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On Geodesics of a Lagrange Space and Associated Riemannian Space

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Abstract. The purpose of the present paper is to study the relation between the geodesics of Lagrange space corresponding to the generalized Lagrange space and associated Riemannian space .

1. Introduction

In 1989 T. Kawaguchi and R. Miron[1] gave a class of generalized Lagrange space $M_n = (M, g_{ij}(x, y))$ where $g_{ij}(x, y) = \gamma_{ij}(x) + (1/c^2)\, y_i y_j$, γ_{ij} being a Riemannian metric on the n-dimensional differential manifold M and $y_i = \gamma_{ij}(x) y^j$.

J. L. Synge[2] and M. C. Chaki and B. Barua[3] used the metric

$$g_{ij}(x, V(x)) = \gamma_{ij}(x) + \left(1 - \frac{1}{\eta^2(x, V(x))}\right) V_i V_j,$$

which occurs in the study of relativistic optics[4]. In this case (x) is a generic point, $V^i(x)$ is the velocity vector of the point and $\eta(x, V(x))$ is the refractive index of the optical medium. If, in particular $\eta(x, V(x)) = 1$, the medium is transparent. Also, if the refractive index is independent of velocity, i.e. if $\eta = \eta(x)$, then the optical medium is called non dispersive.

The purpose of the present paper is to study the relation between the geodesics of a Lagrange space corresponding to the generalized Lagrange space $M^n = (M, g_{ij}(x, y))$, where

$$g_{ij}(x, y) = \gamma_{ij}(x) + \left(1 - \frac{1}{\eta^2(x)}\right) y_i y_j,$$

and the associated Riemannian space $V^n = (M, \gamma_{ij}(x))$.

2. Preliminaries

Let M be an n-dimentional differentiable manifold, (TM, π, M) its tangent bundle and (x^i, y^i) (i, j, ...= 1, 2, ..., n) the canonical coordinates of the point u$\in$ TM, $\pi(u) = x$ in a coordinate neighbourhood $\pi^{-1}(U)$, where U is a coordinate neighbourhood of M at x i.e. $x \in U \subset M$. Let us consider a Riemannian metric $\gamma_{ij}(x)$ on M and the Riemannian space $V^n = (M, \gamma_{ij}(x))$.

The Liouville vector field $y = y^i\, \partial/\partial y^i$ is globally defined on the total space TM. Thus the covector field

(2.1)
$$y_i = \gamma_{ij}(x)\, y^j$$

is globally defined on TM, and also the square of the norm of y and the functions a_σ is defined respectively by

(2.2)
$$\|y\|^2 = \gamma_{ij}(x) y^i y^j,$$

(2.3)
$$a_\sigma(x,y) = 1 + \sigma\left(1 - \frac{1}{\eta^2(x)}\right)\|y\|^2. \quad \sigma \in N \text{ (the set of natural numbers).}$$

On TM we can consider the d-tensor field

(2.4)
$$g_{ij}(x,y) = \gamma_{ij}(x) + \left(1 - \frac{1}{\eta^2(x)}\right) y_i y_j,$$

Proposition (2.1). *The d-tensor field $g_{ij}(x, y)$ defined by (2.4) is symmetric non degenerate (i.e. rank $\| g_{ij}(x, y)\| = n$) and globally defined on TM.*

Indeed , the matrix $\| g_{ij}(x, y)\|$ has the inverse matrix $\| g^{ij}(x, y)\|$ given by

(2.5)
$$g^{ij}(x,y) = \gamma^{ij}(x) - \frac{1}{a_1(x,y)}\left(1 - \frac{1}{\eta^2(x)}\right) y^i y^j,$$

By the virtue of proposition (2.1) we can cosider the generalized Lagrange space $M^n = (M, g_{ij}(x, y))$, with the fundamental metric function $g_{ij}(x, y)$ given by (2.4).

We say that the generalized Lagrange space $M^n = (M, g_{ij}(x, y))$ is reducible to a Lagrange space if there is a Lagrangian L: TM\{0}$\to$ R with the property $g_{ij}(x, y) = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^i \partial y^j}$.

When the Lagrangian L(x, y) is 2-homogeneous with respect to y^i we say that M^n is reducible to a Finsler space. Finally if $g_{ij}(x, y)$ does not depend on y^i we say that M^n is reducible to a Riemannian space.

Theorem (2.1). *The generalized Lagrange space M^n is not reducible to a Lagrange space, neither to a Finslerian space nor to a Riemannian space.*

Proof. Let C_{jhk} be the d-tensor field

$$C_{jhk} = g_{hr}C^r_{jk} = \frac{1}{2}\left(\frac{\partial g_{jh}}{\partial y^k} + \frac{\partial g_{kh}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h}\right).$$

By (2.4) we get

$$C_{jhk} = \left(1 - \frac{1}{\eta^2(x)}\right)\gamma_{jk}y_h \tag{2.6}$$

From here we find that the d-tensor field C_{jhk} is not ever totally symmetric for $y^i \neq 0$. So M^n is not reducible to a Riemannian space or to a Finslerian or a Lagrange space.

Now considering the d-tensor field $C^i_{jk} = g^{ih}C_{jhk}$ from (2.5) and (2.6) we have

$$C^i_{jk} = \frac{1}{a}\left(1 - \frac{1}{\eta^2(x)}\right)\gamma_{jk}y^i . \tag{2.7}$$

Thus we have

Proposition (2.2) (a). *The coefficients of the vertical part of the canonical metrical d-connection are given by (2.7).*

(b) The following identity also holds good:

$$g_{ij}|_k = 0.$$

(c) The d-tensor field of torsion S^i_{jk} vanishes where $S^i_{jk} = C^i_{jk} - C^i_{kj}$.

3. Absolute energy of the space M^n

Let $\varepsilon(x, y)$ be absolute energy of the metric $g_{ij}(x, y)$

(3.1) $$\varepsilon(x, y) = g_{ij}(x, y)\, y^i y^j = a_1\, \|y\|^2.$$

Proposition (3.1) *The absolute energy is globally defined on TM.*

Theorem (3.1) *The space $\varepsilon^n = (M^n, \ \varepsilon(x, y))$ is a Lagrange space.*

Proof. We shall show that d-tensor field

$$g^*_{ij}(x,y) y^i y^j = \frac{1}{2}\frac{\partial^2 \varepsilon}{\partial y^i \partial y^j}$$

is non-degenerate. Indeed by (2.4) and (3.1), we have

(3.2) $$g^*_{ij} = a_2 \gamma_{ij} + 4\left(1 - \frac{1}{\eta^2}\right) y_i y_j$$

and taking into account $a_\sigma(x,y) \neq 0, \ \forall \sigma \in N$ the inverse matrix of $\|g^*_{ij}(x,y)\|$ has the elements

(3.3) $$g^{*ij} = \frac{1}{a_2}\left[\gamma^{ij} - \frac{4}{a_6}\left(1 - \frac{1}{\eta^2}\right) y^i y^j\right]$$

Now one can consider the action integral on a curve C:[0, 1]→M as

(3.4) $$I(C) = \int_0^1 \varepsilon\left(x, \frac{dx}{dt}\right) dt,$$

solution of which will give the Euler-Lagrange equation

(3.5) $$-\frac{\partial \varepsilon}{\partial x^k} + \frac{d}{dt}\left(\frac{\partial \varepsilon}{\partial y^k}\right) = 0, \qquad \left(y^k = \frac{dx^k}{dt}\right).$$

The equations (3.5) are of geodesics of ε^n and these are given by the system of differential equations

(3.6) $$\frac{d^2x^i}{dt^2} - 2G^{*i}\left(x, \frac{dx}{dt}\right) = 0.$$

where

(3.7) $$G^{*i} = \frac{1}{4}g^{*ij}\left(\frac{\partial^2 \varepsilon}{\partial y^j \partial x^k}y^k - \frac{\partial \varepsilon}{\partial x^j}\right)$$

Now from equation (3.1) we have

(3.8) $$\frac{\partial \varepsilon}{\partial x^j} = 2a_2 \frac{\gamma_{rs}}{\partial x^j} y^r y^s + \frac{2}{\eta^3}\eta_j \| y \|^2 .$$

(3.9) $$\frac{\partial^2 \varepsilon}{\partial y^j \partial x^k} = 8\left(1 - \frac{1}{\eta^2}\right)\frac{\gamma_{rs}}{\partial x^k} y^r y^s + 2a_2\left\{[j,sk]y^s + [r,jk]y^r\right\} + \frac{8}{\eta^3}\eta_k y_j \| y \|^2 .$$

wherer $\eta_j = \frac{\partial \eta(x)}{\partial x^j}$. From (3.8) and (3.9) we have

(3.10) $$\left(\frac{\partial^2 \varepsilon}{\partial y^j \partial x^k}y^k - \frac{\partial \varepsilon}{\partial x^j}\right) = 8\left(1 - \frac{1}{\eta^2}\right)[r,sk]y^r y^s y^k + 2a_2 [j,sk]y^s y^k$$

$$+ \frac{8}{\eta^3}\eta_k y^k y_j \| y \|^2 - \frac{2}{\eta^3}\eta_j \| y \|^4$$

From equation (3.3), (3.7) and (3.10) we have

(3.11) $$2G^{*i} = \left\{ {}^{\,i}_{j,k} \right\} y^j y^k + \frac{1}{a_2 \eta^3}\| y \|^2 \left(\frac{4a_3}{a_6}(\eta_k y^k) y^i - \eta^i \| y \|^2\right) .$$

Using (3.6) and (3.11) it follows that the geodesics of Lagrange space $\varepsilon^n = (M^n, \varepsilon(x, y))$ will be the geodesic of the associated Riemannian space $V^n = (M, \gamma_{ij}(x))$ if and only if

(3.12) $$\frac{1}{a_2 \eta^3}\| y \|^2 \left(\frac{4a_3}{a_6}(\eta_k y^k) y^i - \eta^i \| y \|^2\right) = 0.$$

Since $||y||^2 \neq 0$, we have $4a_3(\eta_k y^k)y^i = a_6\eta^i ||y||^2$, which after contraction with y_i gives $4a_3 = a_6$ or $\eta_k y^k = 0$. But $4a_3 = a_6$ but $a_2 = 0$, which is not possible, therefore $\eta_k y^k = 0$. Differentiating this with respect to y^i we have $\eta_i = 0$ i.e. η is constant.

On the other hand if η is constant then (3.12) will be satisfied identically. Hence we have the following theorem:

Theorem (3.2). *The geodesics of the Lagrange space $\varepsilon^n = (M^n, \varepsilon(x, y))$ will be the geodesic of the associated Riemannian space $V^n = (M, \gamma_{ij}(x))$ if and only if η is constant.*

References

1. T. Kawaguchi, and R. Miron: On the generalized Lagrange spaces with the metric $g_{ij}(x, y) = \gamma_{ij}(x) + (1/c^2) y_i y_j$, *Tensor, N.S.*, **48** (1989) 153-168.

2. J.L. Synge:Relativity: The general theory, North Holland Publishing Comp., Amsterdam.

3. M.C.Chaki and B. Barua: Conformal symmetries of Synge metric in generalized Lagrange space, *Bull. Cal. Math. Soc.*, **94(4)** (2002) 325-328.

4. R. Miron and T. Kawaguchi: Relativistic geometric optics, *Int. Jour. Theor. Phys.* **30** (1991) 1521.

5 R. Miron and M. Anastasiei: Vector bundles and Lagrange spaces with application to relativity, *Edtura Academiei R.S.Romania*, 1987.