Scalar Curvature of Two-dimensional Cubic Finsler space

T. N. Pandey and V. K. Chaubey

Department of Mathematics & Statistics, D. D. U. Gorakhpur University, Gorakhpur
E-mail: vkcoct@rediffmail.com, vkcoct@gmail.com

B. N. Prasad

C-10, Surajkuand Coloney, Gorakhpur (U.P.)

(Received August 18, 2008)

Abstract: In the present paper we obtain scalar curvature R of Two-dimensional Finsler space with cubic metric. Some special cubic metrics have been considered and explicit expression for scalar curvature R has been found. Variation of scalar curvature has been shown in various figures for x and y.

Key Words: Finsler space with cubic root metric, scalar curvature R.

2000 AMS Subject Classification Number: **53B40, 53C60**.

1. Preliminaries

There are few papers on cubic Finsler spaces where certain properties[1-6] have been studied. There are various papers on the geometry of spaces with a cubic metric as a generalization of Euclidean or Riemannian geometry. In 1995 M. Matsumoto and K. Okubo[7], introduced the Christoffel symbols of m-th order, they obtained connection coefficients of Berwald and the differential equations of geodesic. They specially treated the two-dimensional Finsler space with cubic and quartic metrics and obtained the main scalar.

In Two-dimensional Finsler space F^2 the main scalar I and curvature R have important roles. The purpose of the present paper is to obtain scalar curvature R of Two-dimensional Finsler space with cubic metric. Some special cubic metrics have been considered and explicit expression for scalar curvature R has been found as given in equations (3.2), (3.3). Variation of scalar curvature for *x* and *y* has been plotted in the figure.

The cubic root Finsler metric *L(x, y)* of an n-dimensional differentiable manifold M^n is defined as

$$L^3(x, y) = a_{ijk}(x) y^i y^j y^k \tag{1.1}$$

where $a_{ijk}(x)$ are components of a symmetric tensor field of (0, 3)-type, depending on the position *x* alone, and a Finsler space with this metric is called the cubic Finsler space.

Let us define $a_{ij}(x, y)$ and $a_i(x, y)$ by

(1.2) $L a_{ij}(x) = a_{ijk}(x) y^k$ and $L^2 a_i(x) = a_{ijk}(x) y^j y^k$.

Then the normalized supporting element $l_i = \dot{\partial}_i L$ the angular metric tensor $h_{ij} = L\dot{\partial}_i \dot{\partial}_j L$ and the fundamental tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2 = h_{ij} + l_i l_j$ and the C-tensor $C_{ijk} = \frac{1}{4}\dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2$ are respectively given by the equations

(1.3) a) $l_i = a_i$, b) $h_{ij} = 2(a_{ij} - a_i a_j)$, c) $g_{ij} = 2a_{ij} - a_i a_j$

d) $LC_{ijk} = (a_{ijk} - a_{jk}a_i - a_{ki}a_j - a_{ij}a_k + 2a_i a_j a_k)$

Let us call $a_{ij}(x, y)$ the basic tensor because this played an important role[1]. The metric L is called regular, if the basic tensor has the non-vanishing determinant. Throughout our theories of cubic root metrics, we would suppose the regularity of the metrics.

By $a^{ij}(x, y)$ we denote the reciprocal of $a_{ij}(x, y)$. Then the reciprocal $g^{ij}(x, y)$ of $g_{ij}(x, y)$ is given as

$$2\, g^{ij} = a^{ij} + a^i a^j \quad \text{and} \quad l^i = a^i ,$$

where $a_i a^i = 1$, $a^i = a^{ij} a_j$, $l^i = y^i / L = g^{ij} l_j$.

2. Scalar Curvature of Two-dimensional cubic Finsler space

In two-dimensional Finsler space the Berwald frame (l_i, m_i) has important role[8]. Any tensor field can be expressed in terms of this frame.

We are concerned with a cubic Finsler space equipped with a Berwald connection BΓ = $(G^i_{jk}, G^i_j, 0)$ where, $G^i_j = \dot{\partial}_j G^i$ and $G^i_j = \dot{\partial}_j G^i_k$, $G_j = g_{ij} G^i = (y^r \dot{\partial}_j \partial_r L^2 - \partial_j L^2)/4$, $\dot{\partial}_j$ represent partial derivative with respect to y^i and ∂_i represent partial derivative with respect to x^i.

There are five torsion tensors and three curvature tensors in the theory of Finsler space equipped with any Finsler connection. If we are concerned with Berwald connection BΓ the only non-vanishing torsion tensor is (v)-h torsion tensor R^i_{jk} given by

(2.1) $R^i_{jk} = \delta_k G^i_j - \delta_j G^i_k$

where $\delta_k = \partial_k - G^r_k \dot{\partial}_r$.

The (v)-h torsion tensor R^i_{jk} in two-dimensional Finsler space[8] may be written as

(2.2) $R^i_{jk} = LRm^i (l_j m_k - l_k m_j)$,

where R is the h-scalar curvature and $m^i = g^{ij} m_j$.

The cubic metric in two-dimensional Finsler space is given by (1.1) with i, j, k =1, 2. If we put (a_{111}, a_{112}, a_{122}, a_{222}) = (c_0, c_1, c_2, c_3) and $(x^1, x^2) = (x, y)$, $(y^1, y^2) = (\dot{x}, \dot{y})$. Then, (1.1) may be written as

$$L^3 = c_0 \dot{x}3 + 3c_1 \dot{x}2 \dot{y} + 3c_2 \dot{x} \dot{y}2 + c_3 \dot{y}^3$$

The Christoffel symbols of third order for a cubic Finsler space has been defined by M. Matsumoto[7], and is given by

(2.3) $$\{ijk, h\} = \frac{1}{4}\left(\partial_i a_{jkh} + \partial_j a_{ikh} + \partial_k a_{ijh} - \partial_h a_{ijk}\right).$$

By putting $\frac{\partial c_a}{\partial x} = c_{a1}$, $\frac{\partial c_a}{\partial y} = c_{a2}$, a = 0, 1, 2, 3, we get eight Christoffel symbols of third order as follows:

$\{111, 1\} = c_{01}/2$, $\{111, 2\} = (3c_{11}-c_{02})/4$, $\{112, 1\} = (c_{02}+c_{11})/4$,
$\{112, 2\} = c_{21}/2$ $\{122, 1\}=c_{12}/2$, $\{122, 2\} = (c_{22}+c_{31})/4$,
$\{222, 1\} = (3c_{22}-c_{31})/4$, $\{222, 2\} = c_{32}/2$

If we put H = L^2det a_{ij}, then from (2.3) and (1.2), we have

(2.4) $$H = A(\dot{x})^2 + B\dot{x}\dot{y} + C(\dot{y})^2 ,$$

where $A = c_0c_2 - c_1^2$, $B = c_0c_3 - c_1c_2$, $C = c_1c_3 - c_2^2$.

The value of G^1, G^2 in two dimensional cubic Finsler space are the solutions of linear equation $a_{ir}G^r = \frac{1}{3L}\{jkh, i\} y^j y^k y^h$ and are given by[7]

(2.5) $$(3H)(2G^1) = a(\dot{x})^4 + b(\dot{x})^3\dot{y} + c(\dot{x})^2(\dot{y})^2 + d(\dot{x})(\dot{y})^3 + e(\dot{y})^4$$

where $a = \frac{1}{2} c_1(c_{02} - 3c_{11}) + c_2c_{01}$, $b = -(3c_1c_{21} - 2c_2c_{02} - c_3c_{01})$,

$$c = -3\{\frac{1}{2}c_1(c_{22} + c_{31}) - c_2(c_{12} - c_{21}) - \frac{1}{2}c_3(c_{02} + c_{11})\},$$

$$d = -(c_1c_{32} + 2c_2c_{31} - 3c_3c_{12}), \qquad e = -\{c_2c_{32} + \frac{1}{2}c_3(c_{31} - 3c_{22})\}$$

and

(2.6) $$(3H)(2G^2) = f(\dot{x})^4 + g(\dot{x})^3\dot{y} + h(\dot{x})^2(\dot{y})^2 + k(\dot{x})(\dot{y})^3 + l(\dot{y})^4$$

where $f = -\{\frac{1}{2}c_0(c_{02} - 3c_{11}) + c_1c_{01}\}$, $g = (3c_0c_{21} - 2c_1c_{02} - c_2c_{01})$,

$$h = 3\{\frac{1}{2}c_0(c_{22} + c_{31}) - c_1(c_{12} - c_{21}) - \frac{1}{2}c_2(c_{02} + c_{11})\},$$

$$k = (c_0c_{32} + 2c_1c_{31} - 3c_2c_{12})), \qquad l = \{c_1c_{32} + \frac{1}{2}c_2(c_{31} - 3c_{22})\}$$

In two-dimensional Finsler space, the non-vanishing components of R^i_{jk} are only $R^1_{12} = -R^1_{21}$, R^2_{12} = - R^2_{21} . We are concerned with R^1_{12} . From (2.1), we have

(2.1)' $$R^1_{12} = \frac{\partial G^1_1}{\partial y} - (\frac{\partial G^1_1}{\partial \dot{x}} G^1_2 + \frac{\partial G^1_1}{\partial \dot{y}} G^2_2) - \frac{\partial G^1_2}{\partial x} + (\frac{\partial G^1_2}{\partial \dot{x}} G^1_1 + \frac{\partial G^1_2}{\partial \dot{y}} G^2_1)$$

and from (2.2), we have

(2.2)' $$R^1_{12} = LRm^1(l_1 m_2 - l_2 m_1) .$$

Firstly, we are concerned with equation (2.2)' for the metric (2.3). The equation (1.2) explicitly may be written as

$$La_{11} = c_0\dot{x} + c_1\dot{y} , \qquad La_{12} = c_1\dot{x} + c_2\dot{y} , \qquad La_{22} = c_2\dot{x} + c_3\dot{y} ,$$

$$L^2 a_1 = c_0(\dot{x})^2 + 2c_1\dot{x}\dot{y} + c_2(\dot{y})^2 , \qquad L^2 a_2 = c_1(\dot{x})^2 + 2c_2\dot{x}\dot{y} + c_3(\dot{y})^2 .$$

With the help of above equations, the expression (1.3c) may be expressed as

(2.7) $$\begin{cases} L^4 g_{11} = c_0^2(\dot{x})^4 - 2c_1^2(\dot{x})^2(\dot{y})^2 + 2(c_0c_3 - c_1c_2)(\dot{x})(\dot{y})^3 + (2c_1c_3 - c_2^2)(\dot{y})^4 \\ L^4 g_{12} = c_0c_1(\dot{x})^4 - (c_1c_2 + c_0c_3)(\dot{x})^2(\dot{y})^2 + c_2c_3(\dot{y})^4 = L^4 g_{21} \\ L^4 g_{22} = (2c_0c_2 - c_1^2)(\dot{x})^4 + 2(c_0c_3 - c_1c_2)(\dot{x})^3(\dot{y}) - 2c_2^2\dot{x}(\dot{y})^4 + c_3^2(\dot{y})^4 \end{cases}$$

Since $l_i = g_{ij} l^j$, we have

(2.8) $$\begin{cases} Ll_1 = g_{11}\dot{x} + g_{12}\dot{y} \\ Ll_2 = g_{21}\dot{x} + g_{22}\dot{y} . \end{cases}$$

Then, in view of (2.7), equation (2.8) reduces to

(2.9) $$\begin{cases} L^5 l_1 = c_0^2(\dot{x})^5 + c_0c_1(\dot{x})^4\dot{y} - 2c_1^2(\dot{x})^3(\dot{y})^2 + (c_0c_3 - 3c_1c_2)(\dot{x})^2(\dot{y})^3 \\ \qquad - (2c_1c_3 - c_2^2)(\dot{x})(\dot{y})^4 + c_2c_3(\dot{y})^5 \\ L^5 l_2 = c_0c_1(\dot{x})^5 + (2c_0c_2 - c_1^2)(\dot{x})^4\dot{y} + (c_0c_3 - 3c_1c_2)(\dot{x})^3(\dot{y})^2 - 2c_2^2(\dot{x})^2(\dot{y})^3 \\ \qquad + c_2c_3(\dot{x})(\dot{y})^4 + c_3^2(\dot{y})^5. \end{cases}$$

The components m^i and m_i (i =1, 2) of Berwald frame are related with l^i and l_i by the relations[8]

(2.10) $$\begin{cases} (m^1, m^2) = \frac{1}{\sqrt{g}}(-l_2, l_1) \\ (m_1, m_2) = \sqrt{g}(-l^2, l^1) \end{cases}$$

where g = det(g_{ij}). In view of (2.10), equation (2.2)' reduces to

(2.11) $$R^1_{12} = LRm^1(l_1 m_2 - l_2 m_1) = -Rl_2(l_1\dot{x} + l_2\dot{y})$$

where l_1 and l_2 are given by (2.9).

Secondly, we are concerned with equation (2.1)'. From (2.4), we have

$$(2.12)\quad \dot{H}_1 = \frac{\partial H}{\partial \dot{x}} = 2A\dot{x} + B\dot{y}, \qquad \dot{H}_2 = \frac{\partial H}{\partial \dot{y}} = B\dot{x} + 2C\dot{y}, \qquad \dot{H}_{11} = \frac{\partial^2 H}{\partial (\dot{x})^2} = 2A$$

$$\dot{H}_{22} = \frac{\partial^2 H}{\partial (\dot{y})^2} = 2C, \qquad \dot{H}_{12} = \frac{\partial^2 H}{\partial \dot{x}\partial \dot{y}} = B = \dot{H}_{21},$$

$$H_1 = \frac{\partial H}{\partial x} = A_1(\dot{x})^2 + B_1(\dot{x})(\dot{y}) + C_1(\dot{y})^2, \quad H_2 = \frac{\partial H}{\partial y} = A_2(\dot{x})^2 + B_2(\dot{x})(\dot{y}) + C_2(\dot{y})^2,$$

$$H_{12} = \frac{\partial \dot{H}_1}{\partial y} = 2A_2(\dot{x}) + B_2(\dot{y}), \quad H_{21} = \frac{\partial \dot{H}_2}{\partial x} = B_1(\dot{x}) + 2C_1(\dot{y})$$

where $A_1 = \frac{\partial A}{\partial x}$, $B_1 = \frac{\partial B}{\partial x}$, $C_1 = \frac{\partial C}{\partial x}$, $A_2 = \frac{\partial A}{\partial y}$, $B_2 = \frac{\partial B}{\partial y}$, and $C_2 = \frac{\partial C}{\partial y}$.

Since $G^i_j = \dot{\partial}_j G^i$ and $G^i_{jk} = \dot{\partial}_j \dot{\partial}_k G^i$, from (2.5), we have

$$(2.13)\quad \begin{cases} 6HG^1_1 = 4a(\dot{x})^3 + 3b(\dot{x})^2\dot{y} + 2c(\dot{x})(\dot{y})^2 + d(\dot{y})^3 - 6\dot{H}_1 G^1 \\ 6HG^1_{11} = 12a(\dot{x})^2 + 6b(\dot{x})\dot{y} + 2c(\dot{y})^2 - 6\dot{H}_{11}G^1 - 12\dot{H}_1 G^1_1 \\ 6HG^1_{12} = 3b(\dot{x})^2 + 4c(\dot{x})\dot{y} + 3d(\dot{y})^2 - 6\dot{H}_{12}G^1 - 6\dot{H}_1 G^1_2 - 6\dot{H}_2 G^1_2 \\ 6HG^1_2 = b(\dot{x})^3 + 2c(\dot{x})^2\dot{y} + 3d(\dot{x})(\dot{y})^2 + 4e(\dot{y})^3 - 6\dot{H}_2 G^1 \\ 6HG^1_{21} = 3b(\dot{x})^2 + 4c(\dot{x})\dot{y} + 3d(\dot{y})^2 - 6\dot{H}_{21}G^1 - 6\dot{H}_2 G^1_1 - 6\dot{H}_1 G^1_2 \\ 6HG^1_{22} = 2c(\dot{x})^2 + 6d(\dot{x})\dot{y} + 12e(\dot{y})^2 - 6\dot{H}_{22}G^1 - 12\dot{H}_2 G^1_2 \end{cases}$$

Also from (2.6), we have

$$(2.14)\quad \begin{cases} 6HG^2_1 = 4f(\dot{x})^3 + 3g(\dot{x})^2\dot{y} + 2h(\dot{x})(\dot{y})^2 + k(\dot{y})^3 - 6\dot{H}_1 G^2 \\ 6HG^2_2 = g(\dot{x})^3 + 2h(\dot{x})^2\dot{y} + 3k(\dot{x})(\dot{y})^2 + 4l(\dot{y})^3 - 6\dot{H}_2 G^2 \end{cases}$$

From (2.13), we have

$$(2.15\quad \begin{cases} 6H\frac{\partial G^1_1}{\partial y} = 4a_2(\dot{x})^3 + 3b_2(\dot{x})^2(\dot{y}) + 2c_2(\dot{x})(\dot{y})^2 + d_2(\dot{y})^3 \\ -6H_{11}G^1 - 6\dot{H}_1 D_2 - 6H_2 G^1_1 \\ 6H\frac{\partial G^1_2}{\partial y} = b_1(\dot{x})^3 + 2c_1(\dot{x})^2(\dot{y}) + 3d_1(\dot{x})(\dot{y})^2 + 4e_1(\dot{y})^3 \\ -6H_{21}G^1 - 6\dot{H}_2 D_1 - 6H_1 G^1_2 \end{cases}$$

where $a_2 = \partial_2 a$, $b_2 = \partial_2 b$, $c_2 = \partial_2 c$, $d_2 = \partial_2 d$, $D_2 = \partial_2 G^1$, $b_1 = \partial_1 b$, $d_1 = \partial_1 d$, $e_1 = \partial_1 e$, $D_1 = \partial_1 G^1$.

In view of equations (2.4), (2.12), (2.13), (2.14), (2.15), the equation (2.1)' reduces to

$$R^1_{12} = \frac{X}{36H^2} \tag{2.16}$$

where

$$\begin{aligned}
X = {} & \{8fc - 3bg + 24a_2A - 6b_1A\}(\dot x)^5 + \{3b^2 - 8bc + 2gc + 24fd - 6bh + 24a_2B + 18b_2A - \\
& 6b_1B - 12c_1A\}(\dot x)^4(\dot y) + \{2bc - 24ad - 4hc + 15gd + 48fe - 9bk + 24a_2C + 18b_2B + 12c_2A \\
& -6_1C - 12c_1B - 18d_1A\}(\dot x)^3(\dot y)^2 + \{6hd + 36ge - 6bd - 10kc - 12bl - 48ae + 6d_2A + 12c_2B \\
& +18b_2C - 12c_1C - 18d_1B - 24e_1A\}(\dot x)^2(\dot y)^3 + \{24eh - 2cd - 3kd - 24be - 16cl + 12c_2C + 6d_2B \\
& -18d_1C - 24e_1B\}(\dot x)(\dot y)^4 + \{3d^2 + 12kl - 16ce - 12dl + 6d_2C - 24e_1C\}(\dot y)^5 - \{24a(\dot H_{21}G^1 + \\
& \dot H_2G^1_1 + \dot H_1G^1_2) + 24f(\dot H_{22}G^1 + 2\dot H_2G^1_2) - 6b(\dot H_{11}G^1 + 2\dot H_1G^1_1) - 6g(\dot H_{12}G^1 + \dot H_1G^1_2 \\
& +\dot H_2G^1_1)\}(\dot x)^3 - \{18b(\dot H_{21}G^1 + \dot H_2G^1_1 + \dot H_1G^1_2) + 18g(\dot H_{22}G^1 + 2\dot H_2G^1_2) - 12c(\dot H_{11}G^1 + \\
& 2\dot H_1G^1_1) - 12h(\dot H_{12}G^1 + \dot H_1G^1_2 + \dot H_2G^1_1)\}(\dot x)^2(\dot y) - \{12c(\dot H_{21}G^1 + \dot H_2G^1_1 + \dot H_1G^1_2) + 12h(\dot H_{22} \\
& G^1 + 2\dot H_2G^1_2) - 18d(\dot H_{11}G^1 + 2\dot H_1G^1_1) - 18k(\dot H_{12}G^1 + \dot H_1G^1_2 + \dot H_2G^1_1)\}(\dot x)(\dot y)^2 - \{6d(\dot H_{21}G^1 \\
& +\dot H_2G^1_1 + \dot H_1G^1_2) + 6k(\dot H_{22}G^1 + 2\dot H_2G^1_2) - 24e(\dot H_{11}G^1 + 2\dot H_1G^1_1) - 24l(\dot H_{12}G^1 + \dot H_1G^1_2 + \\
& \dot H_2G^1_1)\}(\dot y)^3 + \{18b\dot H_1G^1 + 12c\dot H_1G^2 - 72a\dot H_2G^1 - 18b\dot H_2G^2 + 36A(H_{12}G^1 - H_{21}G^1 + \dot H_1D_2 \\
& +H_2G^1_1 - \dot H_2D_1 - H_1G^1_2)\}(\dot x)^2 - \{24c\dot H_1G^1 + 36d\dot H_1G^2 - 36b\dot H_2G^1 - 24c\dot H_2G^2 + 36B(H_{12}G^1 \\
& -H_{21}G^1 + \dot H_1D_2 + H_2G^1_1 - \dot H_2D_1 - H_1G^1_2)\}(\dot x)(\dot y) - \{18d\dot H_1G^1 + 72e\dot H_1G^2 - -24c\dot H_2G^1 \\
& -18d\dot H_2G^2 + 36C(H_{12}G^1 - H_{21}G^1 + \dot H_1D_2 + H_2G^1_1 - \dot H_2D_1 - H_1G^1_2)\}(\dot y)^2 + 36(\dot H_{21}G^1 \\
& +\dot H_2G^1_1 + \dot H_1G^1_2)(\dot H_1G^1 - \dot H_2G^2) + 36\dot H_1G^2(\dot H_{22}G^1 + 2\dot H_2G^1_2) - 36\dot H_2G^1(\dot H_{11}G^1 + 2\dot H_2G^1_1)
\end{aligned}$$

Now in view of (2.11) and (2.16), we have

$$R = -\frac{X}{36H^2 l_2(l_1\dot x + l_2\dot y)} \tag{2.17}$$

where l_1 and l_2 are given by (2.9).

Theorem 1: *The (h)-scalar curvature R of a two-dimensional Finsler space with a cubic metric is given by* (2.17).

The expression of R given in (2.17) is very lengthy. Due to this fact it will be very difficult to study any more properties of two-dimensional Finsler spaces with cubic metric. Therefore in the next article we shall consider special cases of cubic Finsler metric.

3. Two-dimensional special cubic Finsler metric

Now, we consider symmetric metric

$$L^3 = c_0(\dot{x}^3 + \dot{y}^3) + 3c_1(\dot{x}^2\dot{y} + \dot{x}\dot{y}^2)$$

where we put $c_0 = c_3$, $c_1 = c_2$.

In order to simplify it a bit more, we take

Case 1: $c_1 = 0$, $L^3 = c_0(\dot{x}^3 + \dot{y}^3)$, $H = c_0^2\dot{x}\dot{y}$

Putting these values in equation (2.17), we get

$$(3.1)\qquad \begin{cases} R = \dfrac{L}{48c_0^3(\dot{y})^2}[2\{c_{01}c_{02} - 2c_0c_{012}\}\dot{x} + \{13c_{02}^2 - 12c_0c_{022}\}\dot{y} \\ +\{13c_{01}^2 - 12c_0c_{011}\}\dfrac{(\dot{y})^2}{\dot{x}} + \{2c_{01}c_{02} - 4c_0c_{012}\}\dfrac{(\dot{y})^3}{(\dot{x})^2} + c_{01}^2\dfrac{(\dot{y})^5}{(\dot{x})^4} + c_{02}^2\dfrac{(\dot{y})^3}{(\dot{x})^2} \end{cases}$$

where $\dfrac{\partial c_{ab}}{\partial x} = c_{ab1}$, $\dfrac{\partial c_{ab}}{\partial y} = c_{ab2}$, a = 0, 1, 2, 3 and b = 1, 2.

Further

(A) For $c_0 = x$, the metric L is given by $L^3 = x((\dot{x})^3 + (\dot{y})^3)$ and the (h)-scalar curvature R from (3.1) reduces to

$$(3.2)\qquad R^3 = R_x(t) = \frac{(t+1)(t+13)^3}{(48)^3 x^8},$$

where $t = \dfrac{(\dot{y})^3}{(\dot{x})^3}$.

Theorem 2: *The (h)-scalar curvature R of a two-dimensional Finsler space with cubic metric* $L^3 = x((\dot{x})^3 + (\dot{y})^3)$ *is given by* (3.2).

In particular, at x = 1, we consider the indicatrix of the metric $L^3 = x((\dot{x})^3 + (\dot{y})^3)$. Fig.1 shows the indicatrix curve in the orthonormal co-ordinates $(\dot{\bar{x}}, \dot{\bar{y}})$, which is obtained from $(\dot{x}, \dot{y})$ by -45^0 rotation. Indicatrix curve is symmetric with respect to $\dot{\bar{y}}$-axis and $\dot{\bar{x}}$-axis is an asymptote to the indicatrix. Variation of indicatrix as a point P moves along the indicatrix is shown in Fig. 2 (M. Matsumoto and K. Okubo[7], page-102, Fig. 2). Further, variation of scalar curvature which is given by (3.2) is shown in Fig. 3 at x = 1. As a point P moves $A \to B \to C \to D$ along the indicatrix curve as shown in Fig. 1, the variation of scalar curvature has also been shown in Fig. 3 with the point P ($A \to B \to C \to D$).

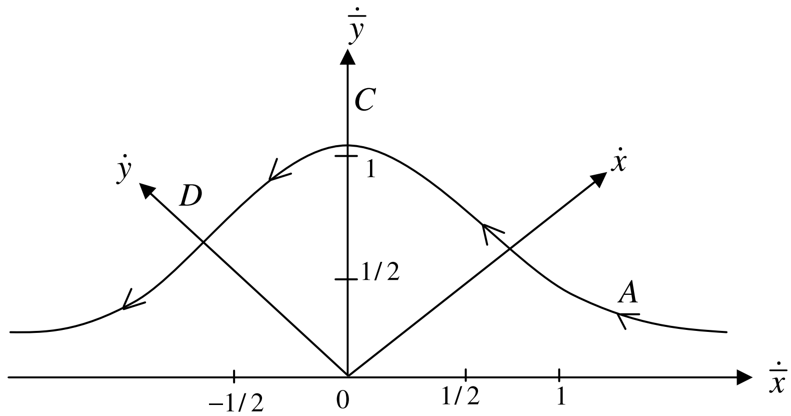

Fig. 1

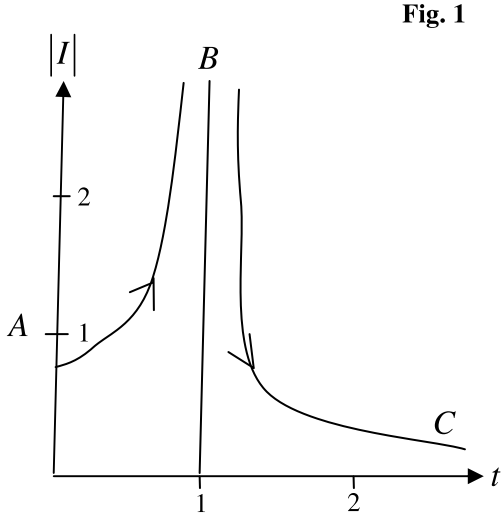

Fig. 2

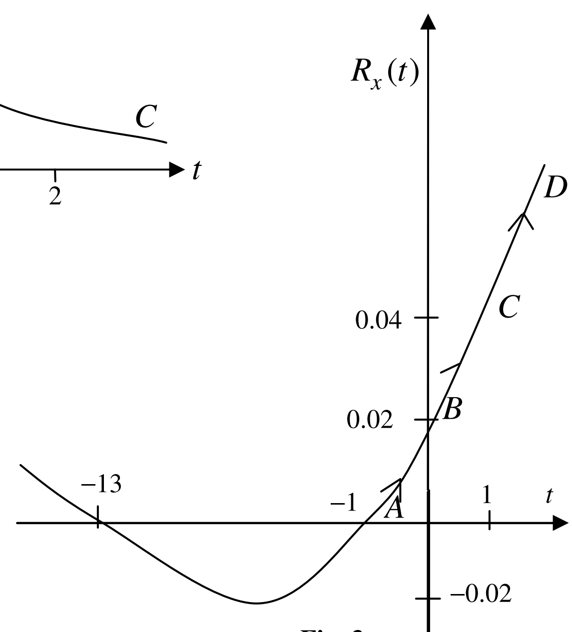

Fig. 3

Analytical analysis is given below:

$$t=-1,\Rightarrow \frac{(\dot{y})^3}{(\dot{x})^3}=-1,\Rightarrow \dot{y}=-\dot{x},\Rightarrow \dot{x}+\dot{y}=0,\Rightarrow \dot{\bar{y}}=0$$, which corresponds to *A*.

$$t=0,\Rightarrow (\dot{y})^3=0,\Rightarrow \dot{\bar{x}}=\dot{\bar{y}}$$, which corresponds to *B*.

$$t=1,\Rightarrow \frac{(\dot{y})^3}{(\dot{x})^3}=1,\Rightarrow \dot{y}=\dot{x},\Rightarrow \dot{x}-\dot{y}=0,\Rightarrow \dot{\bar{x}}=0$$, which corresponds to *C*.

$$t\to\infty,\Rightarrow (\dot{x})=0,\Rightarrow \dot{\bar{y}}=-\dot{\bar{x}}$$, which corresponds to *D*.

From the equation (3.2) it is obvious that $R_x(t)<0$ when $-13<t<-1$ and $R_x(t)=0$ for t = -13 and t = -1. $R_x(t)>0$ when $t<-13$ or $t>-1$. $R_x(t)\to\infty$ when $t\to\pm\infty$. $R_x(0)\approx\frac{0.0199}{x^8}$ and $R_x(0)_{\min.}\approx-\frac{0.0199}{x^8}$ at $t=-4$.

In Fig. 2, $t=\frac{\dot{y}}{\dot{x}}$ and in Fig. 3, $t=\frac{(\dot{y})^3}{(\dot{x})^3}$.

(B): For $c_0=y$, the metric *L* becomes $L^3=y((\dot{x})^3+(\dot{y})^3)$ and the (h)-scalar curvature *R* from (3.1) reduces to

$$R^3=R_y(t)=\frac{(\frac{1}{t}+1)(\frac{1}{t}+13)^3}{(48)^3y^8}. \tag{3.3}$$

Theorem 3: *The (h)-scalar curvature R of a two-dimensional Finsler space with cubic metric* $L^3=y((\dot{x})^3+(\dot{y})^3)$ *is given by* (3.3).

In particular, at *y* = 1, we consider the indicatrix of the metric $L^3=y((\dot{x})^3+(\dot{y})^3)$. Fig. 1 shows the indicatrix curve in the orthonormal co-ordinates $(\dot{\bar{x}},\dot{\bar{y}})$, which is obtained from $(\dot{x},\dot{y})$ by -45^0 rotation. Variation of scalar curvature which is given by (3.3), is shown in Fig. 4 for *y* = 1 graphically. As a point P moves $A\to B\to C\to D$ along the indicatrix curve as shown in Fig. 1. The variation of scalar curvature has also been shown in Fig. 4 with the point P ($A\to B\to C\to D$). Analytical analysis is given below:

$$t=-1,\Rightarrow \frac{(\dot{y})^3}{(\dot{x})^3}=-1,\Rightarrow \dot{y}=-\dot{x},\Rightarrow \dot{x}+\dot{y}=0,\Rightarrow \dot{\bar{y}}=0$$, which corresponds to *A*.

$$t=0,\Rightarrow (\dot{y})^3=0,\Rightarrow \dot{\bar{x}}=\dot{\bar{y}},\Rightarrow R_y(t)\to\infty$$, which corresponds to *B*.

$$t=1,\Rightarrow \frac{(\dot{y})^3}{(\dot{x})^3}=1,\Rightarrow \dot{y}=\dot{x},\Rightarrow \dot{x}-\dot{y}=0,\Rightarrow \dot{\bar{x}}=0$$, which corresponds to *C*.

$$t\to\infty,\Rightarrow (\dot{x})=0,\Rightarrow \dot{\bar{y}}=-\dot{\bar{x}}$$, which corresponds to *D*.

From the equation (3.3) it is obvious that $R_y(t) < 0$ when $-13 < 1/t < -1$ and $R_y(t) = 0$ for 1/ t = -13 and t = -1. $R_y(t) > 0$ when $\frac{1}{t} < -13$ or $t > -1$. $R_y(t) \to \infty$ when $\frac{1}{t} \to \pm\infty$.

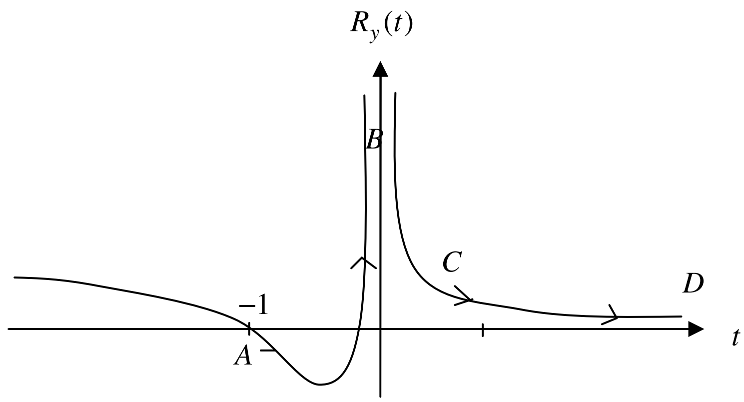

Fig. 4

(C): For $c_0 =$ constant, the metric L becomes, $L^3 = \text{constant}\,((\dot{x})^3 + (\dot{y})^3)$, and the (h)-scalar curvature R from (3.1), reduces to $R = 0$. Thus

Theorem 4: *The (h)-scalar curvature R of a two-dimensional Finsler space with cubic metric* $L^3 = constant\,((\dot{x})^3 + (\dot{y})^3)$ *vanishes.*

Remark: When c_0 is a constant, c_{01} is zero and hence the expression for R in (3.1) vanishes. On the other hand when c_{01} is calculated for x = 1 it comes out to be one and the value of scalar curvature R is different from zero. This case is different from the case when in the metric c_0 is taken as a constant.

Case 2:- $c_0 = 0$, $\quad L^3 = 3c_1((\dot{x})^2(\dot{y}) + (\dot{x})(\dot{y})^2)$, $\quad H = -c_1^2((\dot{x})^2 + \dot{x}\dot{y} + (\dot{y})^2)$,

Putting these values in equation (2.17) we get

$$R = -\frac{X_1}{36H^2 l_2(l_1\dot{x} + l_2\dot{y})} \tag{3.4}$$

where

$$\begin{aligned}
X_1 = {} & \{18c_1^2(2M_2 - M_1)\}(\dot{x})^5 + \{18c_1^2(c_{11}^2 - 5M_2 + 2M_1)\}(\dot{x})^4(\dot{y}) + \{9c_1^2(3c_{11}^2 - 12M_2 \\
& -10M_1)\}(\dot{x})^3(\dot{y})^2 - \{18c_1^2(4c_{11}^2 - 3M_2 + 2M_1)\}(\dot{x})^2(\dot{y})^3 - \{18c_1^2(2c_{11}^2 - 3M_2)\}(\dot{x})(\dot{y})^4 \\
& \{9c_1^2(3c_{11}^2 - 2M_2)\}(\dot{y})^5 - \{24a(\dot{H}_{21}G^1 + \dot{H}_2G_1^1 + \dot{H}_1G_2^1) - 6b(\dot{H}_{11}G^1 + 2\dot{H}_1G_1^1)\}(\dot{x})^3 \\
& -\{18b(\dot{H}_{21}G^1 + \dot{H}_2G_1^1 + \dot{H}_1G_2^1) - 12c(\dot{H}_{11}G^1 + 2\dot{H}_1G_1^1) - 12h(\dot{H}_{12}G^1 + \dot{H}_1G_1^1 + \\
& \dot{H}_2G_1^1)\}(\dot{x})^2(\dot{y}) - \{12c(\dot{H}_{21}G^1 + \dot{H}_2G_1^1 + \dot{H}_1G_2^1) + 12h(\dot{H}_{22}G^1 + 2\dot{H}_2G_2^1) - 18d(\dot{H}_{11}G^1
\end{aligned}$$

$$2\dot H_1 G_1^1) - 18k(\dot H_{12}G^1 + \dot H_1 G_2^1 + \dot H_2 G_1^1)\}(\dot x)(\dot y)^2 - \{6d(\dot H_{21}G^1 + \dot H_2 G_1^1 + \dot H_1 G_2^1) +$$
$$6k(\dot H_{22}G^1 + 2\dot H_2 G_2^1) - 24l(\dot H_{12}G^1 + \dot H_1 G_2^1 + \dot H_2 G_1^1)\}(\dot y)^3 + \{18b\dot H_1 G^1 + 12c\dot H_1 G^2$$
$$-72a\dot H_2 G^1 - 18b\dot H_2 G^2 + 36A(H_{12}G^1 - H_{21}G^1 + \dot H_1 D_2 + H_2 G_1^1 - \dot H_2 D_1 - H_1 G_2^1)\}(\dot x)^2$$
$$-\{24c\dot H_1 G^1 + 36d\dot H_1 G^2 - 36b\dot H_2 G^1 - 24c\dot H_2 G^2 + 36B(H_{12}G^1 - H_{21}G^1 + \dot H_1 D_2$$
$$+H_2 G_1^1 - \dot H_2 D_1 - H_1 G_2^1)\}(\dot x)(\dot y) - \{18d\dot H_1 G^1 + 72e\dot H_1 G^2 - -24c\dot H_2 G^1 - 18d\dot H_2 G^2$$
$$+36C(H_{12}G^1 - H_{21}G^1 + \dot H_1 D_2 + H_2 G_1^1 - \dot H_2 D_1 - H_1 G_2^1)\}(\dot y)^2 + 36(\dot H_{21}G^1 + \dot H_2 G_1^1$$
$$+\dot H_1 G_2^1)(\dot H_1 G^1 - \dot H_2 G^2) + 36\dot H_1 G^2(\dot H_{22}G^1 + 2\dot H_2 G_2^1) - 36\dot H_2 G^1(\dot H_{11}G^1 + 2\dot H_2 G_1^1)$$

$$\begin{cases} L^5 l_1 = -2c_1^2(\dot x)^3(\dot y)^2 - 3c_1^2(\dot x)^2(\dot y)^3 + c_1^2(\dot x)(\dot y)^4 \\ L^5 l_2 = -c_1^2(\dot x)^4(\dot y) - 3c_1^2(\dot x)^3(\dot y)^2 - c_1^2(\dot x)^2(\dot y)^3 \end{cases},$$

$$(3H)(2G^1) = -\frac{3}{2}c_1 c_{11}[(\dot x)^4 + 2(\dot x)^3(\dot y) + (\dot x)^2(\dot y)^2 - 2(\dot x)(\dot y)^3] \text{ and}$$

$$(3H)(2G^2) = -\frac{3}{2}c_1 c_{11}[(\dot x)^2(\dot y)^2 + 2(\dot x)(\dot y)^3 + (\dot y)^4],$$

$$2a = b = 2c = -d = -3c_1 c_{11} = 2h = k = 2l,$$

$$M_1 = \partial_1 c_1 c_{11}, \qquad M_2 = \partial_2 c_1 c_{11}.$$

Theorem 5: *The (h)-scalar curvature R of a two-dimensional Finsler space with cubic metric* $L^3 = 3c_1((\dot x)^2(\dot y) + (\dot x)(\dot y)^2)$ *is given by* (3.4).

References

1. M. Matsumoto and S. Numata, On Finsler spaces with a cubic metric, *Tensor, N. S.,* **33** (1979).
2. V. K. Kropina, Projective two-dimensional Finsler spaces with special metric, (Russian), *Trudy sem. Vektor. Tenzor. Anal.*, **11** (1962) 277-292.
3. V. V. Wagner, Two-dimensional space with the metric defined by a cubic differential form, (Russian and English), *Abh. Tschern, Staatuniv, Saratow*, **1** (1938) 29-40.
4. V. V. Wagner, On generalized Berwald spaces, *C. R. Dokl. Acad. Sci. URSS, N.S.*, **39** (1943) 3-5.
5. J. M. Wegener, Untersuchung der zwei- und dreidimensionalen Finslershen Raume mit der Grundform $L^3 = a_{ikl}\,\dot x^i \dot x^k \dot x^l$, *Akad. Wetensch Proc.*, **38** (1935) 949-955.
6. J. M. Wegener, Untersuchung uber Finslersche Raume, *Lotos Prag*, **84** (1936) 4-7.
7. M. Matsumoto and K. Okubo, Theory of Finsler spaces with m-th Root metric: Connections and Main Scalars, *Tensor, N.S.*, **56** (1995) 93-104.
8. M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Saikawa, Otsu, Japan, 1986.