Three-Dimensional Finsler Spaces with m-th Root Metric

T. N. Pandey and V. K. Chaubey
Department of Mathematics & Statistics, D. D. U. Gorakhpur University, Gorakhpur
E-mail: vkcoct@rediffmail.com, vkcoct@gmail.com

B. N. Prasad
C-10, Surajkuand Coloney, Gorakhpur (U.P.)

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Abstract: In the present paper we have worked out the non-linear connection of Berwald and Cartan connection of a Finsler space with m-th root metric. We have also obtained main scalars of m-th root metric, in particular, of cubic and quartic metrics of a three-dimensional Finsler space.

Key Words: Finsler space with m-th root metric, Berwald connection, Cartan connection.

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1. Preliminaries

The theory of *m*-th root metrics was developed by H. Shimada[1] as an interesting example of Finsler metrics, immediately after M. Matsumoto and S. Numata's theory of cubic metrics[2]. By introducing the regularity of the metric various fundamental quantities as a Finsler metric could be found. In particular, the Cartan connection of a Finsler space with m-th root metric could be discussed from the theoretical standpoint. M. Matsumoto and K. Okubo[3] studied Berwald connection of Finsler spaces with m-th root metric and gave main scalars in two-dimensional case and also defined higher order Christoffel symbols.

In 1992-93 the m-th root metrics have begun to be applied to theoretical physics[4, 5], but the results of our investigations are not yet ready for acceding to the demands of various applications. The purpose of present paper is to study three-dimensional case and give the main scalars, in particular, of cubic metrics and quartic metrics. Since the scalars make clear the essential difference from the Riemannian structure, we have good reason to anticipate physical meanings of those scalars.

The m-th root Finsler metric $L(x, y)$ of an n-dimensional differentiable manifold M^n was defined by H. Shimada[1] as

$$L^m(x, y) = a_{i_1.........i_m}(x)\, y^{i_1}.........y^{i_m}$$

where the coefficients $a_{i_1.........i_m}(x)$ are components of a symmetric tensor field covariant of order *m*. Consequently the second root metric is, of course, a Riemannian metric and we shall restrict $m > 2$ throughout the paper. The third and fourth root metrics are especially interesting and have the well known names:

$$L^3(x,y) = a_{ijk}(x) y^i y^j y^k \quad \ldots\ldots \text{ Cubic metric}^{[2,4]}$$

$$L^4(x,y) = a_{hijk}(x) y^h y^i y^j y^k \quad \ldots\ldots \text{ Quartic metric}^{[5]}.$$

We shall sketch some fundamental part of the theory of Finsler spaces $F^n = (M^n, L)$ with m-th root metric $L(x, y)$ for the later use.

Let us first define the tensors $a_i(x, y)$, $a_{ij}(x, y)$ and $a_{ijk}(x, y)$ as follows:

$$(1.1)\qquad \begin{cases} L^{(m-1)} a_i = a_{ij_1 \ldots j_{(m-1)}} y^{j_1} \ldots\ldots\ldots y^{j_{(m-1)}} \\ L^{(m-2)} a_{ij} = a_{ijk_1 \ldots k_{(m-2)}} y^{k_1} \ldots\ldots\ldots y^{k_{(m-2)}} \\ L^{(m-3)} a_{ijk} = a_{ijkl_1 \ldots l_{(m-3)}} y^{l_1} \ldots\ldots\ldots y^{l_{(m-3)}} \end{cases}$$

Then, the normalized supporting element $l_i = \dot\partial_i L$, the angular metric tensor, $h_{ij} = L \dot\partial_i \dot\partial_j L$, the fundamental tensor $g_{ij} = \dot\partial_i \dot\partial_j L^2/2$ and the (h)hv-torsion tensor $C_{ijk} = \dot\partial_i \dot\partial_j \dot\partial_k L^2/4$ of the Cartan connection CΓ of $F^n = (M^n, L)$ are written as

$$(1.2)\qquad l_i = a_i, \qquad h_{ij} = (m-1)(a_{ij} - a_i a_j), \quad g_{ij} = (m-1)a_{ij} - (m-2)a_i a_j,$$

$$2LC_{ijk} = (m-1)(m-2)(a_{ijk} - a_{jk}a_i - a_{ki}a_j - a_{ij}a_k + 2a_i a_j a_k).$$

We have the following relations among $a_i(x, y)$, $a_{ij}(x, y)$ and $a_{ijk}(x, y)$:

(1.3)

1. $a_i y^i = L, \qquad a_{ij} y^j = L a_i, \qquad a_{ijk} y^k = L a_{ij}$
2. $(a_{ij} - a_i a_j) y^j = 0, \quad (a_{ijk} - a_{ij} a_k) y^k = 0$
3. $(a_{ijk} - a_{ij} a_k) y^k = L(a_{jk} - a_j a_k)$
4. $L(\dot\partial_k a_{ij}) = (m-2)(a_{ijk} - a_{ij} a_k)$

Let us call $a_{ij}(x, y)$ as the basic tensor because this played an important role[1,2]. The metric L is called regular if the basic tensor has the non-vanishing determinant. Throughout our discussion of m-th root metrics we would suppose the regularity of the metrics.

By $a^{ij}(x, y)$ we denote the reciprocal of $a_{ij}(x, y)$, then the reciprocal $g^{ij}(x, y)$ of $g_{ij}(x, y)$ is given as

$$(m-1)\, g^{ij} = a^{ij} + (m-2)\, a^i a^j, \qquad l^i = a^i,$$

where $a_i a^i = 1$, $a^i = a^{ij} a_j$, $l^i = y^i / L = g^{ij} l_j$.

We shall also consider the Berwald connection[4,6] $B\Gamma = (G_j{}^i{}_k, G^i_j, 0)$ and Cartan connection $C\Gamma = (F_j{}^i{}_k, G^i_j, C_j{}^i{}_k)$ of F^n.

2. Finsler connections of m-th root metrics

The coefficients $(F_j{}^i{}_k, G^i_j)$ of the Berwald connection and the Cartan connection of a Finsler space with m-th root metric are given by[3]

$$(2.1)\qquad a_{jr} F_k{}^r{}_i = f_{ijk} - \frac{m-2}{2L}(a_{ijr} - a_{ij}a_r)G^r_k - \frac{m-2}{2L}(a_{jkr} - a_{jk}a_r)G^r_i$$

$$+ \frac{m-2}{2L}(a_{kir} - a_{ki}a_r)G^r_j$$

(2.2) $$\frac{2(m-1)}{m-2} a_{ir}G^r_j = f_{ij0} + \frac{m}{m-2} f_{ji0} + \frac{1}{L^2} f_{000}a_{ij} - \frac{1}{L}(f_{i00}a_{ij} - f_{j00}a_i + f_{0r0}a_i{}^r{}_j)$$

or

(2.3) $$\frac{2(m-1)}{m-2} a_{ir}G^r_j = \frac{1}{2(m-2)L^{(m-2)}}\{(m-2)A_{ij0} + mA_{ji0}\} - \frac{1}{2mL^{(m-1)}} a_i{}^r{}_j\{(m-2)A_{r00} + mA_{0r0}\}$$

(2.4) $$2(m-1)a_{jr}G^r = f_{0j0} + \frac{m-2}{L} f_{000}a_j,$$

(2.5) $$2(m-1)a_{jr}G^r = \frac{1}{2L^{(m-2)}}(A_{ojo} + \frac{m-2}{m}A_{j00}),$$

where $G^i = G^i_r y^r/2$,

$$A_{ijk}(x, y) = (\partial_k a_{ijr_1r_2.......r_{m-2}} + \partial_i a_{jkr_1r_2.......r_{m-2}} - \partial_j a_{ikr_1r_2.......r_{m-2}})\ y^{r_1}y^{r_2}.......y^{r_{m-2}},$$

and $2f_{ijk} = \partial_k a_{ij} + \partial_i a_{jk} - \partial_j a_{ki}$. The symbol 0 denotes contraction by y^i except in the quantities c_o and $c_{o\alpha}$ which occur latter.

K. Okubo[3,7-9] generalized the Christoffel symbols as follows:

Definition: For a symmetric covariant tensor field $a_{i_1.........i_m}(x)$ of order m the Christoffel symbols of m-th order are defined by

$$\{i_1i_2.......i_m, j\} = \frac{1}{2(m-1)}\left(\partial_{i_1}a_{i_2....i_mj} + \partial_{i_2}a_{i_3....i_1j} + \partial_{i_3}a_{i_4....i_2j} + ... - \partial_j a_{i_1....i_m}\right)$$

where the cyclic permutation is applied to $\{i_1i_2.......i_m\}$ in the first m terms of the right hand side. From the definition we get immediately

(2.6) $$\partial_j a_{i_1....i_m} = \{i_2.......i_m j, i_1\} + ...\{i_1.......i_{m-1} j, i_m\} - \{i_1.......i_m, j\}$$

where the first m terms of the right-hand side are constructed by cyclic permutation of $(i_1i_2.......i_m)$.

From (2.5), we have

$$(\partial_k a_{ijr_1r_2.......r_{m-2}})\ y^{r_1}y^{r_2}.......y^{r_{m-2}} = \{jk0.......0, i\} + \{ki0.......0, j\} + (m-2)[\{ijk0.....0, 0\} - \{ij0....0, k\}]$$

Consequently A_{ijk} can be written in terms of Christoffel symbols of m-th order as

$$A_{ijk} = m\{ki0......0, j\} - (m-2)[\{ij0......0, k\} + \{jk0.....0, i\} - \{ijk0.....0, 0\}]$$

Therefore $a_{ir}G^r_j$ of (2.3) can be written as

(2.7) $$a_{ir}G^r_j = \frac{1}{L^{(m-2)}}\{j0......0, i\} - \frac{m-2}{mL^{(m-1)}} a_i{}^r{}_j\{0..........0, r\}$$

while (2.5) is written in the form

(2.8) $a_{ir}G^r = \frac{1}{mL^{(m-2)}}\{0........0, i\}$.

As a consequence of (2.8) the geodesics can be given by the differential equations

(2.9) $a_{ij}\frac{d^2x^i}{ds^2} + \frac{2}{m}\{j_1........j_m, i\}\frac{dx^{j_1}}{ds}.........\frac{dx^{j_m}}{ds} = 0$

where $y^i = \frac{dx^i}{ds}$ in a_{ij}.

Theorem 1: *The nonlinear connection $G^i_j(x, y)$ of the Berwald connection and Cartan connection of a Finsler space with m-th root metric is given by* (2.7). *The equations of geodesic are written in the form* (2.9).

Remark: In the Riemannian case (m=2), the equation (2.9) reduces to the usual equation of geodesic.

3. Three-dimensional Finsler space with special m-th root metric

Example 1: We are concerned with a cubic metric (m = 3) of dimension three. For brevity, we shall write (x^i) and (y^i) as (x, y, z) and ($\dot{x}, \dot{y}, \dot{z}$) respectively, and $L(x, y, z; \dot{x}, \dot{y}, \dot{z})$ in the form

(3.1) $L^3 = c_0\dot{x}^3 + 3c_1\dot{x}^2\dot{y} + 3c_2\dot{x}\dot{y}^2 + c_3\dot{y}^3 + 3c_4\dot{x}^2\dot{z} + 3c_5\dot{x}\dot{z}^2 + 3c_6\dot{y}\dot{z}^2$
$+ 3c_7\dot{y}^2\dot{z} + 6c_8\dot{x}\dot{y}\dot{z} + c_9\dot{z}^3$

where

(a_{111}, a_{112}, a_{122}, a_{222}, a_{113}, a_{133}, a_{233}, a_{223}, a_{123}, a_{333}) = (c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9)

which are functions of (x, y, z). We put $\frac{\partial c_a}{\partial x^i} = c_{ai}$, a = 0,1,2,…...,9 and i = 1, 2, 3. Then we get thirty Christoffel symbols of third order as follows:

{111, 1} = $c_{01}/2$	{111, 2} = $(3c_{11}-c_{02})/4$	{111, 3} = $(3c_{41}-c_{03})/4$
{112, 1} = $(c_{02}+c_{11})/4$	{112, 2} = $c_{21}/2$	{112, 3} = $(2c_{81}+c_{42}-c_{13})/4$
{122, 1}=$c_{12}/2$	{122, 2} = $(c_{22}+c_{31})/4$	{122, 3} = $(c_{71}+2c_{82}-c_{23})/4$
{222, 1} = $(3c_{22}-c_{31})/4$	{222, 2} = $c_{32}/2$	{222, 3} = $(3c_{72}-c_{33})/4$
{113, 1} = $(c_{03}+c_{41})/4$	{113, 2} = $(2c_{81}+c_{13}-c_{42})/4$	{113, 3} = $c_{51}/2$
{133, 1} = $c_{43}/2$	{133, 2} = $(2c_{83}+c_{61}-c_{52})/4$	{133, 3} = $(c_{91}+c_{53})/4$
{233, 1} = $(2c_{83}+c_{52}-c_{61})/4$	{233, 2} = $c_{73}/2$	{233, 3} = $(c_{92}+c_{63})/4$
{223, 1} = $(2c_{82}+c_{23}-c_{71})/4$	{223, 2} = $(c_{72}+c_{33})/4$	{223, 3} = $c_{62}/2$
{123, 1} = $(c_{42}+c_{13})/4$	{123, 2} = $(c_{71}+c_{23})/4$	{123, 3} = $(c_{61}+c_{52})/4$
{333, 1}=$(3c_{53}-c_{91})/4$	{333, 2}=$(3c_{63}-c_{92})/4$	{333, 3}=$c_{93}/2$.

Now $H = L^3\det(a_{ij}) = A\dot{x}^3 + B\dot{x}^2\dot{y} + C\dot{x}\dot{y}^2 + D\dot{y}^3 + E\dot{x}^2\dot{z} + F\dot{x}\dot{z}^2 + K\dot{y}^2\dot{z} + M\dot{y}\dot{z}^2 + N\dot{z}^3 + P\dot{x}\dot{y}\dot{z}$,

where

$A = c_0c_2c_5 - c_0c_8^2 + 2c_1c_4c_8 - c_1^2c_5 - c_4^2c_2$

$B = c_0c_2c_6 - c_1c_2c_5 + c_0c_5c_3 - 2c_0c_8c_7 + 2c_1c_4c_7 - c_1^2c_6 + c_1c_8^2 - c_4^2c_3$

$C = c_1c_3c_5 + c_3c_6c_0 - c_6c_1c_2 - c_6c_7^2 + 2c_2c_7c_4 + c_2c_8^2 - c_2^2c_5 - 2c_4c_8c_3$

$D = c_1c_3c_6 - c_1c_7^2 + 2c_2c_7c_8 - c_2^2c_6 - c_8^2c_3$

$E = c_0c_2c_9 - c_2c_5c_4 + c_0c_5c_7 - 2c_8c_6c_0 + 2c_1c_4c_6 - c_1^2c_9 + c_4c_8^2 - c_4^2c_7$

$F = c_7c_8c_0 - c_4c_7c_5 + c_4c_9c_2 - c_0c_6^2 + 2c_6c_5c_1 - 2c_1c_8c_9 + c_5c_8^2 - c_5^2c_2$

$K = c_1c_3c_9 + c_3c_6c_4 - c_1c_6c_7 - c_7^2c_4 + 2c_2c_7c_5 + c_7c_8^2 - c_2^2c_9 - 2c_3c_5c_8$

$M = c_7c_9c_1 - c_4c_7c_6 + c_4c_9c_3 - c_6^2c_1 + c_8^2c_6 + 2c_2c_5c_6 - 2c_2c_8c_9 - c_5^2c_3$

$N = c_4c_7c_9 - c_6^2c4 + 2c_5c_6c_8 - c_8^2c_9 - c_5^2c_7$

$P = c_0c_3c_9 + 3c_1c_7c_5 + 3c_2c_4c_6 - 2c_4c_7c_8 - c_0c_6c_7 - 2c_1c_6c_8 + 2c_8^3 - c_1c_2c_9 - 2c_2c_8c_5 - c_3c_4c_5.$

Then from (2.8), we get

$$(3.2)\qquad \begin{cases} a_{11}G^1 + a_{12}G^2 + a_{13}G^3 = \dfrac{1}{3L}\{000,1\} \\ a_{21}G^1 + a_{22}G^2 + a_{23}G^3 = \dfrac{1}{3L}\{000,2\} \\ a_{31}G^1 + a_{32}G^2 + a_{33}G^3 = \dfrac{1}{3L}\{000,3\} \end{cases}$$

By equation (3.2) we get

$$(3.3)\qquad \begin{cases} (3H)(2G^1) = L^2\begin{vmatrix} \{000,1\} & a_{12} & a_{13} \\ \{000,2\} & a_{22} & a_{23} \\ \{000,3\} & a_{32} & a_{33} \end{vmatrix} \\ (3H)(2G^2) = L^2\begin{vmatrix} a_{11} & \{000,1\} & a_{13} \\ a_{21} & \{000,2\} & a_{23} \\ a_{31} & \{000,3\} & a_{33} \end{vmatrix} \\ (3H)(2G^3) = L^2\begin{vmatrix} a_{11} & a_{12} & \{000,1\} \\ a_{21} & a_{22} & \{000,2\} \\ a_{31} & a_{13} & \{000,3\} \end{vmatrix} \end{cases}$$

Theorem 2: *The equations* (3.3) *give G^i and hence the Berwald connection $B\Gamma$ of cubic metric in F^3.*

Example 2: In quartic metric theories of gravity, I. W. Roxburgh[5] paid special attention to the strongly spherically symmetric metric

$$ds^4 = Adt^4 + Bdt^2 d\bar{x}^2 + Cd\bar{x}^4 ,$$

where A, B, C are functions of the Newtonian potential U = m/r, where $r^2 = x^2 + y^2 + z^2$ and $d\bar{x}^2 = dx^2 + dy^2 + dz^2$. Though this metric is of four-dimension, we shall reduce it to three-dimension and consider a special quartic metric of the form

(3.4) $$L^4 = c_0 \dot{x}^4 + 6c_2 \dot{x}^2 \dot{y}^2 + c_4 \dot{y}^4 + 6c_6 \dot{x}\, 2\, \dot{z}^2 + 6c_8 \dot{y}^2 \dot{z}^2 + c_{14} \dot{z}^4$$

where $(c_0, c_2, c_4, c_6, c_8, c_{14}) = (a_{1111}, a_{1122}, a_{2222}, a_{1133}, a_{2233}, a_{3333})$. The surviving components of the Christoffel symbols of fourth order are as follows:

$\{1111, 1\} = c_{01}/2$	$\{1111, 2\} = - c_{02}/6$	$\{1111, 3\} = - c_{03}/6$
$\{1112, 1\} = c_{02}/6$	$\{1112, 2\} = c_{21}/2$	$\{1112, 3\} = 0$
$\{1122, 1\} = c_{21}/6$	$\{1122, 2\} = c_{22}/6$	$\{1122, 3\} = - c_{23}/6$
$\{1222, 1\} = c_{22}/2$	$\{1222, 2\} = c_{41}/6$	$\{1222, 3\} = 0$
$\{2222, 1\} = - c_{41}/6$	$\{2222, 2\} = c_{42}/2$	$\{2222, 3\} = - c_{43}/6$
$\{1113, 1\} = c_{03}/6$	$\{1113, 2\} = 0$	$\{1113, 3\} = c_{61}/2$
$\{1133, 1\} = c_{61}/6$	$\{1133, 2\} = - c_{62}/6$	$\{1133, 3\} = c_{63}/6$
$\{1333, 1\} = c_{63}/2$	$\{1333, 2\} = 0$	$\{1333, 3\} = c_{141}/6$
$\{2233, 1\} = - c_{81}/6$	$\{2233, 2\} = c_{82}/6$	$\{2233, 3\} = c_{83}/6$
$\{2333, 1\} = 0$	$\{2333, 2\} = c_{83}/2$	$\{2333, 3\} = c_{142}/6$
$\{2223, 1\} = 0$	$\{2223, 2\} = c_{43}/6$	$\{2223, 3\} = c_{82}/2$
$\{1233, 1\} = c_{62}/6$	$\{1233, 2\} = c_{81}/6$	$\{1233, 3\} = 0$
$\{1223, 1\} = c_{23}/6$	$\{1223, 2\} = 0$	$\{1223, 3\} = c_{81}/6$
$\{1123, 1\} = 0$	$\{1123, 2\} = c_{23}/6$	$\{1123, 3\} = c_{62}/6$
$\{3333, 1\} = - c_{141}/6$	$\{3333, 2\} = - c_{142}/6$	$\{3333, 3\} = c_{143}/2.$

Next, we refer to the arc length s and put $(\dot{x}, \dot{y}, \dot{z}) = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$.Then we have

$L(x, y, z;\ \dot{x}, \dot{y}, \dot{z}) = 1$ and $a_{11} = c_0 \dot{x}^2 + c_2 \dot{y}^2 + c_6 \dot{z}^2$, $a_{12} = a_{21} = 2c_2 \dot{x}\ \dot{y}$,
$a_{13} = a_{31} = 2c_6 \dot{x}\ \dot{z}$, $a_{23} = a_{32} = 2c_8 \dot{y}\ \dot{z}$, $a_{22} = c_2 \dot{x}^2 + c_4 \dot{y}^2 + c_8 \dot{z}^2$,
$a_{33} = c_4 \dot{x}^2 + c_8 \dot{y}^2 + c_{14} \dot{z}^2$.

Thus we obtain the equations of geodesic as follows:

$$\begin{cases} (c_0(\dot{x})^2 + c_2(\dot{y})^2 + c_6(\dot{z})^2)\ddot{x} + (2c_2\dot{x}\dot{y})\ddot{y} + (2c_6\dot{x}\dot{z})\ddot{z} + \{\frac{c_{01}}{4}(\dot{x})^4 + \frac{c_{02}}{3}(\dot{x})^3\dot{y} \\ +\frac{c_{21}}{12}(\dot{x})^2(\dot{y})^2 + \frac{c_{22}}{4}(\dot{x})(\dot{y})^3 - \frac{c_{41}}{12}(\dot{y})^4 + \frac{c_{03}}{12}(\dot{x})^3\dot{z} + \frac{c_{61}}{12}(\dot{x})^2(\dot{z})^2 + \frac{c_{63}}{4}(\dot{x})(\dot{z})^3 \\ -\frac{c_{81}}{12}(\dot{y})^2(\dot{z})^2 + \frac{c_{62}}{12}(\dot{x})(\dot{y})(\dot{z})^2 + \frac{c_{23}}{12}(\dot{x})(\dot{y})^2(\dot{z}) - \frac{c_{141}}{12}(\dot{z})^4\} = 0 \\ (2c_2\dot{x}\dot{y})\ddot{x} + (c_2(\dot{x})^2 + c_4(\dot{y})^2 + c_8(\dot{z})^2)\ddot{y} + (2c_8\dot{x}\dot{z})\ddot{z} + \{-\frac{c_{02}}{12}(\dot{x})^4 + \frac{c_{21}}{4}(\dot{x})^3\dot{y} \\ +\frac{c_{22}}{12}(\dot{x})^2(\dot{y})^2 + \frac{c_{41}}{12}(\dot{x})(\dot{y})^3 + \frac{c_{42}}{4}(\dot{y})^4 - \frac{c_{62}}{12}(\dot{x})^2(\dot{z})^2 + \frac{c_{82}}{12}(\dot{y})^2(\dot{z})^2 + \frac{c_{83}}{4}(\dot{y})(\dot{z})^3 \\ +\frac{c_{43}}{12}(\dot{y})^3(\dot{z}) + \frac{c_{81}}{12}(\dot{x})(\dot{y})(\dot{z})^2 + \frac{c_{23}}{12}(\dot{x})^2(\dot{y})(\dot{z}) - \frac{c_{142}}{12}(\dot{z})^4\} = 0 \\ (2c_6\dot{x}\dot{z})\ddot{x} + (2c_8\dot{x}\dot{y})\ddot{y} + (c_4(\dot{x})^2 + c_8(\dot{y})^2 + c_{14}(\dot{z})^2)\ddot{z} + \{-\frac{c_{03}}{12}(\dot{x})^4 - \frac{c_{23}}{12}(\dot{x})^2(\dot{y})^2 \\ -\frac{c_{43}}{12}(\dot{y})^4 + \frac{c_{61}}{4}(\dot{x})^3(\dot{z}) - \frac{c_{63}}{12}(\dot{x})^2(\dot{z})^2 + \frac{c_{141}}{12}(\dot{x})(\dot{z})^3 + \frac{c_{83}}{12}(\dot{y})^2(\dot{z})^2 + \frac{c_{142}}{4}(\dot{y})(\dot{z})^3 \\ +\frac{c_{82}}{12}(\dot{y})^3(\dot{z}) + \frac{c_{81}}{12}(\dot{x})(\dot{y})^2(\dot{z}) + \frac{c_{62}}{12}(\dot{x})^2(\dot{y})(\dot{z}) + \frac{c_{143}}{4}(\dot{z})^4\} = 0 \end{cases}$$

4. Main Scalars of three-dimensional Finsler space with m-th root metric

We shall consider a three-dimensional Finsler space F^n with m-th root metric. In the following the symbols (x, y, z) without dot will be used instead of (y^1, y^2, y^3).

If we put

$$A_{ij} = a_{ij} - a_i a_j, \qquad A_{ijk} = a_{ijk} - a_{ij}a_k - a_{jk}a_i - a_{ki}a_j + 2a_i a_j a_k.$$

We have from (1.2)

$$h_{ij} = (m-1)A_{ij}, \qquad 2LC_{ijk} = (m-1)(m-2)A_{ijk} \tag{4.1}$$

It follows from (1.3) that A_{ij} and A_{ijk} satisfy

$$A_{ij}y^j = A_{i1}x + A_{i2}y + A_{i3}z = 0, \qquad A_{ijk}y^k = A_{ij1}x + A_{ij2}y + A_{ij3}z = 0$$

Then we have

(4.2 a) $\quad A_{13} = uA_{11} + vA_{12}, \quad A_{23} = uA_{12} + vA_{22}, \quad A_{33} = uA_{13} + vA_{23}$

and

(4.2 b) $$\begin{cases} A_{113} = uA_{111} + vA_{112}, \quad A_{123} = uA_{112} + vA_{122}, \quad A_{133} = uA_{113} + vA_{123}, \\ A_{233} = uA_{123} + vA_{223}, \quad A_{223} = uA_{122} + vA_{222}, \quad A_{333} = uA_{133} + vA_{233}. \end{cases}$$

where we put $u = -x/z$, $v = -y/z$.

Since there are three equations and six unknown terms, so in general there are no solutions or infinitely many solutions. So, we are putting

$$A_{11} = K_1, \qquad A_{22} = K_2, \qquad A_{33} = K_3,$$

where K_1, K_2 and K_3 are scalars. Then

$$A_{12} = \frac{z^2K_3 - x^2K_1 - y^2K_2}{2xy}, \quad A_{23} = \frac{z^2K_3 - x^2K_1 + y^2K_2}{-2yz},$$

$$A_{13} = \frac{z^2K_3 + x^2K_1 - y^2K_2}{-2xz}.$$

Thus we have

(4.3)
$$A_{11} = K_1, \qquad A_{22} = K_2, \qquad A_{33} = K_3,$$
$$A_{12} = \frac{z^2K_3 - x^2K_1 - y^2K_2}{2xy}, \quad A_{23} = \frac{z^2K_3 - x^2K_1 + y^2K_2}{-2yz},$$
$$A_{13} = \frac{z^2K_3 + x^2K_1 - y^2K_2}{-2xz}.$$

Similarly by putting scalars G_1, G_2, G_3 G_4 such as

$$A_{111} = G_1 \qquad A_{222} = G_2 \qquad A_{333} = G_3 \qquad A_{113} = G_4$$

we have

(4.4) $A_{111} = G_1 \qquad A_{222} = G_2 \qquad A_{333} = G_3 \qquad A_{113} = G_4$

$$A_{123} = \frac{z^3G_3 + x^3G_1 + y^3G_2}{2xyz}, \quad A_{122} = \frac{z^3G_3 - 2x^3G_1 + y^3G_2 - 3x^2zG_4}{-3xy^2}$$

$$A_{223} = \frac{z^3G_3 - 2x^3G_1 - 2y^3G_2 - 3x^2zG_4}{3y^2z}, \qquad A_{133} = \frac{3x^3zG_4 + z^3G_3 + x^3G_1 + y^3G_4}{-2xz^2},$$

$$A_{233} = \frac{2z^3G_3 - x^3G_1 - y^3G_2 - 3x^2zG_4}{-3yz^2}$$

$\det(a_{ij}) = \det(A_{ij} + a_i a_j)$.

Proposition 1: *The regularity of m-th root metric of dimension three is equivalent to* $K_1 \neq 0$, $K_2 \neq 0$, $K_3 \neq 0$ *in* (4.3).

Now it is usual to use the Berwald frame (l, m, n) for considering three dimensional Finsler spaces, where l is the normalized supporting element given now by $l_i = a_i$ and m, n are unit vectors orthogonal to l given by $h_{ij} = m_i m_j + n_i n_j$. From

(4.5)
$$m_i l^i = (m_1 x + m_2 y + m_3 z)/L$$
$$n_i l^i = (n_1 x + n_2 y + n_3 z)/L$$

then by putting

(4.5)' $m_1 = p, \quad m_2 = q, \quad n_1 = p', \quad n_2 = q',$

$m_3 = um_1 + vm_2, \quad n_3 = un_1 + vn_2$

where p, q, p' and q' are scalars.

Next, the main scalars H, J, I of F^3 are defined by[4]

(4.6)
$$LC_{ijk} = Hm_i m_j m_k - J\pi_{\{ijk\}}(m_i m_j n_k) + I\pi_{\{ijk\}}(m_i n_j n_k) + Jn_i n_j n_k$$

where $\pi_{\{ijk\}}$ represent cyclic sum of the terms obtained by cyclic permutation of *i, j, k.*

According to (4.4) for instance

$$Hm_1^3 - J(3m_1^2 n_1 - n_1^3) + 3\,\mathrm{Im}_1\, n_1^2 = \frac{(m-1)(m-2)}{2} A_{111}$$

$$Hm_2^3 - J(3m_2^2 n_1 - n_2^3) + 3\,\mathrm{Im}_2\, n_2^2 = \frac{(m-1)(m-2)}{2} A_{222}$$

$$Hm_3^3 - J(3m_3^2 n_3 - n_3^3) + 3\,\mathrm{Im}_3\, n_3^2 = \frac{(m-1)(m-2)}{2} A_{333}$$

we have

$$(4.7)\quad \begin{cases} H = \dfrac{(m-1)(m-2)}{6Z}\begin{vmatrix} A_{111} & n_1(n_1^2-3m_1^2) & 3m_1n_1^2 \\ A_{222} & n_2(n_2^2-3m_2^2) & 3m_2n_2^2 \\ A_{333} & n_3(n_3^2-3m_3^2) & 3m_3n_3^2 \end{vmatrix} \\[2ex] J = \dfrac{(m-1)(m-2)}{6Z}\begin{vmatrix} m_1^3 & A_{111} & 3m_1n_1^2 \\ m_2^3 & A_{222} & 3m_2n_2^2 \\ m_3^3 & A_{333} & 3m_3n_3^2 \end{vmatrix} \\[2ex] I = \dfrac{(m-1)(m-2)}{6Z}\begin{vmatrix} m_1^3 & n_1(n_1^2-3m_1^2) & A_{111} \\ m_2^3 & n_2(n_2^2-3m_2^2) & A_{222} \\ m_3^3 & n_3(n_3^2-3m_3^2) & A_{333} \end{vmatrix} \end{cases}$$

where m_1, m_2, m_3, n_1, n_2, n_3 and A_{111}, A_{222}, A_{333} are given by (4.5)' and (4.4), and

$$Z = \begin{vmatrix} m_1^3 & n_1(n_1^2-3m_1^2) & 3m_1n_1^2 \\ m_2^3 & n_2(n_2^2-3m_2^2) & 3m_2n_2^2 \\ m_3^3 & n_3(n_3^2-3m_3^2) & 3m_3n_3^2 \end{vmatrix}.$$

Proposition 2: *The main scalars H, J, I of a three dimensional Finsler space with m-th root metric are given by* (4.7).

Special cases:

1) Cubic metric: When we treat with cubic metric (m = 3) then equation (4.7) can be rewritten as:

$$(4.8)\quad \begin{cases} H = \dfrac{1}{Z}\begin{vmatrix} A_{111} & n_1(n_1^2-3m_1^2) & 3m_1n_1^2 \\ A_{222} & n_2(n_2^2-3m_2^2) & 3m_2n_2^2 \\ A_{333} & n_3(n_3^2-3m_3^2) & 3m_3n_3^2 \end{vmatrix}, \qquad J = \dfrac{1}{Z}\begin{vmatrix} m_1^3 & A_{111} & 3m_1n_1^2 \\ m_2^3 & A_{222} & 3m_2n_2^2 \\ m_3^3 & A_{333} & 3m_3n_3^2 \end{vmatrix}, \\[2ex] I = \dfrac{1}{Z}\begin{vmatrix} m_1^3 & n_1(n_1^2-3m_1^2) & A_{111} \\ m_2^3 & n_2(n_2^2-3m_2^2) & A_{222} \\ m_3^3 & n_3(n_3^2-3m_3^2) & A_{333} \end{vmatrix}. \end{cases}$$

Proposition 3: *The main scalars H, J, I of a three dimensional Finsler space with cubic metric are given by* (4.8).

2) Quartic metric: When we treat with quartic metric (m = 4) then equation (4.7) can be rewritten as

$$(4.9)\qquad \begin{cases} H = \dfrac{3}{Z}\begin{vmatrix} A_{111} & n_1(n_1^2 - 3m_1^2) & 3m_1 n_1^2 \\ A_{222} & n_2(n_2^2 - 3m_2^2) & 3m_2 n_2^2 \\ A_{333} & n_3(n_3^2 - 3m_3^2) & 3m_3 n_3^2 \end{vmatrix} \\[2ex] J = \dfrac{3}{Z}\begin{vmatrix} m_1^3 & A_{111} & 3m_1 n_1^2 \\ m_2^3 & A_{222} & 3m_2 n_2^2 \\ m_3^3 & A_{333} & 3m_3 n_3^2 \end{vmatrix} \\[2ex] I = \dfrac{3}{Z}\begin{vmatrix} m_1^3 & n_1(n_1^2 - 3m_1^2) & A_{111} \\ m_2^3 & n_2(n_2^2 - 3m_2^2) & A_{222} \\ m_3^3 & n_3(n_3^2 - 3m_3^2) & A_{333} \end{vmatrix} \end{cases}$$

Proposition 4: *The main scalars H, J, I of a three dimensional Finsler space with quartic metric are given by* (4.9).

5. Result reducible to two-dimensional space

If we put c_4, c_5, c_6, c_7, c_8, c_9 zero and c_0, c_1, c_2, c_3 are function of (*x, y*) then (3.1) reduces to

$$(5.1)\qquad L^3 = c_0 \dot{x}^3 + 3c_1 \dot{x}^2 \dot{y} + 3c_2 \dot{x} \dot{y}^2 + c_3 \dot{y}^3,$$

and thirty Christoffel symbols of third order reduces to only eight symbols such as

$\{111, 1\} = c_{01}/2$, $\{111, 2\} = (3c_{11} - c_{02})/4$, $\{112, 1\} = (c_{02} + c_{11})/4$,

$\{112, 2\} = c_{21}/2$, $\{122, 1\} = c_{12}/2$, $\{122, 2\} = (c_{22} + c_{31})/4$,

$\{222, 1\} = (3c_{22} - c_{31})/4$, $\{222, 2\} = c_{32}/2$.

We put

$$H = L^2 \det a_{ij} = L^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

then the equation (3.2) reduces to

$$a_{11}G^1 + a_{12}G^2 = \frac{1}{3L}\{000, 1\}, \qquad a_{21}G^1 + a_{22}G^2 = \frac{1}{3L}\{000, 2\}.$$

Solving this equation, we get

$$(3H)(2G^1) = \{\tfrac{1}{2}c_1(c_{02} - 3c_{11}) + c_2 c_{01}\}\dot{x}^4 - (3c_1 c_{21} - 2c_2 c_{02} - c_3 c_{01})\dot{x}^3\dot{y} - 3\{\tfrac{1}{2}c_1(c_{22} + c_{31}) - c_2(c_{12} - c_{21}) - \tfrac{1}{2}c_3(c_{02} + c_{11})\}\dot{x}^2\dot{y}^2 - (c_1 c_{32} + 2c_2 c_{31} - 3c_3 c_{12})\dot{x}\dot{y}^3 - \{c_2 c_{32} + \tfrac{1}{2}c_3(c_{31} - 3c_{22})\}\dot{y}^4$$

$$(3H)(2G^2) = -\{\tfrac{1}{2}c_0(c_{02} - 3c_{11}) + c_1c_{01}\}\,\dot{x}^4 + (3c_0c_{21} - 2c_1c_{02} - c_2c_{01})\,\dot{x}^3\dot{y} + 3\{\tfrac{1}{2}c_0(c_{22} + c_{31}) - c_1(c_{12} - c_{21}) - \tfrac{1}{2}c_2(c_{02} + c_{11})\}\,\dot{x}^2\dot{y} + (c_0c_{32} + 2c_1c_{31} - 3c_2c_{12}))\,\dot{x}\,\dot{y}^3 - \{c_1c_{32} + \tfrac{1}{2}c_2(c_{31} - 3c_{22})\}\,\dot{y}^4$$

These are required Berwald connection coefficients for a two dimensional Finsler space[3].

Again, if we put c_6, c_8, c_{14} as zero and c_o, c_2, c_4 are function of (x, y) only then, (3.4) reduces to

(5.2) $$L^4 = c_0\,\dot{x}^4 + 6c_2\,\dot{x}^2\,\dot{y}^2 + c_4\,\dot{y}^4$$

and forty five Christoffel symbols reduces to the following ten symbols

$\{1111, 1\} = c_{01}/2$, $\{1111, 2\} = -c_{02}/6$, $\{1112, 1\} = c_{02}/6$,

$\{1112, 2\} = c_{21}/2$, $\{1122, 1\} = c_{21}/6$, $\{1122, 2\} = c_{22}/6$,

$\{1222, 1\} = c_{22}/2$, $\{1222, 2\} = c_{41}/6$, $\{2222, 1\} = -c_{41}/6$,

$\{2222, 2\} = c_{42}/2$,

and $a_{11} = c_0\,\dot{x}^2 + c_2\,\dot{y}^2$, $a_{12} = 2c_2\,\dot{x}\,\dot{y}$, $a_{22} = c_2\,\dot{x}^2 + c_4\,\dot{y}^2$

Thus, we obtain the equations of geodesic as follows[3]

$$(c_0(\dot{x})^2 + c_2(\dot{y})^2)\ddot{x} + (2c_2\dot{x}\dot{y})\ddot{y} + \frac{c_{01}}{4}(\dot{x})^4 + \frac{c_{02}}{3}(\dot{x})^3\dot{y} + \frac{c_{21}}{2}(\dot{x})^2(\dot{y})^2 + c_{22}(\dot{x})(\dot{y})^3 - \frac{c_{41}}{12}(\dot{y})^4 = 0$$

$$(2c_2\dot{x}\dot{y})\ddot{x} + (c_2(\dot{x})^2 + c_4(\dot{y})^2)\ddot{y} - \frac{c_{02}}{12}(\dot{x})^4 + c_{21}(\dot{x})^3\dot{y} + \frac{c_{22}}{2}(\dot{x})^2(\dot{y})^2 + \frac{c_{41}}{3}(\dot{x})(\dot{y})^3 + \frac{c_{42}}{4}(\dot{y})^4 = 0$$

Next Putting A_{13}, A_{23}, A_{33} zero in (4.2a), then we have

(5.3) $$(A_{11}, A_{12}, A_{22}) = X\,(y^2, -xy, x^2)$$

where X is a scalar. Putting A_{113}, A_{123}, A_{133}, A_{233}, A_{223}, A_{333} zero in (4.2b), we get

(5.4) $$(A_{111}, A_{112}, A_{122}, A_{222}) = 2Y(\,y^3, -xy^2, x^2y, -x^3)$$

where Y is a scalar. Since n_i does not appear in two-dimensional case and m_3 vanishes; from equation (4.5), we have $(m_1, m_2) = k\,(-y, x)$.

Thus, from equations (4.1) and (5.3), we have

(5.5) $$(m\text{-}1)\,X = k^2$$

and in equation (4.6), J and I vanish. Therefore $LC_{ijk} = Hm_im_jm_k$

Further, in view of (4.1) and (5.4), we have

(5.6) $$-Hk^3 = (m\text{-}1)(m\text{-}2)Y.$$

Hence from (5.5) and (5.6), we have

$$-H^2 = \frac{(m-2)^2Y^2}{(m-1)X^3}.$$

This is the main scalar of cubic metric in two-dimensional Finsler space[3].

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