On the Range of the Berezin Transform

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Abstract: Let D be the open unit disk in the complex plane C. Let $L_a^2(D)$ be the Bergman space of holomorphic functions on D that are in $L^2(D,dA)$ where $dA(z)=\frac{1}{\pi}dxdy$. Let $\mathcal{L}(L_a^2(D))$ be the space of all bounded linear operators from $L_a^2(D)$ into itself. Define a map σ from $\mathcal{L}(L_a^2(D))$ into $L^\infty(D)$ as $\sigma(T)(z)=\langle Tk_z,k_z\rangle$ where $\{k_z\}_{z\in D}$ are the normalized reproducing kernel for the Bergman space. The function $\sigma(T)$ is called the Berezin transform of T. It is not known what is the range of σ. In this paper we have shown if ξ is a bounded linear functional on S_p, the Schetten p-class, $1 \leq p < \infty$, then there exists a bounded linear operator $S\in\mathcal{L}(L_a^2(D))$ such that $\xi(P_z)\in$ Range σ and $\xi(P_z)=\langle Sk_z,k_z\rangle$ for all $z\in D$ where $P_zf=\langle f,k_z\rangle k_z$. Further $S\in S_q, \frac{1}{p}+\frac{1}{q}=1$ and $\|\xi\|=\|S\|_q$.

1. Introduction

Let $D=\{z\in C:|z|<1\}$ be the open unit disk in the complex plane C and $dA(z)=\frac{1}{\pi}dxdy$. Let $L_a^2(D)$ be the Bergman space of holomorphic functions on D that are in $L^2(D,dA)$. The reproducing kernel K(z,ω) of $L_a^2(D)$ is holomorphic in z and anti-holomorphic in ω, and

$$\int_D |K(z,\omega)|^2\, dA(\omega)=K(z,z)>0,$$

for all $z\in D$. Thus we can define for each $z\in D$, a unit vector $k_z\in L_a^2(D)$ by

$$k_z(\omega) = \frac{K(\omega, z)}{\sqrt{K(z,z)}}.$$

Let $£(L_a^2(D))$ be the set of all bounded linear operators from $L_a^2(D)$ into itself. Let $L^\infty(D)$ be the space of all essentially bounded measurable functions on D. Define a map σ from $£(L_a^2(D))$ into $L^\infty(D)$ as $\sigma(T)(z) = \langle Tk_z, k_z \rangle = \widetilde{T}(z)$, $z \in D$. Since $|\widetilde{T}(z)|$ = $|\sigma(T)(z)|$ = $|\langle Tk_z, k_z\rangle|$ ≤ $\|T\|$, the map σ is well defined. It is not known what is the range of σ. The function $\sigma(T)$ is called the Berezin transform of T.

Given $1 \le p < \infty$, we define the Schetten p-class of the Hilbert space H, denoted by $S_p(H)$ or simply S_p, to be the space of all compact operators T on H with its singular value sequence $\{\lambda_n\}$ belonging to l^p (the p-summable sequence space). It is known that S_p is a Banach space with the norm[1].

$$\|T\|_p = \left[\sum_n |\lambda_n|^p\right]^{1/p}.$$

The space S_I is also called the trace class of H. If T is in S_I, then the series $\sum_{k=1}^{\infty}\langle Te_k, e_k\rangle$ converges absolutely for any orthonormal basis $\{e_k\}$ of H and the sum is independent of the choice of the orthonormal basis. We call this value the trace of T and denote it by $tr(T)$.

Let $P_z(f) = \langle f, k_z\rangle k_z, f \in L_a^2(D)$. Then P_z is the rank one projection onto the span of k_z. For $T \in £(L_a^2(D)), TP_z$ is also of rank one and, therefore, belong to the trace class. Moreover,

$$tr(TP_z) = \langle TP_z k_z, k_z\rangle = \widetilde{T}(z).$$

Additivity of the trace now shows that

$$\widetilde{T}(z) - \widetilde{T}(\omega) = tr[T(P_z - P_\omega)].$$

2. The range of the map σ

The linear map $\sigma(T) = \widetilde{T}$ is defined and one-to-one on $£(L_a^2(D))$. The range of σ is not well understood. By our earlier discussion, it is a linear space of bounded functions on D.

Let Aut(D) be the Lie group of all automorphisms (biholomorphic mappings) of D. We can define for each $a \in D$ an automorphism φ_a in Aut(D) such that

(i) $\varphi_a o \varphi_a(z) \equiv z$;

(ii) $\varphi_a(0) = a, \varphi_a(a) = 0$;

(iii) φ_a has a unique fixed point in D. In fact $\varphi_a(z) = \dfrac{a-z}{1-\bar{a}z}$ for all a and z in D.

Theorem 2.1. *If $\varphi \in L^\infty(D)$ belongs to Range of σ then $\varphi o \varphi_a \in$ Range σ for all $a \in D$.*

Proof. Given $a \in D$ and $f \in L_a^2(D)$, we define a function $U_a f$ on D by

$$(U_a f)(z) = k_a(z) f(\varphi_a(z)).$$

Notice that U_a is a bounded linear operator on $L_a^2(D)$ for all $a \in D$. Further one can check[2] that $U_a^* = U_a$ and U_a is unitary. Now $\varphi \in$ Range σ implies there exists $T \in £(L_a^2)$ such that $\varphi(z) = \langle Tk_z, k_z \rangle$.

If $f \in L_a^2$, then

$$\langle f, U_a K_z \rangle = \langle U_a f, K_z \rangle = (U_a f)(z)$$

$$= (f o \varphi_a)(z) \varphi_a'(z) = \left\langle f, \overline{\varphi_a'(z)}\, K_{\varphi_z(z)} \right\rangle.$$

Thus

$$U_a K_z = \overline{\varphi_a'(z)} K_{\varphi_a(z)}.$$

Rewriting this in terms of the normalized reproducing kernels, we have

$$U_a k_z = \alpha k_{\varphi_a(z)}$$

for some complex constants α. Since $\|k_z\|_2 = \|k_{\varphi_a(z)}\| = 1$ and U_a is unitary, we have $|\alpha| = 1$. Thus $U_a k_z = \alpha k_{\varphi_a(z)}$ where $|\alpha| = 1$. Hence for all $z \in D$,

$$\langle U_a T U_a k_z, k_z \rangle = \langle T U_a k_z, U_a k_z \rangle = \left\langle T\alpha k_{\varphi_a(z)}, \alpha k_{\varphi_a(z)} \right\rangle$$

$$= |\alpha|^2 \left\langle T\alpha k_{\varphi_a(z)}, k_{\varphi_a(z)} \right\rangle = \varphi(\varphi_a(z)) = (\varphi o \varphi_a)(z).$$

Thus $\widetilde{U_a} \widetilde{T} \widetilde{U_a}(z) = (\varphi o \varphi_a)(z)$ for all $z \in D$. Hence $\varphi o \varphi_a \in$ Range σ.

We need to calculate $\|P_z - P_\omega\|_{trace}$, where $\|A\|_{trace} = trace(\sqrt{A^* A})$ to verify whether $\tilde{T} \in B \subset (D)$, the space of bounded continuous functions on D.

Theorem 2.2. *For $z, \omega \in D$,*

$$\|P_z - P_\omega\|_{trace} = 2\{1 - |\langle k_z, k_w \rangle|^2\}^{1/2}.$$

Proof. Let $(g\otimes h)f=\langle f,h\rangle g$, a rank one operator. Now

$$k_{\omega}=\langle k_{\omega},k_z\rangle k_z+h_{z,\omega}$$

with $\langle h_{z,\omega},k_z\rangle=0$ and

$$\begin{aligned}\|h_{z,\omega}\|^2&=\langle h_{z,\omega},h_{z,\omega}\rangle\\&=\langle k_{\omega}-\langle k_{\omega},k_z\rangle k_z,k_{\omega}-\langle k_{\omega},k_z\rangle k_z\rangle\\&=\|k_{\omega}\|^2-\overline{\langle k_{\omega},k_z\rangle}\langle k_{\omega},k_z\rangle-\langle k_{\omega},k_z\rangle\langle k_z,k_{\omega}\rangle+|\langle k_{\omega},k_z\rangle|^2\|k_z\|^2\\&=1-|\langle k_{\omega},k_z\rangle|^2.\end{aligned}$$

For $f\in L_a^2(D)$,

$$\begin{aligned}P_zf-P_{\omega}f&=\langle f,k_z\rangle k_z-\langle f,k_{\omega}\rangle k_{\omega}\\&=\langle f,k_z\rangle k_z-\langle f,\langle k_{\omega},k_z\rangle k_z-h_{z\omega}\rangle(\langle k_{\omega},k_z\rangle k_z-h_{z\omega})\\&=\langle f,k_z\rangle k_z-[\langle f,\langle k_{\omega},k_z\rangle k_z\rangle\langle k_{\omega},k_z\rangle k_z-\langle f,\langle k_{\omega},k_z\rangle k_z\rangle h_{zw}\\&\quad+\langle f,h_{z\omega}\rangle\langle k_{\omega},k_z\rangle k_z-\langle f,h_{z\omega}\rangle h_{z\omega}]\\&=(k_z\otimes k_z)f-|\langle k\omega,k_z\rangle|^2(k_z\otimes k_z)f+\overline{\langle k_{\omega},k_z\rangle}\langle f,k_z\rangle h_{z\omega}\\&\quad+\langle k_{\omega},k_z\rangle\langle f,h_{z\omega}\rangle k_z-\langle f,h_{z\omega}\rangle h_{z\omega}\\&=(1-|\langle k_{\omega},k_z\rangle|^2)(k_z\otimes k_z)f+\langle k_z,k_{\omega}\rangle(h_{z\omega}\otimes k_z)f\\&\quad+\langle k_{\omega},k_z\rangle(k_z\otimes h_{z\omega})f-(h_{z\omega}\otimes h_{z\omega})f\\&=\|h_{z\omega}\|^2(k_z\otimes k_z)f+\langle k_z,k_{\omega}\rangle(h_{z\omega}\otimes k_z)f\\&\quad+\langle k_{\omega},k_z\rangle(k_z\otimes h_{z\omega})f-(h_{z\omega}\otimes h_{z\omega})f.\end{aligned}$$

Since P_z-P_{ω} is self adjoint, another calculation shows that

$$(P_z-P_{\omega})^2=\|h_{z\omega}\|^2k_z\otimes k_{\omega}+h_{z\omega}\otimes h_{z\omega}.$$

Thus $P_z-P_{\omega}=0$ if and only if $h_{z\omega}=0$. For $h_{z\omega}\neq 0$ implies $(P_z-P_{\omega})^2$ is diagonal in any orthonormal basis including k_z and $\frac{h_{z\omega}}{\|h_{z\omega}\|}$ and has two nonzero eigenvalues, both equal to

$\|h_{z\omega}\|^2$. The positive square root of $(P_z - P_\omega)^2$ is then diagonal in the same basis and it follows that

$$\|P_z - P_\omega\|_{trace} = 2\|h_{z\omega}\| = 2\{1-|\langle k_z, k_\omega\rangle|^2\}^{1/2}.$$

Remarks 2.3. Since $P_z - P_\omega = 0$ implies that $|\langle k_z, k_\omega\rangle| = 1$ it is an easy consequence of the Cauchy-Schwarz inequality that $K(u,z) = \lambda K(u,\omega)$ for all $u \in D$ for some complex number λ.

Thus $\dfrac{1}{(1-\bar{z}u)^2} = \dfrac{\lambda}{(1-\bar{\omega}u)^2}$ for all $u \in D$.

Hence $z = \omega$. Subadditivity of $\|.\|_{trace}$ now implies that $d(z,\omega) = \|P_z - P_\omega\|_{trace}$ is a topological metric on D. Further notice that for any $\varphi_a \in Aut(D)$, we have

$$\|P_z - P_\omega\|_{trace} = \|P_{\varphi_a(z)} - P_{\varphi_z(\omega)}\|_{trace}.$$

This happens since

$$|\langle k_{\varphi_a(z)}, k_{\varphi_z(\omega)}\rangle|^2 = |\langle k_z, k_\omega\rangle|^2.$$

Theorem 2.4. *For $T \in \pounds(L_a^2(D))$,*

$$|\widetilde{T}(z) - \widetilde{T}(\omega)| \le 2\sqrt{2}\,\|T\|\,\rho(z,\omega),$$

where $\rho(z,\omega) = \left|\dfrac{z-\omega}{1-\bar{z}\omega}\right|$, the pseudohyperbolic metric on D.

Proof. We have already observed that $\widetilde{T}(z) - \widetilde{T}(\omega) = trace[T(P_z - P_\omega)]$. It is known[1], for $X \in S_1, T \in \pounds(L_a^2), TX \in S_1$ and $|trace(TX)| \le \|T\|\,\|X\|_{trace}$.

Thus $|\widetilde{T}(z) - \widetilde{T}(\omega)| \le 2\|T\|\,\{1-|\langle k_z, k_\omega\rangle|^2\}^{1/2}$.

By direct calculation, using $K(z,a) = \dfrac{1}{(1-\bar{a}z)^2}$, we see that

$$1-|\langle k_z, k_\omega\rangle|^2 = 1-\left|\left\langle k_z, \frac{K_\omega}{\|K_\omega\|}\right\rangle\right|^2 = 1-\frac{1}{\|K_\omega\|^2}|k_z(\omega)|^2 = 1-\frac{\left(1-|\omega|^2\right)^2\left(1-|z|^2\right)^2}{|1-\bar{z}\omega|^4}$$

$$= 1-\left(1-\frac{|z-\omega|^2}{|1-\bar{\omega}z|^2}\right)^2 = 1-\left(1+\frac{|z-\omega|^4}{|1-\bar{\omega}z|^4} - 2\frac{|z-\omega|^2}{|1-\bar{\omega}z|^2}\right) = -\frac{|z-\omega|^4}{|1-\bar{\omega}z|^4} + \frac{|z-\omega|^2}{|1-\bar{\omega}z|^2}$$

$$=\frac{|z-\omega|^2}{|1-\bar{\omega}z|^2}\left(2-\frac{|z-\omega|^2}{|1-\bar{\omega}z|^2}\right)=\frac{|z-\omega|^2}{|1-\bar{\omega}z|^2}\left(1+\frac{\left(1-|\omega|^2\right)\left(1-|z|^2\right)}{|1-\bar{\omega}z|^2}\right)\le 2\frac{|z-\omega|^2}{|1-\bar{\omega}z|^2}=2(\rho(z,\omega))^2.$$

Thus $\left\{1-|\langle k_z,k_\omega\rangle|^2\right\}^{1/2}\le\sqrt{2}\rho(z,\omega).$

Hence $|\widetilde{T}(z)-\widetilde{T}(\omega)|\le 2\sqrt{2}\,\|T\|\,\rho(z,\omega).$

Let $h^\infty(D)$ be the space of bounded harmonic functions on D and define the Toeplitz operator T_ψ from $L_a^2(D)$ into itself as $T_\psi f=P(\psi f)$, $\psi\in h^\infty(D)$, $f\in L_a^2(D)$ and P is the orthogonal projection from $L^2(D)$ onto $L_a^2(D)$. It is not so difficult to verify that $\|T_\psi\|\le\|\psi\|_\infty$ and $\widetilde{T}_\psi(z)=\langle T_\psi k_z,k_z\rangle=\psi(z)$ as $\psi\in h^\infty(D)$.

Corollary 2.5. *If* $\psi\in h^\infty(D)$, then for $z,\omega\in D$, we have

$$|\psi(z)-\psi(\omega)|\le 2\sqrt{2}\|\psi\|_\infty\rho(z,\omega).$$

Proof. Since $\widetilde{T}_\psi(z)=\psi(z)$, we have

$$|\psi(z)-\psi(\omega)|=|\widetilde{T}_\psi(z)-\widetilde{T}_\psi(\omega)|$$

$$\le 2\sqrt{2}\,\|T_\psi\|\,\rho(z,\omega)\le 2\sqrt{2}\,\|\psi\|_\infty\rho(z,\omega).$$

Theorem 2.6. *If* $1\le p<\infty,\frac{1}{p}+\frac{1}{q}=1,\xi$ *is a bounded linear functional on* S_p *then there is an operator* S *in* S_q *such that* $\xi(P_z)=\langle Sk_z,k_z\rangle$ *for all* $z\in D$. *Further,* $\|\xi\|=\|S\|_q$.

Proof. For $f\in L_a^2, z\in D$, let $P_zf=\langle f,k_z\rangle k_z$ for any f and g in $L_a^2(D)$, let $L_{f,g}$ be the rank one operator defined by $L_{f,g}h=\langle h,g\rangle f, h\in L_a^2(D)$. Let $\Omega(f,g)=\xi(L_{f,g})$. It is easy to see that L is linear in f and conjugate linear in g and $|\Omega(f,g)|\le\|\xi\|\,\|L_{f,g}\|_p=\|\xi\|\,\|f\|\,\|g\|$. It follows[1] that there is a bounded linear operator S on $L_a^2(D)$ such that $\Omega(f,g)=\langle Sf,g\rangle$ for all f and g in $L_a^2(D)$. Hence

$$\xi(P_z)=\xi(L_{k_z,k_z})=\langle Sk_z,k_z\rangle=tr(P_zS).$$

This implies that $\xi(T)=tr(TS)$ for all finite rank operator T and hence

$$\|S\|_q=\sup\{|tr(TS)|:\|T\|_p=1,\ T\text{ has finite rank }\}$$

$$= \sup\{|\xi(T)| : \|T\|_p = 1, \ T \text{ has finite rank }\} = \|\xi\|$$

since S_p is generated by finite rank operators.

References

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