Mathematical Analysis of Four Species Model of Mutualism in Competitive Systems

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Abstract: Mutualism arises in many qualitatively different ways. We present a model of mutualism in which interactions among four species lead to mutualism with prey .Our model involves interactions among a prey, a prey mutualist and two predators competing for the same prey. Giving ecologically reasonable constraints upon the functions existing in the model, we describe (a) the conditions for bounded ness of solutions, (b) the equilibria and their local stability, and (c) conditions for global stability of the interior equilibrium.

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1. Introduction

An interaction among organisms of different species is called mutualistic if the presence of each species increases growth rate of the other. Much of classic ecological theory (e.g. natural selection, niche separation, meta population dynamics) has focused on negative interactions, such as predation and competition, but now positive interactions (mutualism) are increasingly receiving focus in ecological research[1-3].Cleaner fish, pollination, seed dispersal, gut flora and nitrogen fixation are some examples of it. The occurrence of mutualism can arise in a great variety of ways in nature[4,5]. Many of these occurrences are because of interaction with a third population in competitive or predator-prey situations. Although most of such mutualisms involving predator-prey systems usually occur with predators and mathematical models have been developed and analysed[6-8].

In the above mentioned papers, the population mutualists to predator are able to survive in the absence of mutualists i.e., the mutualism is obligate on, at most, one of the two populations, since isolated communities are rare in nature and most of the time there is migration from one community to other. Also there are instances where mutualism occurs with prey. Original models of mutualism were two dimensional (Dean[9], Freedman[10], Freedman et. al.[11]).Three dimensional mutualistic models, where the mutualism arises due to presence of a third population were first analysed in Rai, et al.[12]. Since then a fair amount of work has been done on three dimensional mutualistic models, where mutualism occur with prey[7,13,14]. Such models are further extended by Kumar and Freedman[8], especially in case of food chains.

One of the difficulties of understanding mutualism is the diversity of qualitatively different types of ecological systems. In addition to degree of obligateness of the interactions, two factors contribute to this diversity:

1. The mechanisms by which one species benefits another.

2. Number of species that must interact in order for there to be mutualism between two species.

The ways in which one species may benefit another are as varied as the types of ecological problems that organism encounter. Examination of those factors thought to be limiting to organism shows that most of these can be modified by one organism to the benefit of other. For example, shelter, organic nutrients, dispersal movement of gametes, competition, and predation are known to be involved in the existence of mutualistic benefits[4]. Since the two mutualist species usually benefit each other in different ways, there are a large number of qualitatively different kinds of mutualistic systems.

Not only mutualism does involve many different mechanisms but the number of species necessary for the interaction to occur may also vary. Mutualistic benefits based upon modification of a biotic environment or the direct transfer of nutrients from one organism to another requires the interaction of just two species. Mutualistic benefits arise from modification of predator-prey or competitive interactions among at least three species. In some cases, benefits may arise indirectly by the interaction of four or more species[15].

In any mutualistic system there may be more species involved in the interaction because a mutualist could simultaneously benefit its partners in more than one ways[16]. A mutualist of prey may decrease the predation of its predators, or compete with its predators. A mutualist of a predator may increase predation on the prey or stimulate the prey to more rapid growth. A mutualist of a species may help it to out compete its predators by adding it directly, competing with competitors or predating on its predators.In view of these, multispecies models are important to be analysed.

Given the complexity and diversity of mutualistic systems, it is not clear that a unified approach to the modelling of mutualism will lead to appropriate results. In the beginning, mutualism has been modelled as a two species phenomenon with no explicit consideration of how benefits arise. Occasionally two species models have been extended to n-species cases but with no additional complexity of the underlying models. Therefore major reasons for considering multispecies complex models are:

1. The results of two species models may not be representative of the qualitative behaviour of more complex systems.

2. Even, if complex systems can be approximated by two-species models, these models will have to be modified to include density dependent mutualism coefficients.

3. Analysis of complex models may suggest appropriate forms for functions appearing in the models.

4. Finally, more complex models may suggest new approaches for field ecologists to look at systems.

Here, in this paper, we consider the cases in which a mutualist modifies predation or competition to the benefit of prey. Out of many many ways that a mutualist may affect predator-prey interactions, we consider just one of these, namely a mutualist deterring predation on prey. This is probably the most obvious and common type of mutualistic benefit. A mutualist can benefit prey in the following manners:

- by decreasing predators functional response,

- by increasing death rates of predators,
- by decreasing the rate of conversion from prey biomass to predator biomass,
- a mutualist can also benefit prey population indirectly by its effect on dynamics of predator i.e. by effecting its death rates.

For example, ants deter herbivorous from feeding on plants[16], and deter predators from feeding on aphids[17,18], enzootic algae deter predators from feeding on protozoa's and crustacean deter star fish from feeding on corals[19].

2. The Model

Here we have modeled an ecological situation arising due to interactions of four species; two competing predators y and z, one prey x and a mutualist u to the prey species living in the same environment. Mathematically this model can be represented by the following system of autonomous differential equations:

$$\begin{aligned}
&\dot{u} = uh(u,x,y,z)\\
&\dot{x} = \alpha x g(u,x) - y p_1(u,x) - z p_2(u,x)\\
(2.1)\qquad &\dot{y} = y[-s_1(y) - q_1(z) + c_1 p_1(u,x)]\\
&\dot{z} = z[-s_2(z) - q_2(y) + c_2 p_2(u,x)]\\
&u(0) > 0, x(0) > 0, y(0) > 0, z(0) > 0
\end{aligned}$$

$. = \frac{\mathrm{d}}{\mathrm{dt}}$ and t represents time.

u = population of mutualist at any time t

x = population of prey at any time t

y = population of predators at any time t

z = population of predators at any time t

$s_i \geq o, c_i \geq 0, i = 1,2\ \alpha \geq 0$ are parameters

Functions h, g ,p ,q are from $R_+ \times R_+ \to R$, are continuous and sufficiently smooth to ensure the existence and uniqueness of solutions of initial value problem (2.1) with initial conditions in R^+ and to allow the stability analysis of any solutions of (2.1). We also require the solutions to be defined on some interval [0.T] where $0 \leq T \leq \infty$ .We further makes the following assumptions:

H_0: Since we are going to consider only the case of predator deterrence by the mutualist, therefore we must assume:

(a) $\dfrac{\partial(u,x,y,z)}{\partial y} = 0,$

(b) $\dfrac{\partial(u,x,y,z)}{\partial z} = 0.$

H_1: The function h (u. x) represents the specific growth rate of mutualist in the absence of prey and satisfies the following assumptions:

$$(a) h(0,x) > 0$$

$$(b) \exists\, L:\mathbb{R}_+ \to \mathbb{R} \text{ such that } h(L(x),x) = 0$$

$$(c) \frac{\partial h(u,x)}{\partial u} < 0$$

$$(d) \frac{\partial h(u,x)}{\partial x} \geq 0$$

Ecologically, the above assumptions impose the following restrictions on mutualist population:

1. The mutualist can grow at low densities with or without the prey (x).This indicates that mutualism is non-obligate for mutualists.

2. The population of mutualists can not grow over a certain population size, which depends on population size of its partner prey; this means that it has carrying capacity L, which is a function of prey population.

3. The population of mutualist is slowed by an increase in its own population, other populations remaining the same .This further implies that mutualist exhibits density dependent growth. Ecologically this is termed as" population effect".

4. Population of mutualists is enhanced by an increase in the prey population for any population of the mutualist.

H_2: The function g(x, u) represent the specific growth rate of prey population. We propose the following hypothesis for this function:

$$(a) g(0,u) > 0,$$

$$(b) \frac{\partial g(x,u)}{\partial x} < 0,$$

$$(c) \exists\, k:\mathbb{R}_+ \to \mathbb{R} \text{ satisfying } g(k(u),u) = 0,$$

$$(d) \frac{\partial g(x,u)}{\partial u} \leq 0.$$

Ecologically the above assumptions, imposes the following restrictions on specific growth rate of prey:

1. The prey can grow at low densities with or without the presence of mutualists, so the mutualism is also non-obligate for prey.
2. The population of prey is slowed by an increase in its own numbers, for a fixed population size of mutualist .In other words the prey exhibit density dependent growth pattern.
3. The population of prey can not grow over a certain size in any environment, In other words the environment has carrying capacity for prey, which depends on population size of the mutualist.

4. There may be a cost to prey; associating with the mutualist, In other words the growth rate of prey is suppressed by an increase in the mutualist population.

H$_3$: The function, $p_i(x,u)$ i=1, 2 represent the predator's response function. We propose the following hypothesis on this function:

$$(a)\, p_i(0,u) = 0$$

$$(b)\, \frac{\partial p_i(x,u)}{\partial x} \geq 0$$

$$(c)\, \frac{\partial p_i(x,u)}{\partial u} \leq 0$$

$$(d)\, p_i(x,u) \geq 0 \qquad i = 1,2$$

Ecologically, these hypotheses impose the following restrictions on the predators response function:

1. The predator's response to the prey density, which refers to change in the density of prey per unit of time per predator as the prey density changes, is assumed always to be non-negative. Also there can not be any predation in the absence of prey.
2. For fixed population of other species, the predation is enhanced with the increase in the number of prey species.
3. The mutualist cuts down the effectiveness of predation on the prey. This may be termed as "Mutualist effect" .This is the main effect incorporated in the model.

H$_4$: The functions $q_1(y)$ and $q_2(z)$ represent competition between predators y and z. We propose the following hypothesizes on these functions

$$(a)\quad q_i(0) = 0; \quad i = 1,2$$

$$(b)\quad \frac{\partial q_1(z)}{\partial y} > 0$$

$$(c)\quad \frac{\partial q_2(y)}{\partial z} > 0$$

Ecologically, these hypotheses impose the following restrictions on the functions q_i ,i=1, 2:

1. In the absence of competing predators there is no competition.
2. Competition increases with the increase in rival densities .Further if $\frac{\partial q_i}{\partial q_j} = 0; \quad \forall i, j$ then also there is no competition.

H$_5$: The functions $s_1(y)$ and $s_2(z)$ are death rates of competing predators. We propose the following hypothesis on these functions:

$$(a) \quad s_i(0) > 0$$

$$(b) \quad \frac{\partial s_1(y)}{\partial y} > 0$$

$$(c) \quad \frac{\partial s_2(z)}{\partial z} > 0$$

Ecologically, these hypotheses impose the following restrictions on the death rates.

1. Initially death rates are positive.
2. Death remains positive for all the time.

The death rates incorporated in the model is a combination of natural death and harvesting of predator by other predators. Obviously our model is valid if a predator is harvested by other predators or they die a natural death.

The above assumptions are ecologically reasonable and exemplified in nature as discussed in previous section.

Theorem 2.1: *Under assumed mathematical conditions on the functions h, g, p, s, q, the solutions {u (t), x (t), y (t), z (t)} of system (2.1) with initial positive conditions are all positive and bounded for* $t \geq t_0$.

Set $\Omega = \{(u(t), x(t), y(t), z(t)) : 0 \leq u \leq \tilde{L};\ 0 \leq x \leq \tilde{K};\ 0 \leq c_1 x + y \leq \tilde{M};$

$$0 \leq c_1 x + c_2 y + z \leq \tilde{N}\}$$

where $\tilde{L} = \lim_{x \to \infty} L(x)$

$$\tilde{K} = \max(x_0, K(0))$$

$$\tilde{M} = \frac{c_1 \tilde{K}}{s_1(0)}[\alpha \overline{g} + s_1(0)] \textit{ with } \overline{g} = \max g(u,0)$$

$$\tilde{N} = \frac{1}{s_2(0)}[c_2(0)\tilde{K}(\alpha \overline{g} + s_2(0)) + \tilde{M}\{s_2(0) + c_1 p_1(0, K)\}]$$

is positively invariant set and attracts all solutions initiating with non-negative initial conditions.

3. The Existence of equilibria

The equilibrium points of the system (2.1) are obtained by equating right hand side of each equation of (2.1) to zero and solving them algebraically.aic equations.

Clearly $E_1(0,0,0,0)$ is equilibrium. From hypothesis (H$_1$-a) and (H$_2$-c), it is clear that $E_2(L(0),0,0,0)$ and $E_3(0,K(0),0,0)$ are also the equilibrium states. The subsystem in the $\mathbb{R}_u{}^+$ is given by

$$\dot{u} = uh(u,0), u(0) > 0 .$$

From hypothesises (H_1-b) it follows that E_2 is equilibrium. Furthermore, conditions (H_1-a), (H_1-b), (H_1-c) imply that

$$\lim_{t\to\infty} u(t) = L_0 ,$$

establishing the uniqueness of E_2.Similar argument followed for E_3 in the light of assumptions (H_2-a), (H_2-b), (H_2-c).Thus system (2.1) has exactly two axial equilibria namely E_2 and E_3.Subsystems in $\mathbb{R}_y{}^+$ and $\mathbb{R}_z{}^+$ has no equilibria, because all of their solutions tend to zero exponentially.

$$E_4(0,\hat{x},\hat{y},0,0): \text{ in } \mathbb{R}^+{}_{xy}$$

Theorem 3.1: *An N-S condition for equilibrium of the form* $E_4(0,\hat{x},\hat{y},0,0)$ *to exist in* $\mathbb{R}^+{}_{xy}$ *is that following conditions must be satisfied*:

1. $\exists \hat{x}$ such that $0 < \hat{x} < K(0)$
2. $\dfrac{s_1(0)}{c_1} = p_1(0,\hat{x})$

Proof: The solutions of the subsystem

$$(3.1) \qquad \begin{aligned} \dot{x} &= \alpha x g(0,x) - y p_1(0,x) \\ \dot{y} &= y[-s_1(y) - q_1(0) + c_1 p_1(o,x)] \end{aligned}$$

in $\mathbb{R}^+{}_{xy}$ plane are bounded for positive time[16], therefore

$$\hat{y} = \frac{\alpha x g(\hat{x},0)}{p_1(\hat{x},0)}$$

In order to make $\hat{y} > 0$, it is necessary to assume condition 1 above.

$E_5(\tilde{u},\tilde{x},0,0)$ in $\mathbb{R}^+{}_{ux}$**:** In order to have a viable mutualistic system in the absence of predation we must have an equilibrium of the type E_5.In order to be an equilibrium for positive values of u and x, the algebraic curves h(u,x) = g(u,x) = 0 must intersect in the first quadrant of u-x plane. This is equivalent to having the curves

$$(3.2) \qquad \begin{aligned} u &= L(x) \\ x &= K(u) \end{aligned}$$

Intersect for positive values. By (H_1), the curve u = L(x) is monotonically increasing function with initial value at (L(0), 0, 0, 0). The curve x = K(u) initiating at (0, K(0), 0, 0) may increase or decrease. If $\dfrac{\partial g(u,x)}{\partial u} \le 0$, then the curve is monotonically decreasing and the two curves will intersect uniquely at $(\tilde{u},\tilde{x},0,0)$ unless the cost to the prey due to the mutualist is so high as to drive the prey extinct before u = L(0). We avoid this case from happening, since then u will become a predator for the prey x, in spite of being a mutualist. If $\dfrac{\partial g(u,x)}{\partial u} > 0$ then by bounded ness of solutions $\lim_{x\to\infty} K(u) = \tilde{K}$. In that cases there will be one or more points of inter sections .Thus in either case, under the assumptions under consideration, there will be equilibrium of the form $E_5(\tilde{u}, \tilde{x}, 0, 0)$ in $\mathbb{R}^+{}_{ux}$.

Similarly we can show the existence of equilibrium of the form $E_6(0, x_2, 0, z_2)$ in $\mathbb{R}^+{}_{xz}$ plane with the following N-S conditions to be satisfied:

1. $\exists x_0 : 0 < x_0 <$ K (0)
2. $\dfrac{s_2(0)}{c_2} = p_2(0, x_0)$

If the above two conditions are satisfied then there exist an equilibrium of the type $E_6(0, x_2, 0, z_2)$, where $z_2 = \dfrac{\alpha x_2 g(x_2, 0)}{p_2(x_2, 0)}$.

The three dimensional interior equilibrium points, if exist are obtained by solving corresponding algebraic equations, we denote them by $E_7(u_3, x_3, y_3, 0)$ in $\mathbb{R}^+{}_{uxy}$ $E_8(u_4, x_4, 0, z_4)$ in $\mathbb{R}^+{}_{uxz}$, $E_9(o, x_5, y_5, z_5)$ $\mathbb{R}^+{}_{xyz}$.

Finally, we will write down conditions for existence of an interior equilibrium in the positive orthant (u > 0, x > 0, y > 0, z > 0). Any equilibrium of this type will be obtained by solving the following system of algebraic equations:

$$(3.3a) \qquad h(u, x) = 0,$$

$$(3.3b) \qquad \alpha x g(u, x) - y p_1(u, x) - z p_2(u, x) = 0,$$

$$(3.3c) \qquad -s_1(y) - q_1(z) + c_1 p_1(u, x)] = 0,$$

$$(3.3d) \qquad -s_2(z) - q_2(y) + c_2 p_2(u, x)] = 0,$$

$$u(0) > 0, x(0) > 0, y(0) > 0, z(0) > 0.$$

From equation (3.3a), we have u = L(x) is a solution, so that in order to solve for x by the equations (3.3b) and (3.3d), we must assume:

$$(3.4a) \qquad \frac{s_1}{c_1} \in \text{ range } p_1(x, L(x)),$$

and

$$(3.4b) \qquad \frac{s_2}{c_2} \in \text{ range } p_2(x, L(x)).$$

Under the above assumptions, the equations

$$-s_1(y) - q_1(z) + c_1 p_1(x, u) = 0$$

$$-s_2(z) - q_2(y) + c_2 p_2(x, u) = 0$$

can have several solutions, giving rise to several interior equilibria. The y and z components of these equilibria are given by

(3.5) $$y^* = \frac{\alpha x^* g(x^*, u^*) - z^* p_2(x^*, u^*)}{p_1(x^*, u^*)} \quad ,$$

(3.6) $$z^* = \frac{\alpha x^* g(x^*, u^*) - y^* p_1(x^*, u^*)}{p_2(x^*, u^*)} .$$

In order to ensure positive y and z-components, it is necessary to assume

(3.7) $$x^* < K(L(x^*)) .$$

In order to have a unique interior equilibrium, we also need to assume

(3.7*a*) $$\begin{aligned} &p_{1x}(x,u) + p_{1u}(L'(x)) > 0 \\ &p_{2x}(x,u) + p_{2u}(L'(x)) > 0 \end{aligned}$$

There fore under the assumptions discussed above system (2.1) posses a unique equilibrium $E_{10}(u^*, x^*, y^*, z^*)$, where x^* is such that

(3.7*b*) $$\begin{aligned} &p_1(x^*, L(x^*)) = \frac{s_1}{c_1}, \\ &p_2(x^*, L(x^*)) = \frac{s_2}{c_2}, \end{aligned}$$

Also u^* is determined by

(3.7c) $$u^* = L(x^*).$$

4. Local Stability of Equilibria

The local stability analysis can be made by computing the eigen values of the variational matrix at the equilibrium points. The signs of the real parts of eigen values evaluated at given equilibrium points determine the stability[9]. The variational matrix for the system (2.1) is given by

(4.1) $M(u, x, y, z)$

$$= \begin{pmatrix} h(u,x) + uh_u(u,x) & uh_x(u,x) & 0 & 0 \\ \begin{array}{c}\alpha x g_u(x,u) - yp_{1u}(x,u) \\ -zp_{2u}(x,u)\end{array} & \begin{array}{c}\alpha[g(x,u) + xg_x(x,u)] \\ -yp_{1x}(x,u) - zp_{2x}(x,u)\end{array} & -p_1(x,u) & -p_2(x,u) \\ yc_1 p_{1u}(x,u) & yc_1 p_{1x}(x,u) & \begin{array}{c}-s_1(y) - q_1(z) + c_1 p_1(x,u) \\ -ys'_1(y)\end{array} & -yq_1'(z) \\ zc_2 p_{2u}(x,u) & zc_2 p_{2x}(x,u) & -zq_2'(y) & \begin{array}{c}-s_2(z) - q_2(y) \\ +c_2 p_2(x,u) - zs_2'(z)\end{array} \end{pmatrix}$$

Now we consider the various equilibrium states separately.

$E_1(0,0,0,0)$: From (3.2), the variational matrix, evaluated at E_1 could be written as

$$M_1(0,0,0,0) = (m_{ij})_{4\times 4} ,$$

where $m_{11} = h(0,0), m_{12} = m_{13} = m_{14} = 0, m_{21} = 0, m_{22} = \alpha g(0,0)$,

$m_{23} = m_{24} = 0, m_{31} = m_{32} = 0 = m_{34}, m_{33} = -s_1(0) - q_1(0) + c_1 p_1(0,0)$,

$m_{41} = m_{42} = m_{43} = 0, m_{44} = -s_2(0) - q_2(0) + c_2 p_2(0,0) - z s_2'(z)$.

Eigen values are given by

$$\lambda = (\lambda_u, \lambda_x, \lambda_y, \lambda_z) = (h(0,0), \alpha g(0,0), -s_1(0) - q_1(0), -s_2(0) - q_2(0)).$$

Since h(0,0)>0 and g(0,0)>0, by assumptions (H_1) and (H_2), the equilibrium E_1 is unstable. Near E_1, *u* and *x* populations grow while *y* and *z* populations decline, because eigen values are positive in *u* and *x* directions whereas those in *y* and *z* directions are negative. Consequently E_1 has a non-empty stable and unstable manifold.

$E_2(L(0),0,0,0)$: From (3.2), the variational matrix evaluated at E_2 could be written as

$$M_2(L(0),0,0,0) = (m_{ij})_{4\times 4}$$

where $m_{11} = L(0) h_u(L(0),0,0,0), m_{12} = L(0) h_x(L(0),0,0,), m_{13} = m_{14} = 0$,

$m_{21} = m_{23} = m_{24} = 0, m_{22} = \alpha g(L(0),0), m_{31} = m_{32} = 0 = m_{34,} m_{33} = -s_1(0)$,

$m_{41} = m_{42} = m_{43} = 0, m_{44} = -s_2(0)$.

Eigen values are given by

$$\lambda = (\lambda_u, \lambda_x, \lambda_y, \lambda_z) = [L(0) h_u(L(0),0), \alpha g(L(0),0), -s_1(0), -s_2(0)]$$

From signs of real parts of eigenvalues, we observe that these are positive in x-direction and negative in *u*, *y* and *z*-directions. Thus we may conclude that E_2 attract in *u*, *y* and *z*-directions and repels in *x*-direction .Here also E_2 has stable and unstable manifolds.

$$M_3(0\,K(0),0,0) = (m_{ij})_{4\times 4},$$

where $m_{11} = h(0,K(0), m_{12} = m_{13} = m_{14} = 0, m_{21} = \alpha K(0) g_u(0,K(0))$,

$m_{22} = \alpha[K(0) g_x(0,K(0)], m_{23} = -p_1(0,K(0)), m_{24} = -p_2(0,K(0))$,

$m_{32} = m_{32} = m_{34} = 0, m_{33} = -s_1(0) - q_1(0) + c_1 p_1(0,K(0))$,

$m_{41} = m_{42} = m_{43} = 0, m_{44} = -s_2(0) - q_2(0) + c_2 p_2(0,K(0))$.

$E_3(0,K(0),0,0)$: From (3.2), the variational matrix evaluated at E_3 could be written as

$$M_3(0\,K(0),0,0) = (m_{ij})_{4\times 4},$$

where $m_{11} = h(0,K(0), m_{12} = m_{13} = m_{14} = 0, m_{21} = \alpha K(0) g_u(0,K(0))$,

$m_{22} = \alpha[K(0) g_x(0,K(0)], m_{23} = -p_1(0,K(0)), m_{24} = -p_2(0,K(0))$,

$m_{32} = m_{32} = m_{34} = 0, m_{33} = -s_1(0) - q_1(0) + c_1 p_1(0,K(0))$,

$m_{41} = m_{42} = m_{43} = 0, m_{44} = -s_2(0) - q_2(0) + c_2 p_2(0,K(0))$.

Eigen values are given by

$$\lambda = (\lambda_u, \lambda_x, \lambda_y, \lambda_z) = [(h(0, K(0)), \alpha K(0) g_x(0, K(0)), \\ -s_1(0) + c_1 p_1(0, K(0)), -s_2(0) + c_2 p_2(0, K(0))]$$

Eigen values are positive in u-direction, negative in x-direction. The eigen values in y and z-directions are given by

$$\lambda_y = -s_1(0) + c_1 p_1(0, K(0))$$

$$\lambda_z = -s_2(0) + c_2 p_2(o, K(0))$$

which are both positive .Thus E_3 is unstable in y and z-directions also. Here we have assumed that

$$p_i(0, K(0)) > \frac{s_i(0)}{c_i(0)}; \qquad i = 1, 2$$

This assumption is ecologically reasonable, because in the absence of mutualist and when the prey population is near its carrying capacity K (0), the predators y and z must multiply.

$E_4(0, x_1, y_1, 0)$ From (3.2), the variational matrix evaluated at E_4 could be written as:

$$M_4(0, K(0), 0, 0) = (m_{ij})_{4\times 4}$$

where

$$m_{11} = h(0, x_1), m_{12} = m_{13} = m_{14} = 0, m_{21} = \alpha x_1 g_u(0, x_1) - y_1 p_{1u}(0, x_1),$$
$$m_{22} = \alpha x_1 g_x(0, x_1) + \alpha g(o, x_1) - y_1 p_{1x}(0, x_1), m_{23} = -p_1(0, x_1), m_{24} = -p_2(0, x_1)$$
$$m_{31} = y_1[-s_1(y_1) + c_1 p_{1u}(0, x_1)], m_{32} = y_1 c_1 p_{1x}(0, x_1), m_{33} = -s_1(y_1) - q_1(0) + c_1 p_1(0, x_1) + \\ y_1 s_1'(y_1), m_{34} = -y_1 q_1'(0), m_{41} = m_{42} = m_{43} = 0, m_{44} = -s_2(0) - q_2(y_1) + c_2 p_2(0, x_1)$$

The eigen value of M_4 in u-direction is $h(0, x_1)$, which is positive by (H_1-a). Thus the equilibrium E_4 is unstable, population of u near E_4 increases. The eigen value in z-direction is given by

(4.2) $$\lambda_z = -s_2(0) - q_2(y_1) + c_2 p_2(0, x_1)$$

The other two eigen values namely in x and y-directions are the roots of the equation

(4.3) $$\lambda^2 - [\alpha x_1 g_x(0, x_1) + \alpha g(0, x_1) - y_1 p_{1x}(0, x_1) - y_1 s_1'(y_1)]\lambda - \\ y_1 p_{1x}(0, x_1)[\alpha x_1 g_x(0, x_1) + \alpha g(0, x_1) - y_1 p_{1x}(0, x_1) + \\ c_1 y_1 p_1(0, x_1) p_{1x}(0, x_1)] = 0$$

From Rough-Hrwitz criteria the roots of equation (4.2) have negative real parts iff

(4.4a) $$\alpha x_1 g_x(0, x_1) + \alpha g(0, x_1) - y_1 p_{1x}(0, x_1) - y_1 s_1'(y_1) < 0,$$

and

(4.4b) $$y_1 p_{1x}(0, x_1)[\alpha x_1 g_x(0, x_1) + \alpha g(0, x_1) - y_1 p_{1x}(0, x_1) \\ + c_1 y_1 p_1(0, x_1) p_{1x}(0, x_1)] < 0$$

E_4 is an interior equilibrium for a competitive predator-prey system, In the absence of mutualist. Freedman[17] has given the graphical analysis of this case and accordingly, we can state the following result for our system. Let us denote

$$H_4(x^*) = \alpha x_1 g_x(0,x_1) + \alpha g(0,x_1) - y_1 p_{1x}(0,x_1) - y_1 s_1'(y_1) \text{ and}$$

$$H_4(x^{**}) = y_1 p_{1x}(0,x_1)[\alpha x_1 g_x(0,x_1) + \alpha g(0,x_1) - y_1 p_{1x}(0,x_1) + c_1 y_1 p_1(0,x_1) p_{1x}(0,x_1)]$$

Also if $H_4(x^{**}) > 0$, then we have the following theorem:

Theorem 4.1: *If $H_4(x^*) < 0$, then E_4 is asymptotically stable and unstable if $H_4(x^*) > 0$.*

$E_5(\tilde{u},\tilde{x},0,0)$ **:** From (3.2), the variational matrix evaluated at E_5 could be written as

$$M_5(\tilde{u},\tilde{x},0,0) = (m_{ij})_{4\times 4} \ ,$$

where

$m_{11} = \tilde{u}h(\tilde{u},\tilde{x}), \quad m_{12} = \tilde{u}h_x(\tilde{u}),\tilde{x}), \quad m_{13} = m_{14} = 0, \quad m_{21} = \alpha\tilde{x}g_u(\tilde{u},\tilde{x}),$

$m_{22} = (\tilde{u}\alpha\tilde{x}g_x,\tilde{x}), \quad m_{23} = -p_1(\tilde{u},\tilde{x}), \quad m_{24} = -p_2(\tilde{u},\tilde{x}), \quad m_{31} = m_{32} = 0 = m_{34} = 0,$

$m_{33} = -s_1(0) - q_1(0) + c_1 p_1(\tilde{u},\tilde{x}), \ m_{41} = m_{42} = m_{43} = 0, m_{44} = -s_2(0) - q_2(0) + c_2 p_2(\tilde{u},\tilde{x})$

The eigen values of M_5 in y and z-directions are given by

(4.5*a*) $$\lambda_y = -s_1(0) + c_1 p_1(\tilde{u},\tilde{x})$$

$$\lambda_z = -s_2(0) + c_2 p_2(\tilde{u},\tilde{x})$$

And other two eigen values in u and x-directions are given by

(4.5*b*) $$\lambda_\pm = \frac{1}{2}[\tilde{u}h_u(\tilde{u},\tilde{x}) + \alpha\,\tilde{x}\,g_x(\tilde{u},\tilde{x}) \pm \frac{1}{2}[(\tilde{u}h_u(\tilde{u},\tilde{x}) + \tilde{x}\alpha g_x(\tilde{u},\tilde{x}))^2 - 4\alpha\tilde{u}\tilde{x}h_u(\tilde{u},\tilde{x})g_x(\tilde{u},\tilde{x}) - g_u(\tilde{u},\tilde{x})h_x(\tilde{u},\tilde{x})]^{\frac{1}{2}}].$$

Thus if we denote $H_5(x^*) = h_u(\tilde{u},\tilde{x})g_x(\tilde{u},\tilde{x}) - g_u(\tilde{u},\tilde{x})h_x(\tilde{u},\tilde{x})$, then we have the following result:

Theorem 4.2: *If $H_5(x^*) > 0$, then the eigen values $\lambda\pm$ have negative real parts and E_5 is asymptotically stable in positive u-x plane. Also if $H(x^*) < 0$, then E_5 has non-empty stable and stable manifolds.*

$E_6(0,x_2,0,z_2)$: From (3.2), the variational matrix evaluated at E_6 could be written as

$$M_6(0,x_2,0,z_2) = (m_{ij})_{4\times 4} \ ,$$

where $m_{11} = h(0,x_2), \quad m_{12} = m_{13} = m_{14} = 0, \quad m_{21} = \alpha x_2 g_u(0,x_2) - z_2 p_{2u}(0,x_2),$

$m_{22} = \alpha x_2 g_x(0,x_2) + \alpha g(0,x_2) - z_2 p_{2x}(0,x_2) \quad m_{23} = -p_1(0,x_2),$

$$m_{24} = -p_2(0,x_2), \quad m_{31} = m_{32} = m_{34} = 0, \quad m_{41} = z_2 c_2 p_{2u}(0,x_2),$$
$$m_{42} = z_2 c_2 p_{2x}(0,x_2), \quad m_{43} = 0, \quad m_{44} = -s_2(z_2) - q_2(0) + c_2 p_2(0,x_2),$$
$$m_{33} = -s_1(0) - q_1(z_2) + c_1 p_1(0,x_2),$$

Eigen values in u-direction is given by

$$\lambda_u = h(0,x_2) > 0 \quad \text{(by H}_1\text{-a)},$$

which implies that E_6 is unstable in u-direction, the population of mutualist increases near E_6. The eigen value in y-direction is given by

$$(4.6a) \qquad \lambda_y = -s_1(0) + c_1 p_1(0,x_2) - q_1(z_2).$$

The other two eigen values namely in x and z-directions have negative real parts if and only if

$$(4.6b) \qquad \alpha x_2 g_x(0,x_2) + \alpha g(0,x_2) - z_2 p_{2x}(0,x_2) - z_2 s_2'(z_2) < 0,$$

and

$$(4.6c) \qquad z_2 s_2'[\alpha x_2 g_x(0,x_2) + \alpha g(0,x_2) - z_2 p_{2x}(0,x_2)] - z_2 c_2 p_2(0,x_2) p_{2x}(0,x_2) < 0.$$

In view of above analysis, we have the following theorem:

Theorem 4.3: If conditions (4.6b) and (4.6c) are satisfied, and then E_6 is asymptotically stable in xz plane, but unstable in u-direction i.e., population of mutualist increases, while that of x and y decreases near point E_6.

$E_7(u_3, x_3, y_3, 0)$: This is interior equilibrium $\mathbb{R}^+_{uxy}$, in the absence of predator z. Detail analysis of this situation may be found in Rai et al[11]. Moreover the variational matrix evaluated at E_7 is given by:

$$M_7(u_3, x_3, y_3, 0) = (m_{ij})_{4\times 4},$$

where $m_{11} = u_3 h_u(u_3, x_3, 0, 0)$, $m_{12} = u_3 h_x(u_3, x_3, 0, 0)$, $m_{13} = m_{14} = 0$,

$$m_{21} = \alpha x_3 g_u(u_3,x_3) - y_3 p_{1u}(u_3,x_3), \quad m_{22} = \alpha x_3 g_x(u_3,x_3) + \alpha g(u_3,x_3) - y_3 p_{1x}(u_3,x_3),$$
$$m_{23} = -p_1(u_3,x_3), m_{24} = -p_2(u_3,x_3), m_{31} = y_3[c_1 p_{1u}(u_3,x_3)], m_{32} = y_3 c_1 p_{1x}(u_3,x_3),$$
$$m_{33} = -s_1(y_3) - q_1(0) + c_1 p_1(u_3,x_3) - y_3 s_1', \quad m_{34} = -y_3 q_1'(0), \quad m_{41} = m_{42} = m_{43} = 0,$$
$$m_{44} = -s_2(0) - q_2(y_3) + c_2 p_2(u_3,x_3).$$

The eigen value in z-direction is given by

$$(4.7a) \qquad \lambda_z = -s_2(0) + c_2 p_2(u_3,x_3) - q_2(y_3).$$

The other three eigen values are the zeros of the polynomial

$$(4.7b) \qquad \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$$

where $A_1 = -\{u_3 h_u(u_3, x_3, y_3, 0) + \alpha x_3 g_x(u_3,x_3) + \alpha g(u_3,x_3) - y_3 p_{1x}(u_3,x_3) + s_1'(y_3)\}$

$$A_2 = u_3 h_u(u_3, x_3, y_3, 0)[\alpha x_3 g_x(u_3, x_3) + \alpha g(u_3, x_3) - y_3 p_{1x}(u_3, x_3) - y_3 s_1'(y_3)]$$
$$+ y_3 c_1 p_1(u_3, x_3) p_{1x}(u_3, x_3) - y_3 s_1'(y_3)[\alpha x_3 g_x(u_3, x_3) + \alpha g(u_3, x_3) - y_3 p_{1x}(u_3, x_3)]$$
$$- u_3 h_x(u_3, x_3, y_3, 0)[\alpha x_3 g_x(u_3, x_3) - y_3 p_{1x}(u_3, x_3)] -$$
$$u_3 y_3 h_x(u_3, x_3, y_3, 0)[-s_1'(y_3) + c_1 p_1(u_3, x_3) + p_{1x}(u_3, x_3)]$$
$$A_3 = -[u_3 y_3[\alpha x_3 g_u(u_3, x_3) - y_3 p_{1u}(u_3, x_3)\{h_x(u_3, x_3, y_3, 0) c_1 p_{1x}(u_3, x_3)$$
$$- h_x(u_3, x_3, y_3, 0) s_1 y_3\}] - h_u(u_3, x_3, y_3, 0)[c_1 p_1(u_3, x_3) p_{1x}(u_3, x_3)$$
$$- s_1'(y_3)\{\alpha x_3 g_x(u_3, x_3) + \alpha g(u_3, x_3) - y_3 p_{1x}(u_3, x_3)\}]$$
$$+ [-s_1(y_3) + c_1 p_1(u_3, x_3) + c_1 p_{1u}(u_3, x_3)]]$$

Therefore, in the light of above analysis and applying Rough-Hurwitz criteria we have the following result:

Theorem 4.3: E_7 *is asymptotically stable if the following conditions hold*

(1) $\lambda_z < 0$ *and*

(2) $A_1 > 0$, $A_3 > 0$ *and* $A_1 A_2 - A_3 > 0$.

In other words, if the above mentioned conditions are satisfied, then solutions of the system, initiating near E_7, *eventually tend to* E_7 *as* $t \to \infty$.

$E_8(u_4, x_4, 0, z_4)$: From (3.2), the variational matrix evaluated at E_8 could be written as

$$M_8(u_4, x_4, 0, z_4) = (m_{ij})_{4\times 4},$$

where $m_{11} = u_4 h_u(u_4, x_4, 0, z_4)$, $m_{12} = u_4 h_x(u_4, x_4, 0, z_4)$, $m_{13} = m_{14} = 0$,

$m_{21} = \alpha x_4 g_u(u_4, x_4) - z_4 p_{2u}(u_4, x_4)$, $m_{22} = \alpha x_4 g_x(u_4, x_4) + \alpha g(u_4, x_4) - z_4 p_{2x}(u_4, x_4)$

$m_{23} = -p_1(u_4, x_4)$, $m_{24} = -p_2(u_4, x_4)$, $m_{31} = m_{32} = 0 = m_{34}$,

$m_{33} = -s_1(0) - q_1(z_4) + c_1 p_1(u_4, x_4)$, $m_{41} = z_4 c_2 p_{2u}(u_4, x_4)$, $m_{42} = z_4 c_2 p_2(u_4, x_4)$,

$m_{43} = -z_4 q_2'(0)$, $m_{44} = -s_2(z_4) - q_2(0) + c_2 p_2(u_4, x_4) - z_4 s_2'(z_4)$

The eigen value of M_8 in y-direction is given by

(4.8*a*) $$\lambda_y = -s_1(0) + c_1 p_1(u_4, x_4) - q_1(z_4).$$

The other three eigen values are the zeros of the polynomial

(4.8*b*) $$\lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3 = 0,$$

where

$$B_1 = -[u_4 h_u(u_4, x_4, 0, z_4) + \alpha x_4 g_x(u_4, x_4) + \alpha g(u_4, x_4) - z_4 p_{2x}(u_4, x_4)]$$

$$B_2 = u_4 h_u(u_4, x_4, 0, z_4)[\alpha x_4 g_x(u_4, x_4) + \alpha g(u_4, x_4) - z_4 p_{2x}(u_4, x_4)]$$
$$+ z_4 c_2 p_2(u_4, x_4) p_{2x}(u_4, x_4) - z_4 s_2'(z_4)[\alpha x_4 g_x(u_4, x_4) + \alpha g(u_4, x_4)$$
$$- z_4 p_{2x}(u_4, x_4)] - u_4 h_x(u_4, x_4, 0, z_4)[\alpha x_4 g_u(u_4, x_4) - z_4 p_{2u}(u_4, x_4)]$$
$$+ z_4 p_2(u_4, x_4)[-s_2(z_4) + c_2 p_2(u_4, x_4) + c_2 p_{2u}(u_4, x_4)]$$

$$B_3 = -[u_4 z_4 [\{\alpha x_4 g_u(u_4, x_4) - z_4 p_{2u}(u_4, x_4)\}\{c_2 h_x(u_4, x_4, 0, z_4) p_{2x}(u_4, x_4)$$
$$- h_x(u_4, x_4, 0, z_4) s_2(z_4)\}] - h_u(u_4, x_4, 0, z_4)[\{c_2 p_2(u_4, x_4) p_{2x}(u_4, x_4) - s_2(z_4)\}$$
$$\{\alpha x_4 g_x(u_4, x_4) + \alpha g(u_4, x_4) - z_4 p_{2x}(u_4, x_4)\}] + [-s_4(z_4) + c_2 p_{2u}(u_4, x_4)]]$$

Therefore, in the light of above analysis and applying Rough-Hurwitz criteria we have the following result:

Theorem 4.4: E_8 *is asymptotically stable if the following conditions hold*

(1) $\lambda_y < 0$ *and*

(2) $B_1 > 0$, $B_3 > 0$ *and* $B_1 B_2 - B_3 > 0$.

$E_9(u^*, x^*, y^*, z^*)$: Finally, we investigate the stability of interior equilibrium.ted in two- and three-dimensional models[13,16]. However mutualistic interactions can have a significant effect on stability, even in the complex systems. In order to conclude ourselves, we follow the technique given in paper[9]. From (3.2), the variational matrix evaluated at E^* could be written as

$$M^*(u^*, x^*, y^*, z^*) = (m_{ij})_{4\times 4},$$

where $m_{11} = u^* h_u(u^*, x^*, y^*, z^*)$, $m_{12} = u^* h_x(u^*, x^*, y^*, z^*)$, $m_{13} = m_{14} = 0$,

$m_{21} = \alpha x^* g_u(u^*, x^*) - z^* p_{2u}(u^*, x^*) - y^* p_{1u}(u^*, x^*)$,

$m_{22} = \alpha x^* g_x(u^*, x^*) + y^* p_{1x}(u^*, x^*) z^* p_{2x}(u^*, x^*)$,

$m_{23} = -p_1(u^*, x^*)$, $m_{24} = -p_2(u^*, x^*)$, $m_{31} = y^* c_1 p_{1u}(u^*, x^*)$,

$m_{32} = y^* c_1 p_{1x}(u^*, x^*)$, $m_{33} = -s_1(y^*) - q_1(z^*) - y^* s_1'(y^*) - c_1 p_1(u^*, x^*)$,

$m_{34} = -y^* q_1(z^*)$, $m_{41} = z^* c_2 p_{2u}(u^*, x^*)$, $m_{42} = z^* c_2 p_2(u^*, x^*)$,

$m_{43} = -z^* q_2'(y^*)$, $m_{44} = -s_2(z^*) - q_2(y^*) + c_2 p_2(u^*, x^*) - z^* s_2'(z^*)$.

The eigen values of the above matrix are the roots of the characteristic polynomial

$$\lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0, \tag{4.9}$$

where $a_1 = -[u^* h_u(u^*, x^*, y^*, z^*) + \alpha\{x^* g_x(u^*, x^*) + g(u^*, x^*)\} -$

$$y^*\{p_{1x}(u^*, x^*) - s_2(y^*) - q_1(z^*)\} - z^* p_2(u^*, x^*) - s_1(y^*) - q_1(z^*)]$$

$$a_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a32 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$$

$$= u^* h_u(u^*, x^*, y^*, z^*)[(\alpha x^* g_x(u^*, x^*) + \alpha g u^*, x^*) - y^* p_{1x}(u^*, x^*) - z^* p_{2x}(u^*, x^*)]$$

$$- u^* h_x(u^*, x^*, y^*, z^*)[\alpha x^* g_u(u^*, x^*) - y^* p_{1u}(u^*, x^*) - z^* p_{2u}(u^*, x^*)]$$

$$- u^* h_u(u^*, x^*, y^*, z^*) p_1(u^*, x^*) - u^* h_u(u^*, x^*, y^*, z^*) p_2(u^*, x^*) + [\alpha x^* g_x(u^*, x^*)$$

$$+ \alpha g u^*, x^*) - y^* p_{1x}(u^*, x^*) - z^* p_{2x}(u^*, x^*)][-s_1(y^*) - q_1(z^*) + c_1 p_1(u^*, x^*) - y^* s_2(y^*)]$$

$$+ y^* c_1 p_{1x}(u^*, x^*) p_1(u^*, x^*) + [\{\alpha x^* g_u(u^*, x^*) - y^* p_{1u}(u^*, x^*) - z^* p_{2u}(u^*, x^*)\}$$

$$\{-s_2(z^*) - q_2(y^*) + c_2 p_2(u^*, x^*) - z^* s_2(z^*) + z^* c_2 p_{2x}(u^*, x^*) p_2(u^*, x^*)\}] +$$

$$\{-s_1(y^*) - q_1(z^*) + c_1 p_1(u^*, x^*) - y^* s_2(y^*)\}$$

$$\{-s_2(z^*) - q_2(y^*) + c_2 p_2(u^*, x^*) - z^* s_2(z^*) - z^* y^* q_2'(y^*) q_1'(z^*)\},$$

$$a_3 = -\{\alpha x^* g_x(u^*, x^*) + \alpha g(u^*, x^*) - y^* p_{1x}(u^*, x^*) - z^* p_{2x}(u^*, x^*)$$

$$[\{-s_1(y^*) - q_1(z^*) + c_1 p_1(u^*, x^*) - y^* s_1'(y^*)\}\{-s_2(z^*) - q_2(y^*) + c_2 p_2(u^*, x^*) - z^* s_2'(z^*)\}$$

$$+ z^* y^* q_1'(y^*) q_2'(z^*) + u^* h_u(u^*, x^*, y^*, z^*)[\{-s_1(y^*) - q_1(z^*) + c_1 p_1(u^*, x^*) - y^* s_1'(y^*)\}$$

$$\{-s_1(y^*) - q_1(z^*) + c_1 p_1(u^*, x^*) - y^* s_1'(y^*)\}] + u^* h_u(u^*, x^*, y^*, z^*)[$$

$$\{\alpha x^* g_x(u^*, x^*) + \alpha g(u^*, x^*) - y^* p_{1x}(u^*, x^*) - z^* p_{2x}(u^*, x^*)\}$$

$$\{-s_2(z^*) - q_2(y^*) + c_2 p_2(u^*, x^*) - z^* s_2'(z^*) + z^* c_2 p_{2x}(u^*, x^*) p_2(u^*, x^*)\}]$$

$$- u^* h_x(u^*, x^*, y^*, z^*)[\{\alpha x^* g_u(u^*, x^*) - y^* p_{1u}(u^*, x^*) - z^* p_{2u}(u^*, x^*)\}$$

$$\{-s_2(z^*) - q_2(y^*) + c_2 p_2(u^*, x^*) - z^* s_2'(z^*) + c_2 z^* p_{2u}(u^*, x^*) p_2(u^*, x^*)\}]$$

$$u^* h_u(u^*, x^*, y^*, z^*)[\{\alpha x^* g_x(u^*, x^*) + \alpha g(u^*, x^*) - y^* p_{1x}(u^*, x^*) - z^* p_{2x}(u^*, x^*)\}$$

$$\{-s_1(y^*) - q_1(z^*) + c_1 p_1(u^*, x^*) - y^* s_1'(y^*) + y^* c_1 p_{1x}(u^*, x^*) p_1(u^*, x^*)\}]$$

$$+ u^* h_x(u^*, x^*, y^*, z^*)[\{\alpha x^* g_u(u^*, x^*) - y^* p_{1u}(u^*, x^*) - z^* p_{2u}(u^*, x^*)\}$$

$$\{-s_1(y^*) - q_1(z^*) + c_1 p_1(u^*, x^*) - y^* s_1'(y^*) + y^* c_1 p_{1u}(u^*, x^*) p_1(u^*, x^*)\}]\},$$

$$a_4 = u^* h_u(u^*, x^*, y^*, z^*)[\{\alpha x^* g_x(u^*, x^*) + \alpha g(u^*, x^*) - y^* p_{1x}(u^*, x^*) - z^* p_{2x}(u^*, x^*)\}$$
$$\{-s_1(y^*) - q_1(z^*) + c_1 p_1(u^*, x^*) - y^* s_2'(z^*)\}\{-s_2(z^*) - q_2(y^*) + c_2 p_2(u^*, x^*) - z^* s_2'(z^*)$$
$$- z^* y^* q_1'(z^*) q_2'(y^*)\} + p_1(u^*, x^*) y^* c_1 p_{1x}(u^*, x^*)\{-s_2(z^*) - q_2(y^*) + c_2 p_2(u^*, x^*)$$
$$- z^* s_2'(z^*) + z^* y^* c_2 p_{2x}(u^*, x^*) q_1'(z^*)\} - p_2(u^*, x^*)\{y^* c_1 p_{1x}(u^*, x^*) z^* q_2'(y^*)$$
$$- z^* c_2 p_{2x}(u^*, x^*)\}\{-s_1(y^*) - q_1(z^*) + c_1 p_1(u^*, x^*) - y^* s_2'(z^*)\}] - u^* h_x(u^*, x^*, y^*, z^*)$$
$$[\alpha x^* g_u(u^*, x^*) - y^* p_{1u}(u^*, x^*) - z^* p_{2u}(u^*, x^*)\{-s_1(y^*) - q_1(z^*) + c_1 p_1(u^*, x^*) - y^* s_2'(z^*)\}$$
$$\{-s_2(z^*) - q_2(y^*) + c_2 p_2(u^*, x^*) - z^* s_2'(z^*) - z^* y^* q_1'(z^*) q_2'(y^*)\} + p_1(u^*, x^*) y^* p_{1u}(u^*, x^*)$$
$$\{-s_2(z^*) - q_2(y^*) + c_2 p_2(u^*, x^*) + c_2 z^* y^* p_{2u}(u^*, x^*) q_1'(z^*)\} - p_2(u^*, x^*)$$
$$\{y^* c_1 p_{1u}(u^*, x^*) q_2(y^*) - c_1 z^* p_{2u}(u^*, x^*)\}\{-s_1(y^*) - q_1(z^*) + c_1 p_1(u^*, x^*) - y^* s_2'(z^*)\}]$$

Now in view of above calculations, we have the following result:

Theorem 4.6: *Equilibrium state E^* is locally asymptotically stable if*

1. $a_4 > 0,\ a_2 > 0,\ a_1 > 0,$
2. $a_3[a_1 a_2 - a_3] > a_1^2 a_4$..

5. Global stability of the interior equilibrium

In this section we will discuss the conditions for the global stability of the equilibrium point $E_9(u^*, x^*, y^*, z^*)$.

For, we assume that E_{10} as defined in the previous section exists. For convenience of notation, we relable E_9 as $E^*(u^*, x^*, y^*, z^*)$. It is the purpose of this section is to derive criterion for E^* to be globally stable i.e. E^* to be asymptotically stable with domain of attraction the positive cone. For this our technique will be to construct a Lyapunov function[10,11], whose domain of validity is the positive cone.

First, we define

$$v_1(u) = u - u* - u* \log\left(\frac{u}{u*}\right), \qquad v_2(x) = x - x* - x* \log\left(\frac{x}{x*}\right),$$

$$v_3(y) = \int_{y*}^{y} \frac{P(u*, x*, \xi, z*)}{q_2(\xi)} d\xi, \qquad v_4(z) = \int_{z*}^{z} \frac{Q(u*, x*, y*, \xi)}{q_1(\xi)} d\xi.$$

where $P(u*, x*, \xi, z*) = -s_2(z*) - q_2(\xi) + c_2 p_2(x*, u*)$,

$Q(u*, x*, y*, \xi) = -s_1(y) - q_1(\xi) + c_1 p_1(x*, u*)$,

$v_i(\chi),\quad \chi = u, x, y, z$ and $i = 1,2,3,4$

We note that

1. $v_i(\chi) > 0, for \chi > 0, \chi \neq \chi^*$
2. $v_i(\chi^*) = 0$
3. $\lim\limits_{\chi \to +\infty} v_i(\chi) = \lim\limits_{\chi \to +0} (\chi) = +\widehat{\infty}$
4. $v_3(y^*) = 0$ *and* $\lim\limits_{y \to 0^+} = +\infty$
5. $v_4(z^*) = 0$ *and* $\lim\limits_{z \to 0^+} = +\infty$

Functions P & Q are assumed to be such that $\lim\limits_{y\to\infty} v_3(y) = +\infty$ and

$v_3(y)$ is $p \lim\limits_{y\to\infty} v_3(y) = +\infty \lim\limits_{y\to\infty} + \infty v_3(y) = +\infty \lim\limits_{y\to\infty} v_3(y) = +\infty$ositive

for $y \leq k + \varepsilon_1$, where ε_1 is any arbitriary small number.

Also Q is such that $\lim\limits_{z\to\infty} v_4(z) = +\infty$ and $v_4(z)$ is positive for $z \leq k + \varepsilon_{2,}$,

where ε_2 is any arbitriary small positive number.

Finally, we define,

(5.1) $$v(u,x,y,z) = v_1(u) + v_2(x) + v_3(y) + v_4(z)$$

And note that v is a positive function in the positive cone with respect to E^* *in* $\mathbb{R}^4_+$.

We now compute $\dot{v}(u,x,y,z)$, the trajectory derivative of v(u,x,y,z) along solutions of system (3.1).

(5.2) $$\dot{v}(u,x,y,z) = v_1'(u)\dot{u} + v_2'(x)\dot{x} + v_3'(y)\dot{y} + v_4'(z)\dot{z}$$

The components of $\dot{v}$ are worked out as:

$$\frac{\partial v_1}{\partial u} = \frac{u - u^*}{u}, \frac{\partial v_2}{\partial x} = \frac{x - x^*}{x}, \frac{\partial v_3}{\partial y} = \frac{P(u^*, x^*, y, z^*)}{q_2(y)}, \frac{\partial v_4}{\partial z} = \frac{Q(u^*, x^*, y^*, z)}{q_1(z)}$$

We now consider the derivative of v along solutions,

(5.3) $$\dot{v}(u,x,y,z) = (u - u^*)h(u,x,y,z) + (x - x^*)[\alpha g(x,u) - \frac{yp_1(x,u)}{x} - \frac{zp_2(x,u)}{x}]$$
$$+ \frac{P(u^*, x^*, y, z^*)}{q_2(y)}[-s_1(y) - q_1(z) + c_1 p_1(x,u)]y$$
$$+ \frac{Q(u^*, x^*, y^*, z)}{q_1(z)}[-s_2(z) - q_2(y) + c_2 p_2(x,u)]z$$

By substitution and some algebraic manipulations, equation (3) can be written as:

(5.4) $$\dot{v}(u,x,y,z) = a_{11} + a_{12} + a_{13} + a_{14} + a_{22} + a_{23} + a_{24} + a_{33} + a_{34} + a_{44}$$

where $a_{11}(u)=(u-u^*)h(u,x^*,y^*,z^*)$,

$$a_{12}(u,x)=(u-u^*)[h(u,x,y,z^*)-h(u,x^*,y^*,z^*)]+$$
$$(x-x^*)[g(x,u)-g(u*,x)-y^*p_1(u,x)-z^*p_2(u,x)],$$

$$a_{13}=(u-u^*)[h(u,x^*,y,z^*)-h(u,x^*,y^*,z^*)]+\frac{Q(u^*,x^*,y,z^*)}{q_1(y)}[(u-u^*)(y-y^*)],$$

$$a_{14}=(u-u^*)[h(u,x^*,y^*,z)-h(u^*,x^*,y^*,z^*)]+\frac{P(u^*,x^*,y^*,z)}{q_2(z)}[(u-u^*)(z-z^*)]$$

$$a_{22}=(x-x^*)[\alpha g(x,u)-y\frac{p_1(x,u)}{x}-z\frac{p_2(x,u)}{x}]$$

$$a_{23}=-\frac{P(u^*,x,y^*,z^*)}{q_2(y)}(x-x^*)(y-y^*)-\frac{Q(u^*,x^*,y,z^*)}{q_1(z)}(x-x^*)(z-z^*)$$

$$a_{24}=\frac{Q(u^*,x,y^*,z^*)}{q_1(z)}(x-x^*)(z-z^*)-\frac{P(u^*,x^*,y^*,z)}{q_2(y)}(x-x^*(y-y^*)$$

$$a_{33}=\frac{P(u^*,x^*,y,z^*)}{q_2(y)}[-s_1(y)-q_1(z)+c_1p_1(x,u)]y$$

$$a_{34}=-\frac{-Q(u^*,x^*,y,z^*)}{q_1(y)}(y-y^*)(z-z^*)+\frac{P(u^*,x^*,y^*,z\ )}{q_2(y)}(x-x^*)(z-z^*)$$

$$a_{44}=\frac{Q(u^*,x^*,y^*,z)}{q_1(z)}[-s_2(z)-q_2(y)+c_2p_2(x,u)]z$$

From above notations and utilizing the techniques given in papers[12,13], we may deduce that a_{ij} 's such that:

$$a_{11}=-b_{11}(u)(u-u^*)^2,\quad a_{12}=-2b_{12}(u,x)(u-u^*)(x-x^*),$$

$$a_{13}=-2b_{13}(u,y)(u-u^*)(y-y^*),\quad a_{14}=-2b_{14}(u,z)(u-u^*)(z-z^*),$$

$$a_{22}=-b_{22}(x)(x-x^*)^2,\quad a_{23}=-2b_{23}(x,y)(x-x^*)(y-y^*),$$

$$a_{24}=-2b_{24}(x,z)(x-x^*)(z-z^*),\quad a_{33}=-b_{33}(y)(y-y^*)^2,$$

$$a_{34}=-2b_{34}(y,z)(y-y^*)(z-z^*),\quad a_{44}=-b_{44}(z)(z-z^*)^2.$$

With above notations, equation (4) can be re written as:

(5.5) $$\dot{v}=-PBP^T$$

where B is a 4×4 matrix B(u,x,y,z), whose ij^{th} term is b_{ij} and $b_{ij}=b_{ji}$

$$P=\left[(u-u^*)\quad (x-x^*)\quad (y-y^*)\quad (z-z^*)\right]$$

and P^T is the transpose matrix of P.

From above, we can state the following theorem:

Theorem: *Let B (u,x,y,z) be a positive matrix for all* $X \in \Omega \cap Int\mathbb{R}_+^4$*.Then* E^* *is globally asymptotically stable equilibrium of the system (3.1) with respect to initial values in* $Int\mathbb{R}_+^4$.

Proof. Let X = {u(t), x(t), y(t), z(t)} be any solution of (3.1) and $X \in \Omega \cap Int\mathbb{R}^4_+$. Since *B* is positive definite, $\dot{v}(X) \le 0$. The set $\{X \in \Omega \cap Int\mathbb{R}_+^4 : v(X) = 0\}$ is a subset of

$S = \{X \in Int\mathbb{R}_+^4 : X = X^*\}$. S is the largest invariant set in S is the equilibrium point E^*, therefore by LaSalle's invariance principle[10], E^* globally asymptotically stable.

6. Example

In order to illustrate the above analysis we consider the following example. All coefficients and functions are taken for mathematical convenience, not exactly from a real ecological system.

$$
\begin{aligned}
&\frac{du}{dt} = u[1 - \frac{u}{3+x}] \\
&\frac{dx}{dt} = 4x(1 - \frac{x}{4}) - xy - xz \\
&\frac{dy}{dt} = y[-\frac{1}{3} - z + 3x] \\
&\frac{dz}{dt} = z[-\frac{1}{3} - y + 3x]
\end{aligned} \tag{6.1}
$$

Clearly system of equations (6.1) satisfies all the mathematical restrictions assumed in hypothesis. After solving the algebraic system,

$$
\begin{aligned}
&u[1 - \frac{u}{3+x}] = 0 \\
&4x(1 - \frac{x}{4}) - xy - xz = 0 \\
&y[-\frac{1}{3} - z + 3x] = 0 \\
&z[-\frac{1}{3} - y + 3x] = 0
\end{aligned} \tag{6.2}
$$

We obtain the following equilibrium states:

$$E_1(0,0,0,0),\ E_2(3,0,0,0),\ E_3(0,4,0,0),\ E_4(0,\frac{13}{12},\frac{35}{12},0),\ E_5(7,4,0,0),$$

$$E_6(0,\frac{13}{12},0,\frac{35}{12}),\ E_7(\frac{49}{12},\frac{13}{12},\frac{35}{12},0),\ E_8(\frac{28}{9},\frac{1}{9}.0,\frac{35}{9}),\ E_9(0,\frac{2}{3},\frac{5}{3},\frac{5}{3})$$

$$E^*(\frac{11}{13},\frac{2}{3},\frac{35}{21},\frac{35}{21})$$

where parameters are taken as

$$L = 3, \alpha = k = 4, \gamma_0 = \delta_0 = 1, s_1 = s_2 = \tfrac{1}{3}, c_1 = c_2 = 3, \delta_1 = \delta_2 = 1, m = 0 .$$

Region of attraction is

$$\Omega = \{(u, x, y, z) : 0 \le u \le 7; 0 \le x \le 4; 0 \le y \le \frac{35}{12}; 0 \le z \le \frac{35}{9}\} .$$

Discussion: In this paper we have considered a system of four autonomous differential equations as a model of four interacting populations, two species competing with each other for a single prey and a prey mutualist. Our main interest was to give criteria for existence of various equilibrium points and their stability in a bounded region. We have been able to obtain such criteria in terms parameters of the system and have illustrated the conclusions with a numerical example.

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