A Control Model in Ecology for the Survival of Resource-Based Industries when its Resource is Depleted by Toxicants/Pollutants and Augmented Precursors: A Qualitative Approach

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Abstract: In this paper, a control mathematical model is proposed to study the depletion of resource-based industry in a forest habitat due to the increase of both industries and pollutants/ toxicants .The toxicants are emitted into the environment by external sources and their concentration is augmented by a precursor produced by industries itself. The densities of resource biomass and resource-based industries decrease as the concentration of the toxicants increases but on applying the control measure, the resource biomass density as well as the density of industries increases. It is found that if the densities of industries and the emission rate of pollutant increase without control, the forestry resource may become extinct. In the view of this a control ecological model in this paper has been proposed using stability theory to obtain the criterion for the survival of resource-based industries. The results are illustrated with the help of a numerical example.

Keywords: Industry, Precursor, Resource biomass, Uptake phase, Control measure.

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1. Introduction

The environmental problem which society faces today is the depletion of resources such as forestry, fertile topsoil, crude oil and minerals etc. due to rise in population, Industrialization and pollution. The increasing amount of toxic elements in the environment caused by industrialization affects the structure and functions of eco-system. An example that suggests the model is that of fisheries and fish-based industries, where the pollutants are emitted from industries, situated in the nearby areas of the water bodies, affecting the regeneration and health of the fishes and therefore pollutants not only affect the resources but also affect the emitter itself (industries). [1-3]

In recent years some investigations have been conducted to study the effects of toxicants (pollutants) on resource-based industries using mathematical models[4-9] but in these studies authors did not consider the effect of the toxicants whose concentration are augmented by precursors(an intermediate product which may get converted into a toxic material in the environment harmful to the producing species as well as to other species living in the same habitat) produced by industries, which in turn has effect on itself. It is therefore necessary that we develop an environmental management system to reduce the emission rate of toxicant in the environment by using some removal mechanism. We have proposed a dynamical model for conservation of resource-based industries by controlling the emission rate of pollutant into the environment.

2. Mathematical Model

We consider a resource-based industry growing logistically in its habitat, which is being affected by a toxicant, introduced in the environment by some external sources and augmented by a precursor produced by industry itself. It is assumed that the rate of emission of pollutant/ toxicant into the environment is a constant. It is also assumed that the concentration of this toxicant in the environment decreases due to its assimilation, absorption, deposition, uptake etc. by resource biomass ,the amount being proportional to resource biomass density as well as environment concentration of the toxicant. It is assumed further that the toxicant in the environment as well as in uptake phase decrease due to natural factors by an amount which is proportional to its concentration in various cases and the amount of precursor produced by the industries is proportional to the density of the producing industry. In view of the above assumptions, we propose the following model, governing the dynamics of the resource-based industries, concentrations of the toxicants/ pollutant, and the control mechanism.

$$
\begin{aligned}
\frac{dI}{dt} &= g\left(B\right) - r_0 I + \alpha_1 IB \\
\frac{dB}{dt} &= s\left(U\right)B - \frac{s_0 B^2}{L(C)} - \alpha_2 IB - k\alpha BC \\
\frac{dP}{dt} &= \lambda I - \lambda_0 P - \theta P \\
\frac{dC}{dt} &= Q + \pi_1 \theta P - \delta C - \alpha BC + \pi \nu UB - \mu F \\
\frac{dU}{dt} &= (1-k)\alpha BC - \phi U - \nu UB \\
\frac{dF}{dt} &= \theta_1\left(C - C_p\right) - \theta_0 F
\end{aligned}
\tag{2.1}
$$

$$I(0) = I_0 \geq 0, \quad B(0) = B_0 \geq 0, \quad P(0) = P_0 \geq 0, \quad C(0) = C_0 \geq 0,$$

$$U(0) = U_0 \geq 0, \quad F(0) = F_0 \geq 0, \quad 0 \leq \mathrm{k} \leq 1, 0 \leq \pi \leq 1, 0 \leq \pi_1 \leq 1.$$

Here I(t) is the density of industries based on resource biomass of density B(t), P(t) is the density of precursor, C(t) and U(t) are the concentrations of the pollutants in the environment and in the uptake phase of the biological species respectively at any time t > 0. Q is the Cumulative rate of production of a toxicant into the environment from the external sources. The constant λ is the growth rate coefficient of precursor produced by industries, λ_0 is the natural depletion rate coefficient of precursor and θ is the fraction of the precursor, part of which is used in forming the toxicant. The constant π_1 is the coefficient of augmentation of the concentration of the same toxicant which is being emitted into the environment, α_1 is the growth rate coefficient of industries and α_2 is the depletion rate coefficient of resource biomass. The constants $\delta > 0$ and $\phi > 0$ are the natural wash out rate coefficients of toxicants and uptake phase respectively, $\alpha > 0$ is the rate of depletion of pollutant in the environment due to uptake of pollutant by the resource biomass. Also some amount of the resource biomass may die out at a rate ν due to excessive and unbearable presence of the toxicant and a fraction of π this may again re-enter into the environment. In (2.1) $k\alpha B$C>0 is a fraction of αBC directly affecting resource biomass and remaining (1-k) αBC of it is up taken by the resource biomass which decreases the intrinsic growth rate of B. F(t) denotes the environmental management system for reducing the concentration of toxicant augmented by precursor by govt./N.G.O's/ Education awareness, reforestation, taxation etc.

μ F is a pollution control device which controls the growth of emission of the toxicant in the environment. Cp is the permissible level of the concentration of toxicant (C), which is harmless to the resource biomass. The term θ_0F is to account for some practical difficulties in implementing the fool proof environmental management system.

$\alpha, \delta, \phi, \nu, \theta_1, \theta, \theta_0, Q, C_p, \lambda, \lambda_0, \alpha_1, \alpha_2, r_0$ are all positive constants.

In the model (2.1) the function $s(U)$ represents the growth rate coefficient of resource biomass which decreases with the increases of U and hence,

(2.1a) $$s(0) = s_0 > 0, \quad \frac{ds(U)}{dU} < 0, \quad \text{for} \quad U \geq 0.$$

Similarly the function $L(C)$ represents the carrying capacity (i.e. the maximum density of resource biomass which the environment can support). We assume that $L(C)$ decreases as C increases hence we have,

(2.1b) $$L(0) = L_0 > 0, \quad \frac{dL(C)}{dC} < 0, \quad \text{for} \quad C \geq 0.$$

Again g(B) denotes the growth rate coefficient of industry which increases as B increases

(2.1c) $$g(0) = g_0 > 0, \quad \frac{dg(B)}{dB} > 0, \text{for} \quad B \geq 0.$$

3. Equilibrium Analysis

The given model (2.1) has two non-negative real equillibria (Feasible equilibrium points) in I-B-P-C-U-F space denoted by $E_0(\hat{I},0,\hat{P},\hat{C},0,\hat{F})$ and $E^*(I^*,B^*,\ P^*,\ C^*,\ U^*,\ F^*)$.

For $E_0(\hat{I},0,\hat{P},\hat{C},0,\hat{F})$,

$$\hat{I}=\frac{g_0}{r_0},$$

$$\hat{P}=\frac{\lambda g_0}{r_0(\lambda_0+\theta)},$$

$$\hat{C}=\frac{\left[Q\theta_0+\pi_1\theta\left(\dfrac{\lambda g_0\theta_0}{r_0(\lambda_0+\theta)}\right)+\mu\theta_1 C_p\right]}{\delta\theta_0+\mu\theta_1},$$

$$\hat{F}=\frac{\left[Q\theta_1+\pi_1\theta\left(\dfrac{\lambda g_0\theta_1}{r_0(\lambda_0+\theta)}\right)-\delta\theta_1 C_p\right]}{\delta\theta_0+\mu\theta_1},$$

provided $Q\theta_1+\pi_1\theta\left(\dfrac{\lambda g_0\theta_1}{r_0(\lambda_0+\theta)}\right)>\delta\theta_1 C_p$,

where $\delta\theta_0+\mu\theta_1\neq 0$.

The other interior equilibrium $E^*(I^*,B^*,\ P^*,\ C^*,\ U^*,\ F^*)$ is the solution of the following system of equations:

(3.1a) $$I=\frac{g(B)}{(r_0-\alpha_1 B)}=i(B),\ \text{(assuming)} \qquad \text{provided } r_0-\alpha_1 B>0$$

(3.1b) $$B=\frac{s(U)L(C)-\alpha_2 IL(C)-k\alpha CL(C)}{s_0},\ \text{provided } s(U)-\alpha_2 I-k\alpha C>0$$

(3.1c) $$P=\frac{\lambda i(B)}{(\lambda_0+\theta)},$$

(3.1d) $$C=\frac{(\phi+\nu B)\left[Q\theta_0+\pi_1\theta\left(\dfrac{\lambda i(B)\theta_0}{(\lambda_0+\theta)}\right)+\mu\theta_1 C_p\right]}{f_1(B)}=e(B),\ \text{(assuming)}$$

(3.1e) $$U=\frac{(1-k)\alpha B\left[Q\theta_0+\pi_1\theta\left(\dfrac{\lambda i(B)\theta_0}{(\lambda_0+\theta)}\right)+\mu\theta_1 C_p\right]}{f_1(B)}=h(B),\ \text{(assuming)}$$

(3.1f) $F = \frac{\theta_1}{\theta_0}(e(B) - C_p) = f(B)$, (assuming) provided $e(B) > C_p$

where $f_1(B) = \phi(\delta\theta_0 + \mu\theta_1) + [\phi\alpha\theta_0 + \nu(\delta\theta_0 + \mu\theta_1)]B + \alpha\nu\theta_0[1 - \pi(1-k)]B^2$

and $i(B) = r_0 I - \alpha_1 IB - g(B)$.

Rewrite equation (3.1b) as

(3.1g) $s_0 B = s(U)L(C) - \alpha_2 IL(C) - k\alpha CL(C)$.

Substituting the value of I, C and U from equations (3.1a),(3.1d)and (3.1e) in the above equation (3.1g), we get

(3.2) $s_0 B = s(h(B))L(e(B)) - \alpha_2 i(B)L(e(B)) - k\alpha e(B)L(e(B))$.

Now we show the existence of the internal equilibrium point $E^*(I^*, B^*, P^*, C^*, U^*, F^*)$, as follows.

Let us consider a function F(B) such that

(3.2a) $F(B) = s_0 B - s(h(B))L(e(B)) + \alpha_2 i(B)L(e(B)) + k\alpha e(B)L(e(B))$.

Putting the value $B=0$ and L_0 in equation (3.2a), we get

(3.2b) $F(0) = -L(e(0))[s_0 - \alpha_2 \frac{g_0}{r_0} - k\alpha e(0)]$

L(e(0)) being carrying capacity is always positive and from equation (3.1b)

$$s(U) - \alpha_2 I - k\alpha C > 0.$$

This gives that at B=0, $s_0 - \alpha_2 \frac{g_0}{r_0} - k\alpha e(0) > 0$,

and hence F(0)<0.

Again

(3.2c) $F(L_0) = L_0[s_0 - s(h(L_0) + \alpha_2 i(L_0)] > 0$.

Thus, there exists a root B^* in the interval $0 < B^* < L_0$ such that $F(B^*) = 0$.

Now for the Uniqueness of B^*, the necessary and sufficient condition is $F'(B) > 0$ in the interval $0 < B < L_0$.

From (3.2a), we get

$$F'(B) = s_0 + \frac{dL}{de}\frac{de}{dB}[-s(h(B) + \alpha_2 i(B) + k\alpha e(B)] + L(e(B)[-\frac{ds}{dh}\frac{dh}{dB} + \alpha_2\frac{di}{dB} + k\alpha\frac{de}{dB}] > 0.$$

Since $B = B^*$ is the root of the equation (3.2a), we get

$$F'(B) = s_0 - \frac{s_0 B}{L(e)}\frac{dL}{de}\frac{de}{dB} + L(e(B))\left(-\frac{ds}{dh}\frac{dh}{dB} + \alpha_2\frac{di}{dB} + k\alpha\frac{de}{dB}\right) > 0.$$

Thus the condition for unique and positive B^* is $F'(B) > 0$ .Therefore B^* has been determined and then I^*, P^*, C^*, U^* and F^* can be found from equations (3.1a)-(3.1f).

In view of the above, we have the following conditions:

Let F(B) and e(B) be given by equations (3.2a) and (3.1d) respectively.

If (i) $r_0 - \alpha_1 L_0 > 0$

(ii) $s(U) - \alpha_2 I - k\alpha C > 0$

(3.2d) (iii) $e(B) > C_p$

(iv)) $F'(B) > 0$ for $0 < B < L_0$,

then there exists a unique interior equilibrium $E^*(I^*, B^*, P^*, C^*, U^*, F^*)$ for the model (2.1).

Now let us examine the effect of Q on B i.e. the cumulative rate of production of the pollutant on the density of the resource biomass.

From equation (3.2),we have

$$s_0 B = s(h(B))L(e(B)) - \alpha_2 i(B)L(e(B)) - k\alpha e(B)L(e(B)).$$

Differentiating with respect to Q, we get

$$\text{(3.3)}\quad s_0 \frac{dB}{dQ} = s(h)\frac{dL}{de}\frac{de}{dQ} + L(e)\frac{dS}{dh}\frac{dh}{dQ} - \alpha_2 L(e)\frac{di}{dQ} - \alpha_2 i\frac{dL}{de}\frac{de}{dQ} - k\alpha L(e)\frac{de}{dQ} - k\alpha e\frac{dL}{de}\frac{de}{dQ}$$

Using the formulae

$$\frac{de}{dQ} = \frac{\partial e}{\partial B}\left(\frac{dB}{dQ}\right) + \frac{\partial e}{\partial Q},$$

$$\text{(3.3a)}\quad \frac{dh}{dQ} = \frac{\partial h}{\partial B}\left(\frac{dB}{dQ}\right) + \frac{\partial h}{\partial Q},$$

$$\frac{di}{dQ} = \frac{\partial i}{\partial B}\left(\frac{dB}{dQ}\right) + \frac{\partial i}{\partial Q},$$

substituting the values from (3.3a) to (3.3) and rearranging the terms, equation (3.3) becomes

$$\text{(3.4)}\quad \frac{dB}{dQ}\left(s_0 - \frac{s_0 B}{L(e)}\frac{dL}{de}\frac{\partial e}{\partial B} - L(e)\frac{dS}{dh}\frac{\partial h}{\partial B} + \alpha_2 L(e)\frac{\partial i}{\partial B} + k\alpha L(e)\frac{\partial e}{\partial B}\right)$$
$$= \frac{s_0 B}{L(e)}\frac{dL}{de}\frac{\partial e}{\partial Q} - k\alpha L(e)\frac{\partial e}{\partial Q} + L(e)\frac{dS}{dh}\frac{\partial h}{\partial Q} - \alpha_2 L(e)\frac{\partial i}{\partial Q}.$$

Using the result of uniqueness i.e. $F'(B) > 0$ in the interval $0 < B < L_0$, in addition to the equations (3.1d), (3.1e) and conditions (2.1a), (2.1b), we analyze equation (3.4) as

$$\frac{dB}{dQ}(\text{+ve funtion}) = (\text{-ve function}).$$

Therefore,

$$\text{(3.4a)}\quad \frac{dB}{dQ} < 0.$$

From this it is concluded that the resource biomass density decreases with the increases in the cumulative rate of production of the pollutants in the environment.

From equation (3.1f),

$$F = \frac{\theta_1}{\theta_0}(e(B) - C_p) = f(B) \quad .$$

Differentiating with respect to F, we get

(3.4b) $$1 = \frac{\theta_1}{\theta_0}\left[\frac{de}{dB}\frac{dB}{dF}\right].$$

From equation (3.1d), we get $\frac{de}{dB} > 0.$

Using this result in the above equation (3.4b), we have

(3.4c) $$\frac{dB}{dF} > 0.$$

Again from equation (3.1a),

$$I = \frac{g(B)}{(r_0 - \alpha_1 B)},$$

which on differentiation with respect to F gives,

$$\frac{dI}{dF} = \left[\frac{(r_0 - \alpha_1 B)\, g'(B) + \alpha_1 g(B)}{(r_0 - \alpha_1 B)^2}\right]\frac{dB}{dF}.$$

Here $(r_o - \alpha_1 B) > 0$ and $g'(B) > 0$ and since $\frac{dB}{dF} > 0$,

(3.4d) $$\frac{dI}{dF} > 0.$$

From (3.4c) and (3.4d) it is clear that with the increase in the control measure (efforts),the resource biomass density as well as the density of the industries based on the resource biomass increases thus the control measure (efforts) has a positive impact in the system and the industries may be saved from going to extinction.

4. Stability Analysis

Here we shall discuss the local as well as global stability of the equilibrium points .The local stability of the equilibria can be studied from variational matrices corresponding to each equilibrium point and for the global stability, suitable Lyapunov functions are found in the interior of some region Ω .

4a. Local Stability via Eigen Value Method

To study the local stability behavior of E_0 and E^*, we compute the variational matrices M_0 and M^* corresponding to E_0 and E^* as follows:

$$M_0 = \begin{bmatrix} -r_0 & g'(0)+\alpha_1 I & 0 & 0 & 0 & 0 \\ 0 & s_0-\alpha_1 I - k\alpha C & 0 & 0 & 0 & 0 \\ \lambda & 0 & -(\lambda_0+\theta) & 0 & 0 & 0 \\ 0 & -\alpha C & \pi_1\theta & -\delta & 0 & -\mu \\ 0 & (1-k)\alpha C & 0 & 0 & -\phi & 0 \\ 0 & 0 & 0 & \theta_1 & 0 & -\theta_0 \end{bmatrix},$$

and

$$M^* = \begin{bmatrix} -r_0+\alpha_1 B^* & g'(B^*)+\alpha_1 I^* & 0 & 0 & 0 & 0 \\ -\alpha_2 B^* & \dfrac{-s_0 B^*}{L(C^*)} & 0 & \dfrac{s_0 B^{*2} L'(C^*)}{\left[L(C^*)\right]^2} - k\alpha B^* & s'(U^*)B^* & 0 \\ \lambda & 0 & -(\lambda_0+\theta) & 0 & 0 & 0 \\ 0 & -\alpha C^* + \pi\nu U^* & \pi_1\theta & -(\delta+\alpha B^*) & \pi\nu B^* & -\mu \\ 0 & (1-k)\alpha C^* - \nu U^* & 0 & (1-k)\alpha B^* & -(\phi+\nu B^*) & 0 \\ 0 & 0 & 0 & \theta_1 & 0 & -\theta_0 \end{bmatrix}.$$

From the matrix M_0, it is clear that $E_0(I,0,P,C,0,F)$ is a saddle point with stable manifold locally in the I- P -U space and unstable manifold locally in the B- direction.

The stability behavior of E^* is not obvious from M*. However, in the following theorem we find sufficient condition for E^* to be locally asymptotically stable.

The following theorem gives the criteria for the local stability of E* which can be proved by constructing a suitable Lyapunov function.

4b. Local Stability via Lyapunov Method:

Theorem 4.1: *If the following inequalities hold*:

(4.1a) $$\left[\left(\frac{g'(B^*)+\alpha_1 I^*}{\alpha_2}\right)\left(\frac{s_0 B^* L'(C^*)}{\left[L(C^*)\right]^2} - k\alpha\right) - C_4\left(\alpha C^* - \pi\nu U^*\right)\right]^2 < \left(\frac{g'(B^*)+\alpha_1 I^*}{\alpha_2}\right)\frac{C_4 s_0 \delta}{L(C^*)},$$

(4.1b) $$\frac{-\left(g'(B)+\alpha_1 I^*\right)s'(U^*)(1-k)\alpha(\pi_1\theta)^2}{\alpha_2\left[(1-k)\alpha C^* - \nu U^*\right]\delta\pi\nu} < \frac{g'(B)}{I^*}\frac{\left(\lambda_0+\theta\right)^2}{\lambda^2},$$

where

(4.1c) $$C_4 = \frac{-\left(g'(B)+\alpha_1 I^*\right)s'(U^*)(1-k)\alpha}{\alpha_2\left[(1-k)\alpha C^* - \nu U^*\right]\pi\nu}$$, *provided* $(1-k)\alpha C^* > \nu U^*$

then $E^*(I^*, B^*, P^*, C^*, U^*, F^*)$ *is locally asymptotically stable.*

Proof.: See the Appendix

5. Global Stability Analysis

Lemma 5.1: *The set*

$$\Omega = \left\{ \begin{array}{l} (I,B,P,C,U,F):\ 0 \le I \le \left(\dfrac{g_0}{r_o - \alpha_1 L_0}\right);\ 0 \le B \le L_0; \\ \qquad 0 \le P + C + U \le \dfrac{\left(Q + \dfrac{g_0 \lambda}{(r_o - \alpha_1 L_0)}\right)}{\phi_1}; \\ \qquad 0 \le F \le \dfrac{\theta_1}{\theta_0} \dfrac{\left(Q + \dfrac{g_0 \lambda}{(r_o - \alpha_1 L_0)}\right)}{\phi_1} \end{array} \right\}$$

is a region of attraction for all solutions initiating in the region

$$R = \{(I,B,P,C,U,F): I > 0, B > 0, P > 0, C > 0, U > 0, F > 0\}$$

where $\phi_1 = \min(\lambda_1, \delta, \phi)$.

The proof is given in Appendix.

Theorem 5.2: *In addition to the assumptions* (2.1a), (2.1b) *and* (2.1c), *let* L_m, p, q and r *be positive constants such that, in the region* Ω,

(5.21) $$L_m \le L(C) \le L_0,\quad 0 \le -L'(C) \le p,\quad 0 \le -s'(U) \le q,\quad 0 \le -g'(B) \le r\ .$$

If the following inequalities hold:

(5.2a) $$\left[\left(\frac{r+\alpha_1 I^*}{\alpha_2}\right)\left(\frac{s_0 B^* p}{L_m^{\ 2}} + k\alpha\right) + C_4\left(\alpha C^* - \pi \nu U^*\right)\right]^2 < \frac{1}{2}\left(\frac{r+\alpha_1 I^*}{\alpha_2}\right)\frac{C_4 s_0 \delta}{L_0},$$

(5.2b) $$\left[q\left(\frac{r+\alpha_1 I^*}{\alpha_2}\right) + \left((1-k)\alpha C^* - \nu U^*\right)\right]^2 < \left(\frac{r+\alpha_1 I^*}{\alpha_2}\right)\frac{s_0 \phi}{L_0},$$

(5.2c) $$\frac{(1-k)\alpha(\pi_1 \theta)^2}{\delta \pi \nu} < \frac{(r_0 - \alpha_1 B)(\lambda_0 + \theta)^2}{\lambda^2},$$

where

$$C_4 = \frac{(1-k)\alpha}{\pi \nu},$$

then E is globally asymptotically stable with respect to all solutions initiating in the interior of the region* Ω .

Proof: See the Appendix.

6. Numerical Example

To explain the applicability of the result we give here numerical simulation of the equillibria and the stability conditions for the model. We assume

(6.1a) $$s(U) = s_0 - \frac{a_1 U}{1 + s_1 U},\quad L(C) = L_0 - \frac{b_1 C}{1 + m_1 C} \quad \text{and} \quad g(B) = g_0 - \frac{c_1 B}{1 + n_1 B}$$

where

$s_0 = 17,\ a_1 = 1,\ s_1 = 4,\ L_0 = 5.8,\ b_1 = 1,\ m_1 = 1.02,\ g_0 = 15,\ c_1 = 1,\ n_1 = 1.04$

We note from the above that

(6.1b) $s'(U) = \dfrac{-a_1}{(1+s_1U)^2}$, $L'(C) = \dfrac{-b_1}{(1+m_1C)^2}$ *and* $g'(B) = \dfrac{-c_1}{(1+n_1B)^2}$

Choosing p, q and r as 1.0 each and

$\alpha_1 = 0.1, \alpha_2 = 0.2, \alpha = 0.01, \delta = 14, k = 0.6, Q = 12, \pi = 0.03, \nu = 0.03, \phi = 16, \mu = 6,$
$C_p = 0.6, \theta_1 = 60, \theta_0 = 0.16, \pi_1 = 0.08, \lambda = 1.4, \lambda_0 = 1.1, \theta = 0.9, r_0 = 5.1.$

It can be checked that the interior equilibrium $E^*\left(I^*, B^*, P^*, C^*, U^*, F^*\right)$ of model (2.1) exists and to find these values using software Mathematica 5.2, we get the equilibrium values I^*, B^*, P^*, C^*, U^* *and* F^* are

$I^* = 3.0996571211706874',\ B^* = 5.22769653540721',\ P^* = 2.169759984819481',$

$C^* = 0.6016452183000792',\ U^* = 0.0007785992306282459',\ F^* = 0.6169568625297188'.$

It can be verified that all the conditions in Theorem (4.1) are satisfied for the above set of parameters and hence E^* is locally asymptotically stable.

We note from (6.1b) that if

(6.2) $\dfrac{-\partial s}{\partial U} = \dfrac{1}{(1+s_1U)^2} \le 1$, $\dfrac{-\partial L}{\partial C} = \dfrac{1}{(1+m_1C)^2} \le 1$, $\dfrac{-\partial g}{\partial B} = \dfrac{1}{(1+n_1B)^2} \le 1$

along with the value of the parameters chosen above then it can be checked that all the conditions of Theorem(5.2) are satisfied and hence E^* is globally asymptotically stable. It is to be noted that B^* is less than its carrying capacity i.e. L_0, as expected from the model study.

In the table-1, we find that at constant pollution control device($\mu = 6$) the equilibrium values of resource biomass and its based industry decrease with increasing of the emission rate Q of the toxicant(see Fig.I and Fig. II) while the equilibrium levels of the environmental and uptake concentrations of the toxicant increase. This suggests that for very large emission rates of the toxicant affecting resources and its dependent industries, there existence will be threatened.

Table-1

Q	I^*	B^*	P^*	C^*	U^*	F^*
12	3.09966	5.2277	2.16976	0.60165	0.000778599	0.616957
24	3.09955	5.2260	2.16969	0.606945	0.000785332	2.60454
36	3.09942	5.22393	2.16959	0.612246	0.000791819	4.59213
48	3.09930	5.22203	2.16951	0.617546	0.000798408	6.57972
100	3.09879	5.2138	2.16915	0.640514	0.000826808	15.1926

In table-2, we see that when the measure of pollution control device i.e. (μ) increases in the same ratio as Q, the equilibrium values of resource biomass and its based industry first decreases slowly as compared to table-1 at a certain level and then increases slowly to get desired level(i.e. nearly pollution free environment),also the equilibrium levels of the environmental and uptake concentrations of the toxicant increase slowly as compared to table-1. This gives that control measure (efforts) has a positive impact in the eco-system and industries may be saved from going to extinction.

Table -2

Q	μ	I^*	B^*	P^*	C^*	U^*	F^*
12	6	3.09966	5.2277	2.16976	0.60165	0.000778599	0.616957
24	12	3.09961	5.227	2.16973	0.603484	0.000780873	1.30633
36	18	3.09954	5.22458	2.16968	0.604099	0.000777202	1.53707
48	24	3.09948	5.22388	2.16964	0.604407	0.00077611	1.65262
100	50	3.09988	5.22617	2.173	0.604888	0.000830436	1.83312

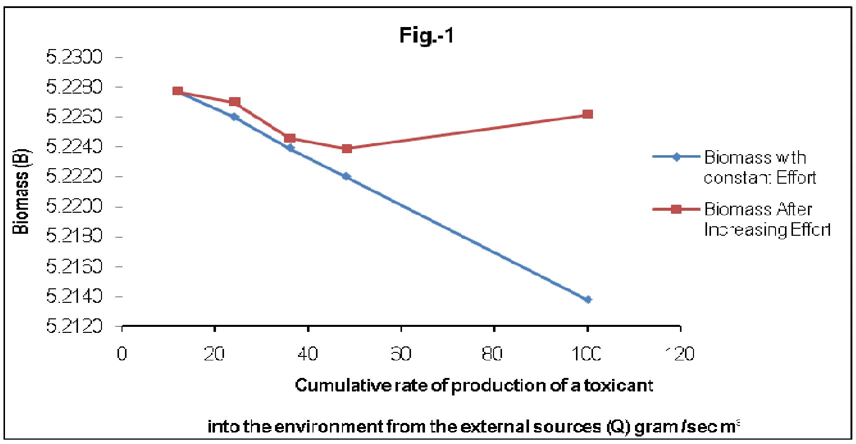

Fig.-1

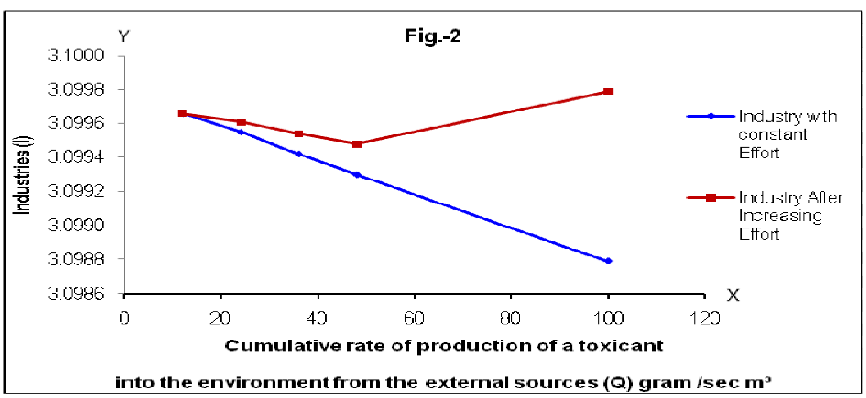

Fig.-2

Summary

In this paper, an ecological model has been proposed and analyzed to study the depletion of resource-based industries in a habitat, which is caused by increase in pollutant emission into the environment by external sources and whose concentration is augmented by a precursor produced by industries itself. The existence of non-trivial equilibrium has been discussed and its local and global stability behavior has been analyzed. Also, a region of attraction has been found for global asymptotic stability of the equilibrium point. It has been shown that the density of the industries dependent on the resource biomass decreases with the increase of cumulative rate of production of the pollutants in the environment. But on application of environmental management system (pollution control device), the resource biomass density as well as density of the industries based on the resource biomass increases. The analysis of the non-linear stability established that the resource settles down to an equilibrium level, which is lower than its initial carrying capacity, the magnitude of which depends upon the toxicity, emission and washout rate of the toxicant. It has been noted that the equilibrium level decreases as the toxicity and emission rates increase but with the increase of washout rates of the toxicants the equilibrium level is controlled to some extent from going down. It has been found that the environmental management systems (control measures) prove as disincentive to the emitters of the pollutants and its emission at source is checked and reduced and the resources and resource-based industries can be maintained at a desired level.

The conclusion drawn here suggests that emission of various kinds of toxicants in the environment must be controlled without further delay otherwise the survival of resources and its dependent industries will be threatened.

Acknowledgement

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Appendix

Proof of Theorem 4.1

Let us linearize the system (2.1) about $E^* \; (I^*, N^*, P^*, C^*, U^*, F^*)$ by using the following transformations

$$I = I^* + i, \quad B = B^* + b, \quad P = P^* + p, \quad C = C^* + c, \quad U = U^* + u, \quad F = F^* + f,$$

where i, b, p, c, u and f are small perturbations around E^*. This results into

$$\frac{di}{dt} = -\frac{g'(B^*)}{I^*} i + \left(g'(B^*) + \alpha_1 I^* \right) b,$$

$$\frac{db}{dt} = \left(-\alpha_2 B^*\right) i - \frac{s_0 B^*}{L\left(C^*\right)} b + \left[\frac{s_0 B^{*2} \quad L'\left(C^*\right)}{\left[L\left(C^*\right)\right]^2} - k\alpha B^* \right] c + \left(s'\left(\mathrm{U}^*\right) B^* \right) u,$$

$$\frac{dp}{dt} = \lambda i - \left(\lambda_0 + \theta\right) p,$$

(4.1d)
$$\frac{dc}{dt}=-\left(\alpha C^*-\pi\nu U^*\right)b+\pi_1\theta p-(\delta+\alpha B^*)c+\left(\pi\nu B^*\right)u-\mu f,$$

$$\frac{du}{dt}=\left(\left(1-k\right)\alpha C^*-\nu U^*\right)b+\left(\left(1-k\right)\alpha B^*\right)c-\left(\phi+\nu B^*\right)u,$$

$$\frac{df}{dt}=\theta_1 c-\theta_0 f.$$

Considering the positive definite function V around E^*, defined as

(4.1e) $$\mathrm{V}=\frac{1}{2}C_1 i^2+\frac{1}{2}C_2\frac{b^2}{B^*}+\frac{1}{2}\mathrm{C}_3\, p^2+\frac{1}{2}C_4c^2+\frac{1}{2}C_5u^2+\frac{1}{2}C_6 f^2$$

where $C_1, C_2, C_3, C_4, C_5, C_6$ are positive constants, we can show that the derivative of V with respect to t along the linearized system (4.1d) is negative definite under the conditions of theorem (4.1).

Hence V is a Lyapunov function with respect to E^*, therefore E^* is locally asymptotically stable.

Proof of lemma 5.1

The second equation of the model (2.1) implies

$$\frac{\mathrm{d}\,B}{\mathrm{dt}}\le s_0B-\frac{s_0B^2}{L_0}\ .$$

Hence $$\lim_{t\to\infty}\ \sup\ \mathrm{B}(t)\le\ \mathrm{L}_0$$

i.e. $0\le B\le L_0$.

The first equation of the model (2.1) implies

$$\frac{dI}{dt}\le g_0-r_0I+\alpha_1IL_0$$

$$\Rightarrow\quad \frac{dI}{dt}\le g_0-I\left(r_0-\alpha_1L_0\right)$$

$$\Rightarrow\quad I(t)\ \le\frac{g_0}{\left(r_0-\alpha_1L_0\right)}\quad ,$$

and hence $$0\le I(t)\ \le\frac{g_0}{\left(r_0-\alpha_1L_0\right)}\ .$$

Adding the third, fourth and fifth equations of the model (2.1), we have

$$\begin{aligned}\frac{dP}{dt}+\frac{dC}{dt}+\frac{dU}{dt}&=\lambda I-\lambda_0P-\theta P+Q+\pi_1\theta P-\delta C-\alpha BC\\&\quad+\pi\nu UB-\mu F+\left(1-k\right)\alpha BC-\phi U-\nu UB\\&=\lambda I+Q-(\lambda_0+\theta-\pi_1\theta)P-\delta C-\phi U\\&\quad-k\alpha BC-(1-\pi)\nu UB-\mu F\end{aligned}$$

$$\le \lambda I + Q - \lambda_1 P - \delta C - \phi U \qquad \text{where } \lambda_1 = \lambda_0 + \theta - \pi_1\theta$$

$$\le \lambda I + Q - \phi_1 (P + C + U)$$

$$\le \frac{\lambda g_0}{r_0 - \alpha_1 L_0} + Q - \phi_1 (P + C + U)$$

$$0 \le P + C + U \le \frac{\left(Q + \lambda \frac{g_0}{(r_0 - \alpha_1 L_0)}\right)}{\phi_1} \quad \text{where } \phi_1 = \min(\lambda_1, \delta, \phi).$$

The sixth equation of (2.1) implies

$$\frac{dF}{dt} \le C\theta_1 - \theta_0 F$$

$$\Rightarrow \quad \frac{dF}{dt} \le \theta_1 \frac{\left(Q + \lambda \frac{g_0}{(r_0 - \alpha_1 L_0)}\right)}{\phi_1} - \theta_0 F$$

$$\Rightarrow \quad F(t) \le \frac{\theta_1}{\theta_0} \frac{\left(Q + \lambda \frac{g_0}{(r_0 - \alpha_1 L_0)}\right)}{\phi_1}$$

$$\text{and hence } 0 \le F(t) \le \frac{\theta_1}{\theta_0} \frac{\left(Q + \lambda \frac{g_0}{(r_0 - \alpha_1 L_0)}\right)}{\phi_1}.$$

Proof of theorem 5.2

Let us consider the positive definite function W around E^*

(5.2d)

$$W(I, B, P, C, U, F) = \frac{1}{2} C_1 (I - I^*)^2 + C_2 \left(B - B^* - B^* \log \frac{B}{B^*}\right) + \frac{1}{2} C_3 (P - P^*)^2 + \frac{1}{2} C_4 (C - C^*)^2$$
$$+ \frac{1}{2} C_5 (U - U^*)^2 + \frac{1}{2} C_6 (F - F^*)^2,$$

where C_1, C_2 C_3, C_4, C_5 and C_6 are positive constants to be chosen such that it becomes a Lyapunov function, and its domain contains the region of attraction as (5.1) defined above. On differentiating W with respect to t, we get

$$\dot{W} = C_1 (I - I^*) \frac{dI}{dt} + C_2 \left(\frac{B - B^*}{B}\right) \frac{dB}{dt} + C_3 (P - P^*) \frac{dP}{dt} + C_4 (C - C^*) \frac{dC}{dt}$$
$$+ C_5 (U - U^*) \frac{dU}{dt} + C_6 (F - F^*) \frac{dF}{dt}.$$

Substituting the values of $\dot{I}, \dot{B}, \dot{P}, \dot{C}, \dot{U}, \dot{F}$ from (2.1), we have

$$\begin{aligned}\dot{W}=&C_1\left(I-I^*\right)\left[g(B)-r_0I+\alpha_1 IB\right]+C_2\left(B-B^*\right)\left[s(U)-\frac{s_0B}{L(C)}-\alpha_2 I-k\alpha C\right]\\&+C_3\left(P-P^*\right)\left[\lambda I-\lambda_0P-\theta P\right]+C_4\left(C-C^*\right)\left[Q+\pi_1\theta P-\delta C-\alpha BC+\pi\nu BU-\mu F\right]\\&+C_5\left(U-U^*\right)\left[(1-k)\alpha BC-\phi U-\nu UB\right]+C_6\left(F-F^*\right)\left[\theta_1\left(C-C_p\right)-\theta_0F\right].\end{aligned}\tag{5.2e}$$

After some algebraic manipulations , it can be written as

$$\begin{aligned}\dot{W}=&-C_1\left(r_0-\alpha_1B\right)\left(I-I^*\right)^2-C_2\frac{s_0}{L(C)}\left(B-B^*\right)^2-C_3\left(\lambda_0+\theta\right)\left(P-P^*\right)^2-C_4\left(\delta+\alpha B\right)\left(C-C^*\right)^2\\&-C_5\left(\phi+\nu B\right)\left(U-U^*\right)^2-C_6\theta_0\left(F-F^*\right)^2\\&+\left(B-B^*\right)\left(I-I^*\right)\left[C_1\left(\eta(B)+\alpha_1 I^*\right)-C_2\alpha_2\right]\\&+\left(B-B^*\right)\left(C-C^*\right)\left[-C_2\left(s_0B^*\xi(C)+k\alpha\right)-C_4\left(\alpha C^*-\pi\nu U^*\right)\right]\\&+\left(I-I^*\right)\left(P-P^*\right)C_3\lambda\\&+\left(B-B^*\right)\left(U-U^*\right)\left[C_2\eta_1(U)+C_5\left((1-k)\alpha C^*-\nu U^*\right)\right]\\&+\left(P-P^*\right)\left(C-C^*\right)C_4\pi_1\theta\\&+\left(C-C^*\right)\left(U-U^*\right)\left[C_4\pi\nu B+C_5(1-k)\alpha B\right]\\&+\left(C-C^*\right)\left(F-F^*\right)\left[-\mu C_4+C_6\theta_1\right].\end{aligned}\tag{5.2f}$$

where

$$\eta(B)=\begin{cases}\dfrac{g(B)-g\left(B^*\right)}{\left(B-B^*\right)} & ;\quad B\neq B^*\\ g'\left(B^*\right) & ;\quad B=B^*\end{cases},$$

$$\xi(\mathrm{C})=\begin{cases}\dfrac{\dfrac{1}{L(C)}-\dfrac{1}{L(C^*)}}{\mathrm{C}-\mathrm{C}^*} & ;\quad C\neq C^*\\ \dfrac{-L'\left(\mathrm{C}^*\right)}{\left[L\left(\mathrm{C}^*\right)\right]^2} & ;\quad C=C^*\end{cases},\tag{5.2g}$$

$$\eta_1(\mathrm{U})=\begin{cases}\dfrac{s(\mathrm{U})-s\left(\mathrm{U}^*\right)}{\left(\mathrm{U}-\mathrm{U}^*\right)} & ;\quad \mathrm{U}\neq\mathrm{U}^*\\ s'\left(\mathrm{U}^*\right) & ;\quad \mathrm{U}=\mathrm{U}^*\end{cases},$$

Choose the value of constants C_1 and C_2 as below

$$C_1 = 1,$$

$$C_2 = \frac{\eta(B)+\alpha_1 I^*}{\alpha_2}. \tag{5.2h}$$

By substituting these values of C_1 and C_2 , equation (5.2e) reduces to

$$\begin{aligned}
\dot{W} = &-(r_0-\alpha_1 B)\left(I-I^*\right)^2 - \left(\frac{\eta(B)+\alpha_1 I^*}{\alpha_2}\right)\frac{s_0}{L(C)}\left(B-B^*\right)^2 - C_3\left(\lambda_0+\theta\right)\left(P-P^*\right)^2 - C_4\left(\delta+\alpha B\right)\left(C-C^*\right)^2 \\
&- C_5\left(\phi+\nu B\right)\left(U-U^*\right)^2 - C_6\theta_0\left(F-F^*\right)^2 \\
&+\left(B-B^*\right)\left(C-C^*\right)\left[-\left(\frac{\eta(B)+\alpha_1 I^*}{\alpha_2}\right)\left(s_0 B^*\xi(C)+k\alpha\right) - C_4\left(\alpha C^* - \pi\nu U^*\right)\right] \\
&+\left(I-I^*\right)\left(P-P^*\right)C_3\lambda \\
&+\left(B-B^*\right)\left(U-U^*\right)\left[\eta_1(U)\left(\frac{\eta(B)+\alpha_1 I^*}{\alpha_2}\right) + C_5\left((1-k)\alpha C^* - \nu U^*\right)\right] \\
&+\left(P-P^*\right)\left(C-C^*\right)C_4\pi_1\theta \\
&+\left(C-C^*\right)\left(U-U^*\right)\left[C_4\pi\nu B + C_5(1-k)\alpha B\right] \\
&+\left(C-C^*\right)\left(F-F^*\right)\left[-\mu C_4 + C_6\theta_1\right].
\end{aligned} \tag{5.2i}$$

Since

$L_m \le L(C) \le L_0$; $0 \le -s'(U) \le q$; $0 \le -L'(C) \le p$; $0 \le -g'(B) \le r$.

from the mean value theorem,

$$\begin{aligned}
&\left|\xi(C)\right| \le \frac{p}{L_m^{\ 2}}, \\
&\left|\eta_1(U)\right| \le q, \\
&\left|\eta(B)\right| \le r.
\end{aligned} \tag{5.2j}$$

for some positive constant K_m, p ,q and r in the region Ω .

Now $\dot{W}$ can further be written as sum of quadratics,

$$\dot{W}=-\frac{1}{2}\left(\frac{r+\alpha_1 I^*}{\alpha_2}\right)\frac{s_0}{L(C^*)}(B-B^*)^2+\left[-\left(\frac{r+\alpha_1 I^*}{\alpha_2}\right)\left(s_0 B^*\frac{p}{L_m{}^2}+k\alpha\right)-C_4\left(\alpha C^*-\pi\nu U^*\right)\right](B-B^*)(C-C^*)$$

$$-C_4\left(\frac{\delta}{4}\right)(C-C^*)^2-\frac{1}{2}\left(\frac{r+\alpha_1 I^*}{\alpha_2}\right)\frac{s_0}{L(C^*)}(B-B^*)^2$$

$$+\left[q\left(\frac{r+\alpha_1 I^*}{\alpha_2}\right)+C_5\left\{(1-k)\alpha C^*-\nu U^*\right\}\right](B-B^*)(U-U^*)$$

(5.2k)
$$-C_5\left(\frac{\phi}{2}\right)(U-U^*)^2-C_4\left(\frac{\delta}{4}+\alpha B\right)(C-C^*)^2+\left[C_4\pi\nu B+C_5(1-k)\alpha B\right](C-C^*)(U-U^*)$$

$$-C_5\left(\frac{\phi}{2}+\nu B\right)(U-U^*)^2-\left(r_0-\alpha_1 B\right)(I-I^*)^2+C_3\lambda(I-I^*)(P-P^*)-C_3\frac{\lambda_0+\theta}{2}(P-P^*)^2$$

$$-C_4\left(\frac{\delta}{4}\right)(C-C^*)^2+C_4\pi_1\theta(C-C^*)(P-P^*)-C_3\frac{\lambda_0+\theta}{2}(P-P^*)^2$$

$$-C_6\theta_0(F-F^*)^2+\left[-\mu C_4+C_6\theta_1\right](C-C^*)(F-F^*)-C_4\left(\frac{\delta}{4}\right)(C-C^*)^2 .$$

For $\dot{W}$ to be negative definite the following inequalities must be satisfied:

(5.2l)
$$\left[\frac{\left(r+\alpha_1 I^*\right)}{\alpha_2}\left(s_0 B^*\frac{p}{L_m{}^2}+k\alpha\right)+C_4\left(\alpha C^*-\pi\nu U^*\right)\right]^2<4\,\frac{\left(r+\alpha_1 I^*\right)}{2\alpha_2}\,\frac{C_4 s_0}{L(C)}\left(\frac{\delta}{4}\right),$$

(5.2m)
$$\left[q\,\frac{\left(r+\alpha_1 I^*\right)}{\alpha_2}+C_5\left(\left(1-k\right)\alpha C^*-\nu U^*\right)\right]^2<4\,\frac{\left(r+\alpha_1 I^*\right)}{2\alpha_2}\,\frac{C_5 s_0}{L(C)}\left(\frac{\phi}{2}\right),$$

(5.2n)
$$\left[C_4\pi\nu B+C_5\left(1-k\right)\alpha B\right]^2<4C_4C_5\left(\frac{\delta}{4}+\alpha B\right)\left(\frac{\phi}{2}+\nu B\right),$$

(5.2o)
$$C_3\left[\lambda\right]^2<4\left(r_0-\alpha_1 B\right)\frac{\left(\lambda_0+\theta\right)}{2},$$

(5.2p)
$$C_4\left(\pi_1\theta\right)^2<4C_3\frac{\left(\lambda_0+\theta\right)}{2}\frac{\delta}{4},$$

(5.2q)
$$\left[-\mu C_4+C_6\theta_1\right]^2<C_4C_6\theta_0\delta.$$

We rewrite the equation (5.2n) as

(5.2r)
$$\left[C_4\pi\nu B-C_5\left(1-k\right)\alpha B\right]^2+4C_4C_5\pi\nu\left(1-k\right)\alpha B^2<4C_4C_5\left(\frac{\delta}{4}+\alpha B\right)\left(\frac{\phi}{2}+\nu B\right).$$

In order to reduce the above inequality, we choose the value of constants

(5.2s)
$$\mathrm{C}_4=\frac{\left(1-\mathrm{k}\right)\alpha}{\pi\nu},\qquad\left(\nu\neq 0\right)$$
$$\mathrm{C}_5=1.$$

From (5.2o) ,we get

(5.2t) $$C_3 < \frac{2\left(r_0 - \alpha_1 B\right)\ \left(\lambda_0 + \theta\right)}{[\lambda]^2}.$$

Keeping the value of C_4 in inequality (5.2n), we get

(5.2u) $$\frac{2(1-k)\alpha\left(\pi_1\theta\right)^2}{\pi\nu\delta\left(\lambda_0 + \theta\right)} < C_3 .$$

In view of (5.2t) and (5.2u), such that

(5.2v) $$\frac{(1-k)\alpha\left(\pi_1\theta\right)^2}{\pi\nu\delta} < C_3 < \frac{\left(r_0 - \alpha_1 B\right)\ \left(\lambda_0 + \theta\right)^2}{[\lambda]^2} .$$

If we choose $C_6 = \frac{\mu C_4}{\theta_1}$, we find that the inequality (5.2q) is automatically satisfied. Such choice of C_6 is always possible since we have assumed a system in which the concentration level of pollutants is much more than the harmful limit and therefore the control measures (efforts) have been taken to reduce the concentration of pollutants in the environment.

Further on keeping the minimum value of the variables on right hand sides of the inequalities, (5.2l) and (5.2m), we get

(5.2w) $$\left[\left(\frac{r + \alpha_1 I^*}{\alpha_2}\right)\left(s_0 B^* \frac{p}{L_m^{\ 2}} + k\alpha\right) + C_4\left(\alpha C^* - \pi\nu U^*\right)\right]^2 < \frac{1}{2}\left(\frac{r + \alpha_1 I^*}{\alpha_2}\right)\frac{C_4 s_0 \delta}{L_0},$$

(5.2x) $$\left[q\left(\frac{r + \alpha_1 I^*}{\alpha_2}\right) + \left((1-k)\alpha C^* - \nu U^*\right)\right]^2 < \left(\frac{r + \alpha_1 I^*}{\alpha_2}\right)\frac{s_o \phi}{L_0},$$

where

$$C_4 = \frac{(1-k)\alpha}{\pi\nu} .$$

Thus we find the inequalities (5.2v), (5.2w) and (5.2x), which are same as mentioned in the Theorem 5.2. Hence W is Lyapunov function with respect to E^* whose domain contains Ω and therefore E^* is non-linearly stable and hence the theorem.

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