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$GS\hat{G}$-Homeomorphism and $\hat{G}GS$-Homeomorphism in Topological Spaces

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Abstract: In the present paper we introduce two new types of mappings called $gs\hat{g}$-homeomorphism and $\hat{g}gs$-homeomorphism and then show that one of these mappings has a group structure. Further we investigate some properties of these two homeomorphisms.

Keywords: Homeomorphism; $\hat{g}gs$-homeomorphism; $gs\hat{g}$-homeomorphism.

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1. Introduction

Levine[1] generalized the concept of closed sets to generalized closed sets in 1970. Bhattacharya and Lahiri[2] generalized the concept of closed sets to semi-generalized closed sets with the help of semi-open sets in 1987 and obtained various topological properties. Arya and Nour[3] defined generalized semi-open sets with the help of semi-openness and used them to obtain some characterizations of s-normal spaces in 1990. In 1995, Devi, Balachandran and Maki[4] defined two new classes of maps called semi-generalized homeomorphisms and generalized semi-homeomorphisms. They also defined two new classes of maps called sgc-homeomorphisms and gsc-homeomorphisms. In 2007, Ahmed and Narli[5] defined two new classes of maps called gsg-homeomorphisms and sgs-homeomorphisms. Garg, Chauhan and Agarwal[6] introduced two new classes of maps namely $gs\psi$-homeomorphisms and ψgs-homeomorphisms in 2007. Garg et al.[7] again in 2007, introduced two new classes of maps called $sg\psi$-homeomorphisms and ψsg-homeomorphisms. In this paper we introduce two new classes of maps called $gs\hat{g}$-homeomorphisms and $\hat{g}gs$-homeomorphisms and study some of their properties.

Throughout the present paper, (X, τ) and (Y, σ) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a topological space (X, τ) the cl(A), int(A) and A^C denote the closure of A, the interior of A and the complement of A in X respectively.

2. Preliminaries

In this section we recall the following definitions.

Definition 2.1: A subset A of a topological space (X, τ) is called semi-open[8] (resp. semi-closed) if A $\subseteq$ cl(int(A)) (resp. int(cl(A)) $\subseteq$ A). Every closed (resp. open) set is semi-closed (resp. semi-open).

Definition 2.2: A subset A of a topological space (X, τ) is called semi-generalized closed[2] (briefly sg-closed) if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open. The complement of sg-closed set is called sg-open set. Every semi-closed set is sg-closed set. The family of all sg-closed sets of any topological space (X, τ) is denoted by sgc (X, τ).

Definition 2.3: A subset A of a topological space (X, τ) is called generalized semi closed[3] (briefly gs-closed) if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open. The complement of gs-closed set is called gs-open set. Every closed (semi-closed, g-closed and sg-closed) set is gs-closed set. The family of all gs-closed sets of any topological space (X, τ) is denoted by gsc (X, τ).

Definition 2.4: A subset A of a topological space (X, τ) is called ψ-closed[9] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is sg-open. The complement of ψ-closed set is called ψ-open set. Every closed (semi-closed) set is ψ-closed set and every ψ-closed set is sg-closed (gs-closed) set. The family of all ψ-closed sets of any topological space (X, τ) is denoted by ψc (X, τ).

Definition 2.5: A subset A of a topological space (X, τ) is called $\hat{g}$-closed[10] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open. The complement of $\hat{g}$-closed set is called $\hat{g}$-open set. Every closed set is $\hat{g}$-closed set and every $\hat{g}$-closed set is ψ-closed (sg-closed, gs-closed, g-closed) set. The family of all $\hat{g}$-closed sets of any topological space (X, τ) is denoted by $\hat{g}$c (X, τ).

Definition 2.6: A map f : (X, τ) $\rightarrow$ (Y, σ) is called semi-closed map[8] (resp. sg-closed map[11], gs-closed map[11], ψ-closed map[12], $\hat{g}$-closed map[13]) if the image of each closed set in (X, τ) is semi-closed set (resp. sg-closed set, gs-closed set, ψ-closed set, $\hat{g}$-closed set) in (Y, σ). Every closed map is semi-closed map. Every semi-closed map is ψ-closed map. Every ψ-closed map is sg-closed map, every sg-closed map is gs-closed map and every $\hat{g}$-closed map is ψ-closed map (sg-closed map, gs-closed map, g-closed map).

Definition 2.7: A map f : (X, τ) $\rightarrow$ (Y, σ) is called $\hat{g}$-continuous[13] (resp. ψ-continuous[9], g-continuous[14], gs-continuous[4], ψ-irresolute[9], sg-irresolute[15], gs-irresolute[4], gsg-irresolute[5], sgs-irresolute[5], gsψ-irresolute[6], ψgs-irresolute[6], sgψ-irresolute[7], ψsg-irresolute[7], $\hat{g}$-irresolute[13]) if the inverse image of every closed (resp. closed, closed, closed, ψ-closed, sg-closed, gs-closed, gs-closed, sg-closed, gs-closed, ψ-closed, sg-closed, ψ-closed, $\hat{g}$-closed) set in (Y, σ) is $\hat{g}$-closed (resp. ψ-closed, sg-closed, gs-closed, ψ-closed, sg-closed, gs-closed, sg-closed, gs-closed, ψ-closed, gs-closed, ψ-closed, sg-closed, $\hat{g}$-closed) set in (X, τ).

Definition 2.8: A bijective map f : (X, τ) $\rightarrow$ (Y, σ) is called

(i) semi-homeomorphism (B)[5] (briefly s.h. (B)) if f is continuous and semi-open map.

(ii) semi-homeomorphism (C.H.)[16] (briefly s.h. (C.H.)) if f is irresolute, presemi-open (i.e. f(U) is semi-open for every semi-open set U of (X, τ)).

(iii) ψ-homeomorphism[12] if f is both ψ-continuous and ψ-open map

(iv) $\hat{g}$ -homeomorphism[13] if f is both $\hat{g}$ -continuous and $\hat{g}$ -open map

(v) semi-generalized homeomorphism[4] (briefly sg-homeomorphism) if f is both sg-continuous and sg-open.

(vi) generalized semi-homeomorphism[4] (briefly gs-homeomorphism) if f is both gs-continuous and gs-open.

(vii) sgc-homeomorphism[4] (resp. gsc-homeomorphism[4], ψ*-homeomorphism[12], $\hat{g}$ c-homeomorphism[13], gsg-homeomorphism[5], sgs-homeomorphism[5], gsψ-homeomorphism[6], ψgs-homeomorphism[6], sgψ-homeomorphism[7], ψsg-homeomorphism[7]) if *f* and f^{-1} are sg-irresolute (resp. gs-irresolute, ψ-irresolute, $\hat{g}$ -irresolute, gsg-irresolute, sgs-irresolute, gsψ-irresolute, ψgs-irresolute, sgψ-irresolute, ψsg-irresolte).

Definition 2.9: A space (X, τ) is called $T_{1/2}$-space[1] (resp. T_b-space[11], $\hat{T}_b$ -space[13]) if every g-closed set (resp. gs-closed set, gs-closed set) is closed set (resp. closed set, $\hat{g}$ -closed set).

Proposition 2.1: *In a $T_{1/2}$-space every gs-closed set is semi-closed set* [11].

3. $GS\hat{G}$ -Homeomorphism

In this section we introduce $gs\hat{g}$ -homeomorphisms and then investigate the group structure of the set of all $gs\hat{g}$ -homeomorphisms.

Definition 3.1*:* A map *f* : (X, τ) → (Y, σ) is called a $gs\hat{g}$ -irresolute map if the set f^{-1} (A) is $\hat{g}$ -closed in (X, τ) for every gs-closed set A of (Y, σ).

Definition 3.2: A bijection *f* : (X, τ) → (Y, σ) is called a $gs\hat{g}$ -homeomorphism if the function *f* and the inverse function f^{-1} are both $gs\hat{g}$ -irresolute maps. If there exists a $gs\hat{g}$ -homeomorphism from X to Y, then the spaces (X, τ) and (Y, σ) are called $gs\hat{g}$ -homeomorphic. The family of all $gs\hat{g}$ -homeomorphisms of any topological space (X, τ) is denoted by $gs\hat{g}$ h(X, τ).

Remark 3.1: The following examples show that the concepts of homeomorphism and $gs\hat{g}$ -homeomorphism are independent of each other.

Example 3.1: Let X = {a, b, c} and τ = {ϕ, {a}, {b}, {a, b}, X}. Define *f* : (X, τ) → (X, τ) as identity mapping then f is a homeomorphism but not a $gs\hat{g}$ -homeomorphism.

Example 3.2: Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b ,c}, X} and σ = {ϕ, Y}. Define *f* : (X, τ) → (Y, σ) as identity mapping, then *f* is a $gs\hat{g}$ -homeomorphism but not homeomorphism.

Proposition 3.1: Every gs$\hat{g}$-homeomorphism is (i) sgc-homeomorphism (ii) sgs-homeomorphism (iii) gsg-homeomorphism (iv) gsc-homeomorphism (v) sgψ-homeomorphism (iv) ψsg-homeomorphism (vii) gsψ-homeomorphism (viii) ψgs-homeomorphism (ix) ψ*-homeomorphism (x) $\hat{g}$c- homeomorphism.

The converse of the above proposition is not true. It can be seen from the following examples.

Example 3.3: Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, X} and σ = {ϕ, {a}, {a, b}, {a, c}, Y}. Define *f* : (X, τ) $\rightarrow$ (Y, σ) by identity mapping then *f* is sgc-homeomorphism but not gs$\hat{g}$ -homeomorphism.

Example 3.4: Let X = Y = {a, b, c}, τ = {ϕ, {a}, X} and σ = {ϕ, {a}, {a, b}, Y}. Define f :(X, τ) $\rightarrow$ (Y, σ) by identity mapping then *f* is sgs-homeomorphism but not gs$\hat{g}$ -homeomorphism.

Example 3.5: Let X = {a, b, c} and τ = {ϕ, {a}, {c}, {a, c}, X}. Define *f* : (X, τ) $\rightarrow$ (Y, σ) by identity mapping then *f* is gsg-homeomorphism but not gs$\hat{g}$-homeomorphism, for *f* and f^{-1} are not gs$\hat{g}$-irresolute maps.

Example 3.6: Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, X} and σ = {ϕ, {a}, {b}, {a, b}, {a, c}, Y}. Define *f*: (X, τ) $\rightarrow$ (Y, σ) by identity mapping then *f* is gsc-homeomorphism but not gs$\hat{g}$-homeomorphism.

Example 3.7: In example 3.3, map *f* is sgψ-homeomorphism but not gs$\hat{g}$-homeomorphism.

Example 3.8: Let X = Y = {a, b, c}, τ = {ϕ, X} and σ = {ϕ, {a, b}, Y}. Define *f* : (X, τ) $\rightarrow$ (Y, σ) by identity mapping then *f* is ψsg-homeomorphism but not gs$\hat{g}$-homeomorphism.

Example 3.9: Let X = {a, b, c} and τ = {ϕ, {a}, {b}, {a, b}, X}. Define *f* : (X, τ) $\rightarrow$ (X, τ) by identity mapping then *f* is gsψ-homeomorphism but not gs$\hat{g}$-homeomorphism.

Example 3.10: In example 3.6, map *f* is ψgs-homeomorphism but not gs$\hat{g}$-homeomorphism.

Example 3.11: In example 3.3, map *f* is ψ*-homeomorphism but not gs$\hat{g}$-homeomorphism.

Example 3.12: Let X = Y = {a, b, c}, τ = {ϕ, {a}, {c}, {a, c}, X} and σ = {ϕ, {a, c}, Y}. Define *f* : (X, τ) $\rightarrow$ (Y, σ) by identity mapping then *f* is $\hat{g}$c-homeomorphism but not gs$\hat{g}$-homeomorphism.

Proposition 3.2: *Every gsg (sgs)-homeomorphism from* $\hat{T}_b$ *-space onto itself is* gs$\hat{g}$ *-homeomorphism.*

Proposition 3.3: *If f : (X, τ) $\rightarrow$ (Y, σ) is* gs$\hat{g}$ *-homeomorphism then every gsc ($\hat{g}$c)-homeomorphism from X to Y is $\hat{g}$c (gsc)-homeomorphism.*

Proof : Straight forward.

Theorem 3.1: *If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are $gs\hat{g}$ -homeomorphisms then their composition $gof : (X, \tau) \to (Z, \eta)$ is also $gs\hat{g}$ -homeomorphism.*

Theorem 3.2: *If $gs\hat{g}\,h(X, \tau)$ is non-empty then the set $gs\hat{g}\,h(X, \tau)$ is a group under the composition of maps.*

Proof: Define a binary operation $*$: $gs\hat{g}\,h(X, \tau) \times gs\hat{g}\,h(X, \tau) \to gs\hat{g}\,h(X, \tau)$ by $f*g$ = gof for all $f, g \in gs\hat{g}\,h(X, \tau)$ and o is the usual operation of composition of maps, then by theorem 3.1 gof $\in gs\hat{g}\,h(X, \tau)$. We know that the composition of maps is associative and the identity element I: $(X, \tau) \to (X, \tau)$ belonging to $gs\hat{g}\,h(X, \tau)$ serves as the identity element. If $f \in gs\hat{g}\,h(X, \tau)$ then $f^{-1} \in gs\hat{g}\,h(X, \tau)$ such that $fof^{-1} = I = f^{-1}of$ and so inverse exists for each element of $gs\hat{g}\,h(X, \tau)$. So $(gs\hat{g}\,h(X, \tau), o)$ is a group under the operation of composition of maps.

Theorem 3.3: *If $f: (X, \tau) \to (Y, \sigma)$ be a $gs\hat{g}$ -homeomorphism then f induces an isomorphism from the group $gs\hat{g}\,h(X, \tau)$ onto the group $gs\hat{g}\,h(Y, \sigma)$.*

Proof: Define θ_f : $gs\hat{g}\,h(X, \tau) \to gs\hat{g}\,h(Y, \sigma)$ by $\theta_f(h) = fohof^{-1}$ for every $h \in gs\hat{g}\,h((X, \tau)$. Then θ_f is a bijection. Further, for all $h_1, h_2 \in gs\hat{g}\,h\ (X, \tau)$, $\theta_f\ (h_1oh_2) = fo(h_1oh_2)of^{-1} = (foh_1of^{-1})$ o $(fo\ h_2of^{-1}) = \theta_f(h_1)$ o $\theta_f\ (h_2)$. So θ_f is a homomorphism and so it is an isomorphism induced by f.

Theorem 3.4: *$gs\hat{g}$ -homeomorphism is an equivalence relation in the collection of all topological spaces.*

Proof: Reflexivity and symmetry are immediate and transitivity followed from Theorem 3.1.

4. $\hat{G}GS$ -Homeomorphism

In this section we introduce $\hat{g}gs$ -homeomorphism and investigate its properties.

Definition 4.1: A map f: $(X, \tau) \to (Y, \sigma)$ is called $\hat{g}gs$ -irresolute map if the set $f^{-1}(A)$ is gs-closed in (X, τ) for every $\hat{g}$ -closed set A of (Y, σ).

Definition 4.2: A bijection $f : (X, \tau) \to (Y, \sigma)$ is called a $\hat{g}gs$ -homeomorphism if the function f and the inverse function f^{-1} are both $\hat{g}gs$ -irresolute maps. If there exists a $\hat{g}gs$ -homeomorphism from X to Y, then the spaces (X, τ) and (Y, σ) are called $\hat{g}gs$ -homeomorphic.

The family of all $\hat{g}gs$ -homeomorphisms of any topological space is denoted by $\hat{g}gs\,h$ (X, τ).

Proposition 4.1: Every (i) homeomorphism (ii) gc-homeomorphism (iii) sgc-homeomorphism (iv) gsc-homeomorphism (v) sgs-homeomorphism (vi) gsg-homeomorphism (vii) sgψ-homeomorphism (viii) ψsg-homeomorphism (ix) gsψ-homeomorphism (x) ψgs-homeomorphism (xi) ψ*-homeomorphism (xii) $gs\hat{g}$ -homeomorphism (xiii) $\hat{g}$ c-homeomorphism, is $\hat{g}$ gs-homeomorphism.

The following examples show that the converse of the above proposition is not true.

Example 4.1: In example 3.2, map *f* is $\hat{g}$ gs-homeomorphism but not homeomorphism.

Example 4.2: In example 3.4, map *f* is $\hat{g}$ gs-homeomorphism but not gc-homeomorphism for f^{-1} is not a g-irresolute map.

Example 4.3: In example 3.6, map *f* is $\hat{g}$ gs-homeomorphism but not sgc-homeomorphism for *f* is not a sg-irresolute map.

Example 4.4: In example 3.3, map *f* is $\hat{g}$ gs-homeomorphism but not gsc-homeomorphism for f^{-1} is not a gs-irresolute map.

Example 4.5: Let X = Y = {a, b, c,}, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$. Define f : $(X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is $\hat{g}$ gs-homeomorphism but not sgs-homeomorphism .

Example 4.6: In example 3.6, map *f* is $\hat{g}$ gs-homeomorphism but not gsg-homeomorphism.

Remark 4.1: In example 3.6, map *f* is $\hat{g}$ gs-homeomorphism but not sgψ-homeomorphism for *f* is not a sgψ-irresolute map.

Example 4.7: Let X = Y = {a, b, c,}, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, Y\}$. Define *f* : $(X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then *f* is $\hat{g}$ gs-homeomorphism but not ψsg-homeomorphism .

Example 4.8: In example 3.6, map *f* is $\hat{g}$ gs-homeomorphism but not gsψ-homeomorphism for *f* is not a gsψ-irresolute map.

Example 4.9: In example 4.5, map *f* is $\hat{g}$ gs-homeomorphism but not ψgs-homeomorphism for f^{-1} is not a ψgs-irresolute map.

Example 4.10: In example 3.2, map *f* is $\hat{g}$ gs-homeomorphism but not ψ*-homeomorphism for f^{-1} is not a ψ-irresolute map.

Example 4.11: Let X = Y = {a, b, c,}, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$. Define *f* : $(X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then *f* is $\hat{g}$gs -homeomorphism but not gs$\hat{g}$ -homeomorphism for *f* is not gs$\hat{g}$ -irresolute map.

Example 4.12: In example 4.11, map *f* is $\hat{g}$ gs-homeomorphism but not $\hat{g}$c -homeomorphism for *f* is not a $\hat{g}$ -irresolute map.

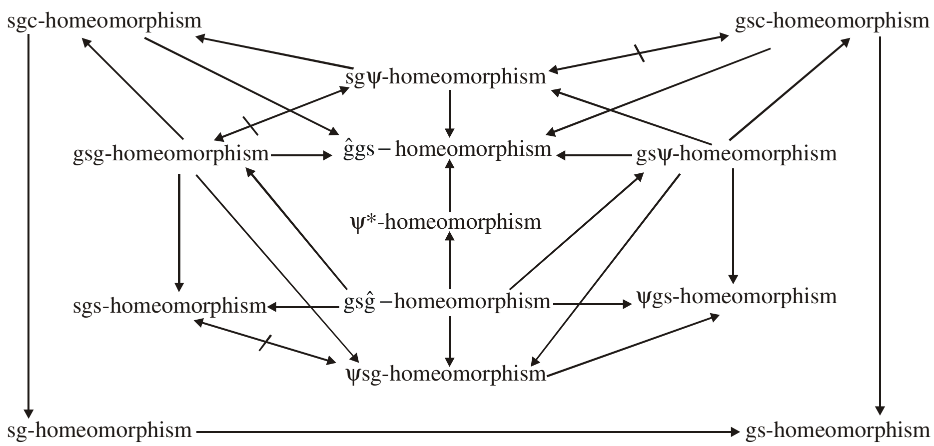
Theorem 4.1: *Every* $\hat{g}$gs *-homeomorphism from a T_b-space onto itself is a homeomorphism. So* $\hat{g}$gs *-homeomorphism is gs-homeomorphism, sgc-homeomorphism, gsc-homeomorphism, sgs-homeomorphism, gsg-homeomorphism, sgψ-homeomorphism, ψsg-homeomorphism, gsψ-homeomorphism, ψgs-homeomorphism, ψ*-homeomorphism,* gs$\hat{g}$ *-homeomorphism and* $\hat{g}$c *-homeomorphism.*

Proof: In view of the fact that in a T_b-space every gs-closed set is closed, the proof is obvious.

Theorem 4.2: *Every* $\hat{g}$gs *-homeomorphism from a* $\hat{T}_b$ *-space onto itself is a* gs$\hat{g}$ *-homeomorphism. So* $\hat{g}$gs *-homeomorphism is sgc-homeomorphism, ψ*-homeomorphism,* *gsg-homeomorphism, sgs-homeomorphism, gsc-homeomorphism, ψgs-homeomorphism, gsψ-homeomorphism, ψsg-homeomorphism, sgψ-homeomorphism, gc-homeomorphism and $\hat{g}$ c-homeomorphism.*

Proof: Since in $\hat{T}_b$ -space every gs-closed set is $\hat{g}$ -closed set so proof is obvious.

All the above discussions of Sections 3 and 4 can be summarized by the following diagram.



Where A → B (resp. A↔B) represents A implies B but not conversely
(resp. A and B are independent)

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