$$

where $\tau = \frac{L}{\beta}$, $m_i = b_i - \frac{1}{\tau} l_i$,
and

(2.2) $$p = \frac{\tau^2(\tau + \rho\tau - 2)}{(\tau-1)^3}, \quad p_1 = \frac{\tau^2(1-\rho\tau)}{(\tau-1)^3}, \quad p_2 = \frac{\tau^3}{(\tau-1)^3}, \quad p_3 = \frac{3\tau^4}{(\tau-1)^4}.$$

The inverse metric tensor $\overline{g}^{ij}$ of the metric tensor g_{ij} is calculated as[1]

(2.3) $$\overline{g}^{ij} = qg^{ij} + q_1 l^i l^j + q_2(l^i m^j + l^j m^i) + q_3 m^i m^j,$$

where

(2.4) $$q = \frac{1}{p}, \quad q_1 = \frac{-1}{2}\left[\frac{p_1 p_3 - p_2^2}{(p_1+p)p_3 - p_2^2} + \frac{2p^2 p_2^2 p_3}{(3p + 2p_3 m^2)\{(p_1+p)p_3 - p_2^2\}^2}\right],$$

$$q_2 = \frac{-2p_2 p_3}{(3p + 2p_3 m^2)\{(p_1+p)p_3 - p_2^2\}}, \quad q_3 = \frac{-2p_3}{p(3p + 2p_3 m^2)}.$$

The Spray coefficients $\overline{G}^i$ of the transformed space $\overline{F}^n$ is given as follows[1]

(2.5) $$\overline{G}^i = G^i + D^i,$$

where

(2.6) $$D^i = \frac{1}{2}\overline{g}^{is}\{(p_2 l_s + p_3 m_s)E_{00} + 2p_2 L F_{s0})\},$$

and

$$E_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad F_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Zero '0' in the subscript stands for contraction with y^j, viz. $F_{s0} = F_{sj}y^j$.

Therefore, we have:

Theorem 2.1.[1] *For the Matsumoto change with an h-vector, the Cartan connection coefficients for both spaces F^n and $\overline{F}^n$ are the same if and only if the h-vector b_i is parallel with respect to the Cartan connection of F^n.*

3. Projective *h*-Matsumoto change

A change $L \to \overline{L}$ of a Finsler metric on the same underlying manifold M^n is called projective if any geodesic in (M^n, L) remains to be a geodesic in $(M^n, \overline{L})$ and vice versa. M. Matsumoto[11] proved that the change $L \to \overline{L}$ is a projective change if and only if there exists a scalar $P(x, y)$ which is positively homogeneous of degree one in y^i and satisfies

$$\overline{G}^i(x, y) = G^i(x, y) + P(x, y)y^i \,, \tag{3.1}$$

where G^i and $\overline{G}^i$ are the Spray coefficients of the spaces F^n and $\overline{F}^n$ respectively, and $P(x, y)$ is called the Projective factor. Now, we find the condition for the h-Matsumoto change to be projective.
The equation (2.6) can be rewritten as

$$\begin{aligned} D^i &= \tfrac{1}{2}\{[(q + q_1)p_2 + q_2p_3m^2]E_{00} + 2q_2p_2LF_{\beta 0}\}l^i \\ &\quad + \frac{1}{2}\{\mu E_{00} + 2q_3p_2LF_{\beta 0}\}m^i + qp_2LF_0^i, \end{aligned} \tag{3.2}$$

where $\mu = qp_3 + p_2q_2 + p_3q_3m^2$ and $F_{\beta 0} = F_{s0}m^s$.

From equation (2.5) and (3.1), it is clear that the h-Matsumoto change is projective if and only if

$$D^i = Py^i. \tag{3.3}$$

In view of (3.2), the above equation can be written as

$$\begin{aligned} Py^i &= \tfrac{1}{2}\{[(q + q_1)p_2 + q_2p_3m^2]E_{00} + 2q_2p_2LF_{\beta 0}\}l^i \\ &\quad + \tfrac{1}{2}\{\mu E_{00} + 2q_3p_2LF_{\beta 0}\}\, m^i + qp_2LF_0^i. \end{aligned} \tag{3.4}$$

Contracting the equation (3.4) by y_i and using $y_iF_0^i = 0 = y_im^i$, we get

$$P = \tfrac{1}{2L}\{[(q + q_1)p_2 + q_2p_3m^2]E_{00} + 2q_2p_2LF_{\beta 0}\}. \tag{3.5}$$

Putting the value of P in equation (3.4), we have

$$\left(\mu E_{00} + 2q_3p_2LF_{\beta 0}\right)m^i + 2qp_2LF_0^i = 0. \tag{3.6}$$

Contracting the above equation by g_{ij}, we obtain

(3.7) $$\left(\mu E_{00} + 2q_3 p_2 L F_{\beta 0}\right) m_j = -2q p_2 L F_{j0} \,.$$

which on transvecting by m^j and using $m^j m_j = m^2$, we immediately get

(3.8) $$\left(\mu E_{00} + 2q_3 p_2 L F_{\beta 0}\right) m^2 = -2q p_2 L F_{\beta 0} \,.$$

Eliminating $F_{\beta 0}$ from equations (3.6) and (3.8), we obtain

(3.9) $$F_{i0} = \frac{-\mu E_{00}}{2p_2 L(q + q_3 m^2)}\, m_i \,,$$

By using $\mu = qp_3 + p_2 q_2 + p_3 q_3 m^2$, the equation (3.9) can be rewritten as

$$F_{i0} = -\frac{1}{2p_2 L}\left[p_3 + \frac{q_2 p_2}{q + q_3 m^2}\right] E_{00} m_i,$$

calculating R.H.S. of the above equation, we get

$$\left[p_3 + \frac{q_2 p_2}{q + q_3 m^2}\right] = \frac{2\tau p_2}{(\tau - 1)},$$

putting this value on the above equation, we obtain

(3.10) $$F_{i0} = \frac{-\tau E_{00}}{L(\tau - 1)}\, m_i \,.$$

The above equation is a necessary condition for the *h*-Matsumoto change to be projective. Conversely, let us suppose that the equation (3.10) is satisfied. Contracting (3.10) by m^i, we get

(3.11) $$F_{\beta 0} = \frac{-\tau E_{00}}{L(\tau - 1)}\, m^2.$$

Substituting the values from equation (3.10) and (3.11) in equation (3.2), we obtain

(3.12) $$D^i = \frac{1}{6}\{3(q + q_1)p_2 + q_2 p_3 m^2\} E_{00}\, l^i + \frac{1}{2}\{\mu - (q_3 m^2 + q)\frac{2p_2\tau}{(\tau - 1)}\} m^i E_{00}$$

which on solving by using $\mu = (qp_3 + p_2 q_2 + p_3 q_3 m^2)$ and the values of the scalars from the equation (2.2), (2.4), the second term of R.H.S. becomes zero.Then the equation (3.12) becomes

(3.13) $$D^i = \frac{1}{6L}\{3(q + q_1)p_2 + q_2 p_3 m^2\} E_{00}\, y^i,$$

which shows that the *h*-Matsumoto change is projective with the projective factor P, *i.e.*

$$P = \frac{1}{6L}\{3(q + q_1)p_2 + q_2 p_3 m^2\} E_{00}.$$

Thus, we have:

Theorem 3.1. *The h-Matsumoto change* (1.1) *is projective if and only if the condition* (3.10) *is satisfied.*

Now differentiating equation (3.10) with respect to y^j, we get

$$F_{ij} + \frac{1}{2L}\rho_0 h_{ij} = \frac{-\tau}{L^2(\tau-1)}\left\{\left(\rho - \frac{1}{\tau}\right)h_{ij} - l_i m_j\right\}E_{00} - \frac{2\tau}{L(\tau-1)}m_i E_{0j} - \frac{\tau}{L^2(\tau-1)^2)}\{m_j\tau - l_j(\tau-1)\}m_i E_{00}, \tag{3.14}$$

where $\rho_0 = \rho_k y^k$. The equation (3.14) can be rewritten as

$$F_{ij} = -\left\{\frac{1}{2L}\rho_0 + \frac{\tau\left(\rho - \frac{1}{\tau}\right)}{L^2(\tau-1)}E_{00}\right\}h_{ij} + \frac{\tau}{L^2(\tau-1)}\left(l_i m_j + l_j m_i\right)E_{00} - \frac{2\tau}{L(\tau-1)}m_i E_{oj} - \frac{\tau^2}{L^2(\tau-1)^2}m_i m_j E_{00}\ ,$$

taking the skew-symmetry part of the above equation, we have

$$F_{ij} = \frac{\tau}{L(\tau-1)}(m_j E_{0i} - m_i E_{oj}). \tag{3.15}$$

Contraction by y^j, the equation (3.15) reduces in (3.10). Therefore (3.15) can also be treated as necessary and sufficient condition for the *h*-Matsumoto change to be projective. Now, let the vector b_i be gradient, *i.e.* $b_{i|j} = b_{j|i}$, then

$$F_{i0} = 0. \tag{3.16}$$

Therefore equation (3.15) and (3.16), gives

$$m_j E_{0i} - m_i E_{oj} = 0, \quad \text{as} \quad \frac{\tau}{L(\tau-1)} \neq 0.$$

Contracting the above equation by y^j and using $m_j y^j = 0$, we have

$$E_{00} = 0, \quad \text{as} \quad m_i \neq 0. \tag{3.17}$$

In view of equation (3.16) and (3.17), we have $F_{ij} = 0 = E_{00}$, which will imply $b_{i|j} = 0$, *i.e.* the *h*-vector b_i is parallel[1]. Therefore by Theorem 2.1, we get the Cartan connection coefficient for both the spaces F^n and $\overline{F}^n$ are identical, which gives $\overline{G}^i = G^i$, *i.e.* the projective transformation is trivial.
Thus, we have:

Theorem 3.2: *There is no non-trivial projective h-Matsumoto change such that the h-vector b_i is gradient.*

Conclusion

Gupta and Pandey[8] have proved that the Randers change with an *h*-vector becomes a projective change if the *h*-vector $b_i(x, y)$ is a gradient. In the present paper, we have derived the condition for which the *h*-Matsumoto change is projective and also shown that there exists no non-trivial Projective *h*-Matsumoto change such that the *h*-vector b_i is gradient. We observe that for both Kropina change[7] and exponential change[12] with an *h*-vector we have the same result as in Matsumoto change with an *h*-vector.

Now the question arises is that, *Is there any specific class of change for which we can get the non-trivial Projective transformation with an h-vector to be gradient* ?

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