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Absolute Nevanlinna Summability of Conjugate Derived Fourier Series

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Abstract: In this paper the following theorem on absolute Nevanlinna summability of conjugate derived Fourier series has been proved, which generalizes various known results.

Theorem: Let $1 < p \leq \alpha$ and the function q_α satisfy the condition

$$\int_0^1 q_\delta(t)\,dt = 1$$

for $\delta > 0, p = [\delta],$ we assume

$$\frac{Q_\delta(t)}{t^{\delta - p + 1}} \in L(0,1)$$

where

$$Q_\delta(t) = \int_{1-t}^{1} q_\delta^{(p)}(x)dx$$

and $\chi(t)$ is of bounded variation in $(0, \pi)$ such that

$$\int_1^u \frac{Q(t)}{t\chi(t)}dt = O(Q(u)) \qquad \text{as } u \to \infty$$

then at $t = x$ the conjugate derived Fourier series of f is summable by the method $\left|N, q(n)\right|$.

Keywords and Phrases: Absolute Nevanlinna summability, Conjugate derived Fourier series.

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1. Definitions and Notations

Given a series $\sum u_n$, let $F(\omega)=\sum_{n<\omega} u_n$. Let $q_\delta = q_\delta(t)$ be defined for $0 \le t < 1$. The $N(q_\delta)$ transform $N(F,q_\delta)$ of F is defined by

$$N(F,q_\delta)(\omega)=\int_0^1 q_\delta(t)\,F(\omega,t)\,dt\,.$$

The series $\sum u_n$ is said to be summable by the method $N(q_\delta)$ to the sum s if

$$\lim_{\omega\to\infty} N(F,q_\delta)(\omega)=s\,.$$

It is said to be absolute summable by the method $N(q_\delta)$ and we shall write

$$\sum u_n \in \left|N(q_\delta)\right| \quad \text{if}$$

$$N(F,q_\delta)(\omega)\in BV(A,\infty)$$

For some $A \ge 0$, which is indeed equivalent to

$$\int_A^\infty \left|\sum_{n<\omega} q_\delta\left(\frac{n}{\omega}\right) n\,u_n\right| \frac{d\omega}{\omega^2} < \infty\,,$$

For the regularity we need

$$\int_0^1 q_\delta(t)dt = 1.$$

The parameter δ will be a non-negative real number. We have further two sets of restriction on q_δ; one for $0 \le \delta < 1$ and the other for $\delta \ge 1$.

In the case $0 \le \delta < 1$, $q_\delta(t)$ is increasing for $0 < t < 1$.

In the case $\delta \ge 1$, q_δ satisfies following:

$q_\delta(t)$ is decreasing for $0 < t < 1$ with $p = [\delta]$, the integral part of δ,

$$\left(\frac{d}{dt}\right)^{p-1} q_\delta(t) \in A \subset [0,1]\,.$$

$$\left[\left(\frac{d}{dt}\right)^{k} q_\delta(t)\right]_{t=1} = 0 \qquad k = 0,1,\ldots\ldots\ldots(p-1),$$

$$(-1)^p \left(\frac{d}{dt}\right)^{p} q_\delta(t) \ge 0 \quad \text{and is increasing.}$$

Also, for $\delta \geq 0, p = [\delta]$, we assume

$$\frac{Q_\delta(t)}{t^{\delta-p+1}} \in L(0,1),$$

where
$$Q_\delta(t) = \int_{1-t}^{1} q_\delta^{(p)}(x)\,dx .$$

2. Introduction

Let $f(t)$ be a continuous function of bounded variation periodic with period 2π and Lebesegue integrable over $(-\pi,\pi)$ and have a derivative $f'(x)$ at $t = x$. Let the Fourier series corresponding to $f(t)$ be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) . \tag{2.1}$$

The series

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) , \tag{2.2}$$

which is obtained by differentiating (2.1) term by term, is called the first derived series or derived Fourier series of $f(t)$.

The series conjugate to (2.2) is

$$\sum_{n=1}^{\infty} n(a_n \cos nt + b_n \sin nt) . \tag{2.3}$$

We write

$$g(t) = g(t,x) = f(x+t) - f(x-t) - 2t\,f'(x) ,$$

$$h(t) = h(t,x) = f(x+t) + f(x-t) - 2f(x)$$

$$G(t) = \int_0^t |dg(u)|$$

$$H(t) = \int_0^t |dh(u)|$$

and

$$T(x) = -\frac{1}{4\pi}\int_0^{\pi} h(t)\cos ec^2\left(\frac{t}{2}\right)dt = \lim_{\varepsilon \to 0}\left\{-\frac{1}{4\pi}\int_{\varepsilon}^{\pi} h(t)\cos ec^2\left(\frac{t}{2}\right)dt\right\} .$$

We denote the integral

$$-\frac{1}{4\pi}\int_{1/n}^{\pi} h(t)\cos ec^2\left(\frac{t}{2}\right)dt$$

by $T_n(x)$, so that, as $n \to \infty$, $T_n(x) \to T(x)$, i.e.

$$\lim_{n\to\infty} T_n(x) = T(x)\,.$$

If the function $f(x)$ is of bounded variation, then $T(x)$ exists for almost all values of x.

Generalizing the theorems of Bosanquet[1,2] and Samal[3] proved the following theorem.

Theorem A. *Let* $1 > c > 0$. *Let the function* q_c *satisfy the conditions*

$$\int_0^1 q_\delta(t)\,dt = 1$$

and $0 \le \delta < 1$, $q_\delta(t)$ *is increasing for* $0 < t < 1$ *and let* $\frac{Q_c(t)}{t^{c+1}} \in L(0,1)$. *Then*

$$\int_0^\pi t^c \left|d\phi(t)\right| < \infty = \sum \left|N(q_c)\right|.$$

In 2000, Dikshit[4] extended the above result for absolute Nevanlinna summability of Fourier series in the following form:

Theorem B. *Let* $\alpha \ge 0$ *and let the functions* q_α *satisfy the conditions*

$$\int_0^1 q_\delta(t)dt = 1$$

for $\delta \ge 0, p = [\delta]$. *We assume* $\frac{Q_\delta(t)}{t^{\delta-p+1}} \in L(0,1)$,

where $Q_\delta(t) = \int_{1-t}^{1} q_\delta^{(p)}(x)\,dx$,

with $\delta = \alpha$. *If* $\phi_\alpha(t) \in BV(0,\pi)$, *then at* $t = x$, *the Fourier series of* f *is summable by the method* $\left|N(q_\alpha)\right|$.

The object of the present paper is to extend the above theorem for absolute Nevanlinna summability of conjugate derived Fourier series.

3. Main Theorem

We shall prove the following theorem.

Theorem: *Let* $1 < p \le \alpha$ *and the function* q_α *satisfy the condition:*

(3.1) $$\int_0^1 q_\delta(t)\,dt = 1$$

for $\delta > 0,\ p = [\delta]$. We assume $\dfrac{Q_\delta(t)}{t^{\delta-p+1}} \in L(0,1)$

where

(3.2) $$Q_\delta(t) = \int_{1-t}^{1} q_\delta^{(p)}(x)\,dx$$

and $\chi(t)$ is of bounded variation in $(0,\pi)$ such that

(3.3) $$\int_1^u \frac{Q(t)}{t\chi(t)}\,dt = O(Q(u)) \quad \text{as } u \to \infty .$$

Then at $t = x$, the conjugate derived Fourier series of f is summable by the method $|N,\, q(n)|$.

Proof: Let $\Im_n^\mu(x)$ denote the sum of the first n terms of the series (2.3) at the point $t = x$, then we have

$$\Im_n^\mu(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\partial}{\partial t}\left\{\sum_1^n \sin v(t-x)\right\} N\big(F, q_\delta\big)\big(\omega\big)\,dt$$

$$= -\frac{1}{\pi}\int_0^\pi \frac{d}{dt}\left[\frac{\cos\left(\frac{t}{2}\right) - \cos\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right]\{f(x+t)+f(x-t)\}\,N(F,q_\delta)(\omega)\,dt\,,$$

$$= -\frac{1}{\pi}\int_0^\pi \left[\frac{\cos\left(\frac{t}{2}\right) - \cos\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right] N(F,q_\delta)(\omega)\,d\,h(t)\,.$$

Therefore

$$\bar{\Im}_v^\mu(x) = -\frac{1}{2\pi}\int_0^\pi\left[\cot\frac{t}{2}(1-\cos vt) + \sin vt\right] N(F,q_\delta)(\omega)\,dh(t)$$

$$-\frac{1}{2\pi}\int_0^\pi \sin vt\, N(F,q_\delta)(\omega)\,dh(t)$$

$$= -\frac{1}{2\pi}\left(N_1 + N_2 + N_3\right) \qquad \text{(say)}$$

But

$$\left|N_1\right| \le \int_0^{1/n} \left| \frac{\cos\frac{1}{2}t}{\sin\frac{1}{2}t} 2\sin^2\frac{vt}{2} \right| \left|N(F, q_\delta)(\omega)\right| \left|dh(t)\right|$$

$$\le 2^v \int_0^{1/n} \left|N(F, q_\delta)(\omega)\right| \left|dh(t)\right|$$

$$= 2^v H\left(\frac{1}{n}\right)$$

$$= o(1).$$

Now

$$-\left(\frac{1}{2\pi}\right) N_2 = -\frac{1}{2\pi}\int_{1/n}^{\pi} \cot\left(\frac{t}{2}\right)\left(1-\cos vt\right) N(F,\, q_\delta)(\omega)\, dh(t)$$

$$= -\frac{1}{2\pi}\int_{1/n}^{\pi} \cos\left(\frac{t}{2}\right) N(F,\, q_\delta)(\omega)\, dh(t)$$

$$+\frac{1}{2\pi}\int_{1/n}^{\pi} \cot\left(\frac{t}{2}\right) \cos vt\, N(F,\, q_\delta)(\omega)\, dh(t)$$

$$= -\frac{1}{2\pi}\left[\cot\left(\frac{t}{2}\right) h(t) N(F,\, q_\delta)(\omega)\right]_{1/n}^{\pi}$$

$$-\frac{1}{2\pi}\int_{1/n}^{\pi} \frac{1}{2}\operatorname{cosec}^2\left(\frac{t}{2}\right) h(t)\, N(F,\, q_\delta)(\omega)\, dt$$

$$+\frac{1}{2\pi}\int_{1/n}^{\pi} \cot\left(\frac{t}{2}\right) \cos vt\, N(F,\, q_\delta)(\omega)\, dh(t)$$

$$= \left(\frac{1}{2\pi}\right) \frac{\left(\cos\frac{1}{2n}\right)\left(\frac{1}{2n}\right) h\left(\frac{1}{n}\right) N(F, q_\delta)(\omega)}{\left(\sin\frac{1}{2n}\right)\left(\frac{1}{2n}\right)} + T_n(x)$$

$$+\frac{1}{2\pi}\int_{1/n}^{\pi} \cot\left(\frac{t}{2}\right) \cos vt\, N(F,\, q_\delta)(\omega)\, dh(t)$$

$$= o(1) + T_n(x) + \frac{1}{2\pi}\int_{1/n}^{\pi} \cot\left(\frac{t}{2}\right)\cos vt\, N(F,\, q_\delta)(\omega)\, dh(t)$$

for $\frac{h(t)}{t}$ tends to zero with t, as $f(x)$ exists.

Hence

$$\bar{\Im}_v^{\mu}(x) - T_n(x) = \frac{1}{2\pi}\int_{1/n}^{\pi} \cos\left(\frac{t}{2}\right)\cos vt\; N(F,\, q_\delta)(\omega)\, dh(t)$$

$$-\frac{1}{2\pi}\int_{0}^{\pi} \sin vt\, N(F, q_\delta)(\omega)\, dh(t) + o(1)$$

$$= \frac{1}{2\pi}\int_{1/n}^{\pi} \cot\left(\frac{t}{2}\right)\cos vt\; N(F,\, q_\delta)(\omega)\, dh(t)$$

$$-\frac{1}{2\pi}\int_{1/n}^{\pi} \sin vt\, N(F, q_\delta)(\omega)\, dh(t)$$

$$-\frac{1}{2\pi}\int_{n}^{1/n} \sin vt\, N(F, q_\delta)(\omega)\, dh(t) + o(1)$$

$$= \frac{1}{2\pi}\int_{1/n}^{\pi} \frac{\cos\left(v + \frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} N(F, q_\delta)(\omega)\, dh(t) + o(1)$$

By use of transformation

$$t_n - T_n(x) = \frac{1}{Q_\delta(t)}\sum_{k=0}^{n} q_\delta^k \left\{\bar{\Im}_{n-k}^{\mu}(x) - T_n(x)\right\} N(F, q_\delta)(\omega)$$

$$= \frac{1}{2\pi\, Q_\delta(t)}\sum_{k=0}^{n} q_\delta^k \int_{1/n}^{\pi} \frac{\cos\left(n - k + \frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} N(F, q_\delta)(\omega)\, dh(t) + o(1)$$

$$= \frac{1}{2\pi\, Q_\delta(t)}\int_{1/n}^{\pi} dh(t) \sum_{k=0}^{n} q_\delta^k \frac{\cos\left(n - k + \frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} N(F, q_\delta)(\omega) + o(1)$$

$$= \frac{1}{2\pi Q_\delta(t)} \int_{1/n}^{\rho} dh(t) \sum_{k=0}^{n} q_\delta^k \frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} N(F,q_\delta)(\omega) + o(1)$$

where ρ is small but fixed.

Now for $\frac{1}{n} \le t \le \rho$, we have

$$\sum_{k=0}^{n} q_\delta^k \frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} N(F,q_\delta)(\omega) = O\left(\frac{Q_\delta\left(\frac{1}{t}\right)}{t\chi(t)}\right)$$

by virtue of condition of theorem.

Thus

$$t_n - T_n(x) = O\left(\frac{1}{Q_\delta(t)}\right) \int_{1/n}^{\rho} |\mathrm{dh(t)}| \left| \sum_{k=0}^{n} q_\delta^k \frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right| |N(F,q_\delta)(\omega)| + o(1)$$

$$= O\left(\frac{1}{Q_\delta(t)}\right) \int_{1/n}^{\rho} |\mathrm{dh(t)}| \frac{Q_\delta\left(\frac{1}{t}\right)}{t} + o(1)$$

$$= o(1) .$$

This completes the proof of the theorem.

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