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Asymptotic Behaviour of Free Convective Boundary Layer Flows of Newtonian Fluids Past a Vertical Flat Plate with Suction/Injection

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Abstract: This paper deals with the asymptotic behaviours of the solutions of Falkner-Skan equations governing the free convective boundary layer flows of Newtonian fluids past a vertical flat plate with suction/injection as the independent variable η tends to infinity; the discussion being based on the asymptotic integrations of second order linear differential equations. It has been found that the principal solutions exhibit asymptotic nature as $\eta \to \infty$, whereas corresponding linearly independent solution does not.

Keywords: Asymptotic Behaviour, Boundary layer flows, Newtonian Fluids, Free Convective.

Nomenclature:

u, v: Velocity components in the x -axis and y -axis, respectively (m/s)

μ: Dynamic viscosity

g: Acceleration due to gravity

T_∞: Ambient Temperature of the fluid in the boundary layer

$\upsilon = \frac{\mu}{\rho}$: Kinematic viscosity of the fluid

ρ: Density of the fluid

T : Temperature of the fluid in the boundary layer

T_w: Wall Temperature of the boundary layer

C_p: Specific heat of the fluid
K: Thermal conductivity of the fluid
a: Constant
b: Constant
f_w: Suction/Injection Parameter
m: Temperature variation index at the wall
ϕ: Coefficient of thermal expansion of the fluid at temperature T_∞
θ: non-dimensional temperature
α: Thermal diffusivity
$V_w(x)$: Suction/Injection velocity
A: Dimensional constant
B: Dimensional constant
f : Non dimensionless stream function
L : Characteristic length
P_r: Prandtl number
η: non-dimensional co-ordinate perpendicular to the wall
ψ: Dimensional stream-function
ε : A constant

1. Introduction

Free convective phenomenon has been the subject of extensive research. The importance of this phenomenon is due to its enhanced concern in science and technology about buoyancy induced motion in the atmosphere, in bodies such as earth. Consequently, free convective flow has been extensively studied by Jaluria[1] and Senoy and Mashelkar[2]. After that the free convective laminar boundary layer flow of power-law fluids have been studied by Sahu and Mathur[3].The study of the asymptotic behaviours of the solutions of the equations governing the problems of the physical significance in boundary layer theory is an interesting feature of discussion in fluid-mechanics. One of the most important problems in the study of the differential equations or of differential-difference equations and their applications is that of describing the nature of the solutions for large positive values of the independent variables and this purpose is completely fulfilled by the study of the asymptotic behaviours. Thus, the asymptotic behaviour plays a particular attention towards a desired problem for finding conditions under which a solution approaches zero as the independent variable tends to infinity, or is small for all independent variables, or is bounded as the independent variable tends to infinity. Because of the above facts, the study of the asymptotic nature of the solutions of the Falkner-Skan[4] equations governing the steady two-dimensional flow of a slightly viscous incompressible fluid past a wedge was initiated by Hartman[5] and Later followed by Singh[6-7],Singh and Singh [8-11],Singh and Kumar[12],Singh and Verma[13], Tiryaki and Yaman[14], Brighi and Hoernel[15] etc.

The objective of the present paper is to study the asymptotic nature of the solutions of equations which govern free convection in boundary flows of Newtonian fluids past a vertical flat plate by taking into account the effects of suction/injection. The discussion here is based on the asymptotic integrations of second order linear differential equations.

Sharma et.al[16] investigated the effects of velocity slip and thermal slip on MHD mixed convective flow along vertical porous plate in the presence of a non-uniform magnetic field. Sharma and Sinha[17] applied Runge-Kutta fourth order scheme with shooting technique for MHD Mixed Convective Slip Flow, Heat and Mass Transfer along a Vertical Porous Plate also Shekar and Shankar[18] analyzed the effect of a suction and Soret number on heat and mass transfer MHD flow past an exponentially stretching sheet with the heat source/sink. Using similarity transformation, the system of PDEs is changed into a system of nonlinear ODEs, which was then solved numerically. Archana Shukla[19] studied an unsteady hydromagnetic free convection flow and mass transfer of an elastic viscous fluid in a rotating porous medium.

2. Formulation of the Problem

The equations governing the two-dimensional plane flow of a Newtonian-fluid in the laminar boundary region close to a vertical flat plate are:

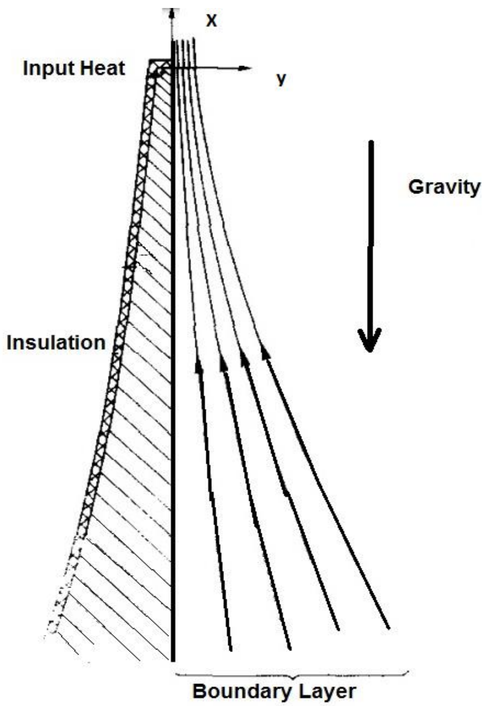

Figure1. Schematics of the problem

(2.1) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(2.2) $$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\left(\frac{\partial^2 u}{\partial y^2}\right)+g\beta(T-T_\infty)$$

(2.3) $$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{K}{\rho c_p}\left(\frac{\partial^2 T}{\partial y^2}\right)$$

The boundary conditions are

(2.4) $y=0:\ u=0,\ v=V_w(\mathrm{x}),\ \mathrm{T}=T_w(\mathrm{x}),\ y\to\infty:\ \mathrm{u}=0,\ \mathrm{T}=T_\infty$

Let

(2.5) $$\begin{cases} u=\psi_y\ ,\ v=-\psi_x, \\ \psi=\mathrm{A}x^a\ \mathrm{f}(\eta),\quad \eta=\mathrm{B\,y\,x^b}, \\ \theta=\dfrac{\mathrm{T}-T_\infty}{T_w(\mathrm{x})-T_\infty}\ ,\ T_w-\mathrm{T}_\infty=\mathrm{Nx^m}, \\ \mathrm{N}=\dfrac{T_w(0)-T_\infty}{L^m} \end{cases}$$

Substituting the expressions (2.5) in equations (2.1)-(2.4), we obtain

(2.6) $\mathrm{A^2\,B^2}\ x^{2a+2b-1}[\,(\mathrm{a+b})\,f'^2-\mathrm{af\,f''}]=\upsilon\ \mathrm{A\,B^3\,f''}\ x^{a+3b}+\mathrm{g}\beta\ \mathrm{Nx^m}\ \theta$

(2.7) $\mathrm{ABNx^{a+b+m-1}}\,[\mathrm{m}\theta'-\mathrm{af}\theta'\,]=\alpha\ \mathrm{N\,B^2\,x^{m+2b}}\,\theta''$

Since equations (2.6) and (2.7) must hold for all values of x if similarity is to exist, we have

(2.8) $$\begin{cases} 2\mathrm{a}+2\mathrm{b}-1=\mathrm{a}+3\mathrm{b}=\mathrm{m}, \\ \mathrm{a}+\mathrm{b}+\mathrm{m}-1=\mathrm{m}+2\mathrm{b} \end{cases}$$

Solving equation (2.8), we get

$$\mathrm{a}=\frac{2}{3}\,,\ \ \mathrm{b}=\mathrm{m}=-\frac{1}{3}$$

Thus, we have

(2.9) $$\begin{cases} \psi=\mathrm{A}\ x^{\frac{2}{3}}\,\mathrm{f}(\eta),\ \eta=\mathrm{By}x^{-\frac{1}{3}} \\ \mathrm{T_w}-\mathrm{T}_\infty=\mathrm{N}x^{-\frac{1}{3}}\ \mathrm{u}=\mathrm{Ab}x^{\frac{1}{3}}\,\mathrm{f}', \\ \mathrm{V}=-\mathrm{A}x^{-\frac{1}{3}}\left(\frac{2}{3}\ \mathrm{f}-\frac{1}{3}\ \eta\mathrm{f}'\right) \\ \mathrm{T}-\mathrm{T}_\infty=\mathrm{N}x^{-\frac{1}{3}}\ \theta\ (\eta) \end{cases}$$

With this, we note that $V_w(x) = \frac{2}{3} Ax^{-\frac{1}{3}}$ fw , where $f_w = f(0)$.
The equations determining f and θ are

(2.10) $$f''' + \frac{2}{3} f f'' - \frac{1}{3} f'^2 + \theta = 0$$

(2.11) $$\theta'' + \left(\frac{2P_r}{3}\right) f\theta' + \left(\frac{P_r f'}{3}\right) \theta = 0 \ ;$$

subject to the conditions

(2.12) $$\begin{cases} \eta = 0 : f = f_w ,\ f' = 0,\ \theta = 1 \\ \eta = \infty : f' = 0,\ \theta = 0 \end{cases}$$

where the primes denote differentiation w. r. t. ‘η’.
The equations (2.10)–(2.12) have been obtained after properly choosing the dimensional constants A, B and α in the following manner :

(2.13) $$\begin{cases} A = \upsilon\, (gN\beta)^{1/4} \\ B = \upsilon^{-1/2} (gN\beta)^{1/4} \\ \alpha = \frac{K}{\rho C_p} \end{cases}$$

(2.14) $$P_r = \frac{\upsilon}{\alpha} = \frac{\mu C_p}{K}$$

3. Mathematical Analysis

If f(η) is the solution of (2.13), let us put

(3.1) $$h(\eta) = \varepsilon + f'$$

where ε is a constant. Then h (η) satisfies the equation

(3.2) $$h'' + \frac{2}{3} f h' + \frac{(\theta - \frac{1}{3} f'^2)}{(\varepsilon + f')} h = 0$$

In order to eliminate the middle term in (3.2), let us put

(3.3) $$h = x \exp\left(-\frac{1}{3} \int_0^{\eta} f(s)\, ds\right)$$

so that x(η) satisfies

(3.4) $$x'' - Q(\eta)\, x = 0,$$

where

$$(3.5)\quad Q(\eta) = \frac{1}{9} f^2 + \frac{1}{3} f' + \frac{\left(\frac{1}{3} f'^2 - \theta\right)}{(\varepsilon + f')} = \frac{1}{9} f^2 \left[\frac{1}{3}\,\frac{f'}{f} + 9\,\frac{\left(\frac{1}{3} f'^2 - \theta\right)}{f^2\,(\varepsilon + f')}\right],$$

Thus

$$Q'(\eta) = \frac{1}{3} f'' + \frac{2}{9}\, f f' + \frac{\frac{2}{3} f' f'' - \theta'}{[f'^2\,(\varepsilon + f')]} - \frac{\left(\frac{1}{3} f'^2 - \theta\right) f''}{(\varepsilon + f')^2}$$

and by (2.13),

$$Q''(\eta) = \left[\frac{1}{3} + \frac{\frac{2}{3} f'}{(\varepsilon + f')} + \frac{\frac{1}{3} f'^2 - \theta}{(\varepsilon + f')^2}\right]\left[-\frac{2}{3} f f'' - \theta + \frac{1}{3} f'^2\right] + \frac{2}{9} f'^2 + \frac{2}{9}\, f f''$$
$$+ \frac{\left(\frac{2}{3} f'' - \theta''\right)}{(\varepsilon + f')} - \frac{f''\left(\frac{2}{3} f' f'' - \theta'\right)}{(\varepsilon + f')^2} + \frac{2\left(\frac{1}{3} f'^2 - \theta\right) f''^2}{(\varepsilon + f')^3}.$$

From (2.12), $f' \sim 0$ as $\eta \to \infty$. Hence $f \sim C$ for large η, where C is a constant. Also $f''(\eta) > 0$ for all $\eta \in (0,\infty)$ by Hartman[20] (P.521). Hence for large η, there exists a constant $C' \neq 0$ such that

$$\frac{Q'^2}{Q^{5/2}} \le C' f'^2$$

$$\frac{|Q''|}{Q^{3/2}} \le C' f''$$

In addition, $\int^{\infty} f''^2\, d\eta$ and $\int^{\infty} f''\, d\eta$ converge absolutely as $f''(\eta) \to 0$ as $\eta \to \infty$.
Hartman[20] (P.521), so that

$$\int^{\infty} \frac{Q'^2}{Q^{5/2}}\, d\eta < \infty$$

$$\text{and} \int^{\infty} \frac{|Q''|}{Q^{3/2}}\, d\eta < \infty .$$

Hence the differential equation (3.4) has the principal solution $x(\eta)$ satisfying, as $\eta \to \infty$,

$$(3.6)\qquad x \sim C Q^{\frac{-1}{4}}(\eta)\ \exp\left(-\int^{\eta} Q^{1/2}(s)\, ds\right);$$

where $C \neq 0$ is a constant, while linearly independent solutions satisfy

(3.7) $$x \sim C Q^{\frac{-1}{4}}(\eta) \exp\left(\int^{\eta} Q^{1/2}(s)\, ds\right);$$

From (3.5) and f ~ C,

$$Q^{1/2}(\eta) = \frac{1}{3} f + \frac{1}{2}\left(\frac{f'}{f}\right) + \frac{3}{2}\left[\frac{\left(\frac{1}{3}f'^2 - \theta\right)}{(\varepsilon + f')}\right] + o(1),$$

$$Q^{1/4}(\eta) \sim \left(\frac{1}{3}\, C\right)^{1/2};$$

Therefore, solutions (3.6) and (3.7) becomes

$$x \sim C_1 \exp\left(-\int^{\eta}\left[\frac{1}{3} f + \frac{1}{2}\left(\frac{f'}{f}\right) + \frac{3}{2}\frac{\left(\frac{1}{3} f'^2 - \theta\right)}{(\varepsilon + f')}\right] ds\right);$$

$$x \sim C_1 \exp\left(\int^{\eta}\left[\frac{1}{3} f + \frac{1}{2}\left(\frac{f'}{f}\right) + \frac{3}{2}\frac{\left(\frac{1}{3} f'^2 - \theta\right)}{(\varepsilon + f')}\right] ds\right);$$

where $C_1 \neq 0$, $\eta \to \infty$.
In view of (3.3), the equation (3.2) has the principal solution satisfying

(3.8) $$\varepsilon + f' = h \sim C_1 \exp\left(-\int^{\eta}\left[\frac{2}{3} f + \frac{1}{2}\left(\frac{f'}{f}\right) + \frac{3}{2}\frac{\left(\frac{1}{3} f'^2 - \theta\right)}{(\varepsilon + f')}\right] ds\right)$$

$C_1 \neq 0$, while linearly independent solutions satisfy

(3.9) $$\varepsilon + f' = h \sim C_1 \exp\left(\int^{\eta}\left[\frac{1}{2}\left(\frac{f'}{f}\right) + \frac{3}{2}\frac{f\left(\frac{1}{3} f'^2 - \theta\right)}{(\varepsilon + f')}\right] ds\right) \text{ as } \eta \to \infty.$$

Taking $f \sim C$ and $f' \to 0$ as $\eta \to \infty$, solutions (3.8) and (3.9) become

(3.10) $$\varepsilon + f' = h \sim C_2 \exp\left(-\frac{2}{3}\, C_\eta\right),$$

(3.11) $$\varepsilon + f' = h \sim C_2;$$

$C_2 \neq 0$ as $\eta \to \infty$.
Similarly, the principal solution satisfying (2.14) will be

(3.12) $$\theta \sim C_3 \exp\left(-\frac{2}{3}\, P_r\, C_\eta\right),$$

$C_3 \neq 0$, while linearly independent solution will satisfy

(3.13) $$\theta \sim C_3;$$

$C_3 \neq 0$, as $\eta \to \infty$.

4. Results and Discussion

For studying the asymptotic behaviour as the similarity variable ‘η’ tends to infinity, we shall impose the conditions f', $\theta \to 0$ as $\eta \to \infty$ on the LHS of the principle and independent solutions. If the LHS behave in the similar fashion as the RHS do, the solutions will exhibit asymptotic nature as $\eta \to \infty$.

The LHS of (3.10) tends to ε as $\eta \to \infty$, where as its RHS tends to zero as $\eta \to \infty$. Hence if we choose ε in such a way that it is very near to zero but not zero, the principle solutions (3.10) show asymptotic nature as $\eta \to \infty$.

Similarly, if we apply the condition $f' \to 0$ as $\eta \to \infty$ on the LHS of (3.11), we get ε. The RHS of (3.11) is $C_2 \neq 0$. So if the constant $\varepsilon = C_2 \neq 0$, the independent solution (3.11) will also satisfy asymptotic nature as $\eta \to \infty$. Since ε is an arbitrary constant, hence it can always be chosen in such a way that $\varepsilon = C_2$. As a result, the independent solution (3.11) will also satisfy asymptotic characteristics as $\eta \to \infty$.

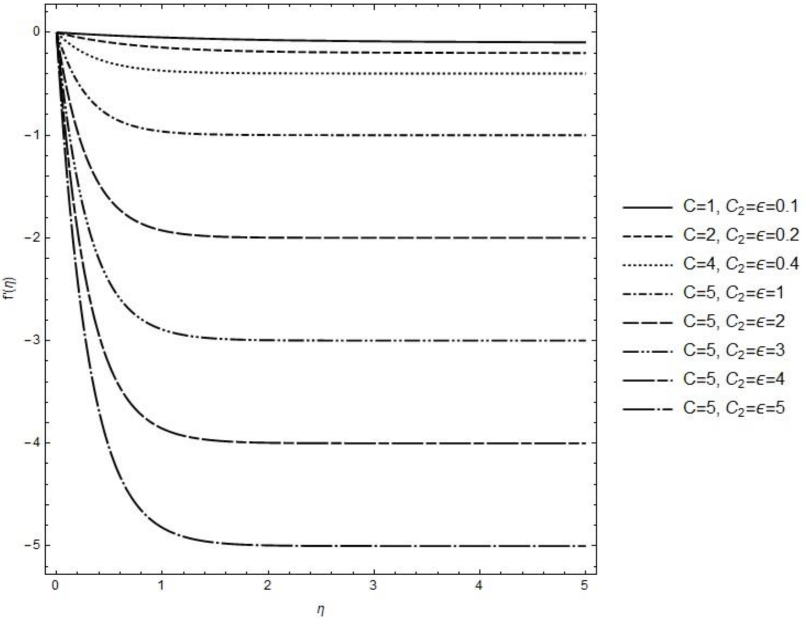

Figure 2. Variation of velocity profiles

If we apply the condition $\theta \to 0$ as $\eta \to \infty$ on the LHS of (3.12), it tends to zero. RHS of (3.12) also tends to zero as $\eta \to \infty$. So, the principle solution (3.12) will show asymptotic nature as $\eta \to \infty$. On the other hand, LHS of the independent solutions (3.13) tends to zero but its RHS is C3 $\neq 0$. Hence (3.13) will not show the asymptotic behavior as $\eta \to \infty$. Hence (3.13) will not show the asymptotic behaviour as $\eta \to \infty$.

In the Figure 2, the velocity profiles have been for different values of C, C2 and ε as $\eta \to \infty$. Figure 3 depicts Variation of temperature profiles for C3 = 1and different values of P_r. The decrease in dimensionless velocity has been observed with increasing values of C , C2 and ε. It is noteworthy to note that C= C2 and C3= 1 is obtained by relations (2.12).

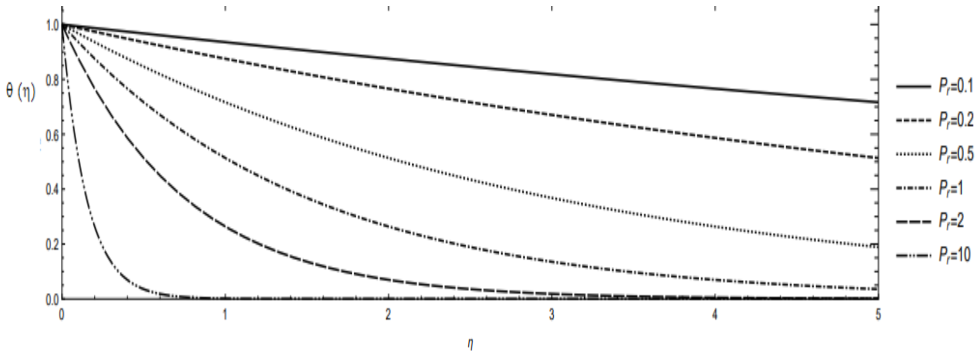

Figure 3. Variation of temperature profiles

5. Conclusions

The asymptotic integration method to find out the solutions of nonlinear boundary layer equations is the corner-stone of Applied Mathematics. This is a method to find the approximate solutions for velocity profiles for very large values of the independent variables. One of the other corner-stones of Applied Mathematics is scientific computing and it is interesting to note that these two subjects have grown together. However, this is not unexpected given their respective capabilities. By using computer, one is capable of solving problems that are non-linear, non-homogeneous and multidimensional. Moreover, it is possible to achieve very high accuracy. The drawbacks are that the computer solutions do not provide much insight into the physics of the problem, particularly for those who do not have access to the appropriate software or computer, and there is always the question as to whether or not the computed solution is correct. On the other hand, the asymptotic integration methods are also capable of dealing with non-linear, non-homogeneous and multidimensional problems. So, the main objective behind the use of the asymptotic integration method, at least as far as the author is concerned, is to provide reasonably accurate expression for the solution for large values of η. By doing this one is able to derive an understanding of the physics of the problem. Also, one can use the result in conjunction with the original problem, to obtain the more efficient numerical procedures for computing the solution.

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