Vol. 9 (2005) **Journal of International Academy of Physical Sciences** pp.39 – 45

Integrability Conditions of a Para Framed Manifold

S.B. Pandey and M. Pant

Department of Mathematics, Kumaun University, S.S.J. Campus, Almora

(Received December 23, 2004)

Abstract: In this paper, the integrability conditions of a Para Framed manifold have been obtained and some related results are derived. Few lemmas have also been discussed which are used in subsequent theorems.

1. Introduction

Let V_n ($n = r + s$ and r even), be a manifold wiht F-structure of rank r. Let there exist on V_n, s vector field $\underset{x}{T}$ and s 1-forms $\overset{x}{A}$, such that

(1.1)a $$F^2 - I_n = -\overset{x}{A} \otimes \underset{x}{T}$$

(1.1)b $$F^2(\underset{x}{T}) = 0$$

(1.1)c $$\overset{x}{A}(\underset{y}{T}) = \overset{x}{\underset{y}{\delta}} = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases}$$

Then we say that F-structure has complemented frames and V_n is said to be a Globally Para Framed F-manifold simply a Para Framed manifold.

2. Integrability Conditions

Theorem (2.1). *The necessary and sufficient conditions for V_n ($n = r + s$ and r even), be a Para Framed manifold is that it contains tangent bundle π_m of dimension m, a tangent bundle $\tilde{\pi}_m$ complex conjugate to π_m and a real line π_s, such that*

$$\pi_m \cap \pi_s = \phi, \quad \pi_m \cap \tilde{\pi}_m = \phi, \quad \tilde{\pi}_m \cap \pi_s = \phi,$$

and $\pi_m \cup \tilde{\pi}_m \cup \pi_s = a$ tangent bundle of dimension $n = 2m+s$,

projections l,m,n on π_m, $\tilde{\pi}_m$, π_s being given by

(2.1)a $$2\ell \overset{def}{=} -F^2 - F$$

(2.1)b $$2m \overset{def}{=} -F^2 + F$$

(2.1)c $$n \overset{def}{=} F^2 - I_n = -\overset{x}{A} \otimes \underset{x}{T}$$

Proof. Let V_n (n = 2m+s) be a Para Framed manifold with the Para Framed structure $\{F, \underset{x}{T}, \overset{x}{A}\}$. Corresponding to the eigen value 1, let $\underset{\alpha}{P}$ (α = 1,2,....m) be m-linealy independaent eigne vectors. $\underset{\alpha}{S}$ be complex conjugate to $\underset{\alpha}{P}$. Corresponding to eigen value 0, let $\underset{x}{T}$ (x = 1,2,....s) be s- linearly independent eigen vectors.

$$\overset{\alpha}{a}\underset{\alpha}{p} = 0 \Rightarrow \overset{\alpha}{a} = 0 \forall \alpha,$$

$$\overset{\alpha}{b}\underset{\alpha}{s} = 0 \Rightarrow \overset{\alpha}{b} = 0 \forall \alpha,$$

$$\overset{x}{h}\underset{x}{T} = 0 \Rightarrow \overset{x}{h} = 0 \forall x,$$

Now

$$\overset{\alpha}{c}\underset{\alpha}{P} + \overset{\alpha}{d}\underset{\alpha}{S} + \overset{x}{e}\underset{x}{T} = 0$$

$$\Rightarrow \overset{\alpha}{c}(\overline{\underset{\alpha}{P}}) + \overset{\alpha}{d}(\overline{\underset{\alpha}{S}}) = 0 \Rightarrow \overset{\alpha}{C}\underset{\alpha}{P} - \overset{\alpha}{d}\underset{\alpha}{S} = 0$$

$$\Rightarrow \overset{\alpha}{c}\underset{\alpha}{P} + \overset{\alpha}{d}\underset{\alpha}{S} = 0 .$$

All these equations imply

$$\overset{x}{e}\underset{x}{T} = \overset{\alpha}{c}\underset{\alpha}{P} = \overset{\alpha}{d}\underset{\alpha}{s} = 0$$

or $$\overset{x}{e} = \overset{\alpha}{c} = \overset{\alpha}{d} = 0 \forall \alpha \text{ and x.}$$

Thus $\{\underset{\alpha}{P}, \underset{\alpha}{S}, \underset{x}{T}\}$ is a linearly independent set. From the euations (2.1), it can be easily seen that

(2.2)a $\ell \underset{\alpha}{P} = -\underset{\alpha}{P}$, (2.2)b $\ell \underset{\alpha}{S} = 0$, (2.2)c $\ell \underset{x}{T} = 0$,

(2.3)a $m \underset{\alpha}{P} = 0$, (2.3)b $m \underset{\alpha}{S} = -\underset{\alpha}{S}$, (2.3)c $m \underset{x}{T} = 0$, (2.4)a $\underset{\alpha}{nP}=0$, (2.4)b $\underset{\alpha}{nS}=0$, (2.4)c $\underset{x}{nT}=-\underset{x}{T}$,

Thus, we prove that on a Para Framed manifold V_{2m+s} there is a tangent bunlde π_m of dimension m, a tangent bundle $\tilde{\pi}_m$ complex conjugate to π_m and a real line π_s, such that

$$\pi_m \cap \tilde{\pi}_s = \phi,\ \pi_m \cap \tilde{\pi}_m = \phi,\ \tilde{\pi}_m \cap \pi_s = \phi,$$

and $\pi_m \cup \tilde{\pi}_m \cup \pi_s$ = a tangent bundle of dimension 2m+s, projections on π_m, $\tilde{\pi}_m$, π_s being ℓ, m and n respectively.

Conversely, suppose that there is a tangent bundle π_m of dimension m, a tangent bundle $\tilde{\pi}_m$ conjugate to π_m and real line π_s, such that

$$\pi_m \cap \tilde{\pi}_m = \phi,\ \pi_m \cap \pi_s = \phi,\ \tilde{\pi}_m \cap \pi_s = \phi,$$

$\pi_m \cup \tilde{\pi}_m \cup \pi_s$ = a tangent bundle of dimension 2m+s.

Let $\underset{\alpha}{P}$ be m-linearly independent vectors in π_m and $\underset{\alpha}{S}$, complex conjugate to $\underset{\alpha}{P}$, be m-linearly independent vectors in $\tilde{\pi}_m$ and $\underset{x}{T}$ be s-linearly independent vactors in π_s. Let $\{\underset{\alpha}{P}, \underset{\alpha}{S}, \underset{x}{T}\}$ span a tangent bundle of dimension 2m+s. Then $\{\underset{\alpha}{P}, \underset{\alpha}{S}, \underset{x}{T}\}$is a linearly independent set. Let us define the inverse set $\{\overset{\alpha}{P}, \overset{\alpha}{S}, \overset{x}{A}\}$, such that

(2.5)a $$I_n = \overset{\alpha}{P} \otimes \underset{\alpha}{P} + \overset{\alpha}{S} \otimes \underset{\alpha}{S} + \overset{x}{A}(x)\underset{x}{T},\ \text{or}$$

(2.5)b $$X = \overset{\alpha}{P}(X)\underset{\alpha}{P} + \overset{\alpha}{S}(X)\underset{\alpha}{S} + \overset{x}{A}(x)\underset{x}{T},$$

Let us put

(2.6)b $$F \overset{def}{=} \{\overset{\alpha}{P} \otimes \underset{\alpha}{P} - \overset{\alpha}{S} \otimes \underset{\alpha}{S}\}$$

(2.6)b $$FX \overset{def}{=} \{\overset{\alpha}{P}(X)\underset{\alpha}{P} - \overset{\alpha}{S}(X)\underset{\alpha}{S}\},\ \text{then}$$

(2.7) $$FFX = \{\overset{\alpha}{P}(FX)\underset{\alpha}{P} - \overset{\alpha}{S}(FX)\underset{\alpha}{S}\},$$

From the equation (2.5)b and (2.6)b, we get

(2.8)a $$\overset{\alpha}{P}(FX) = \overset{\alpha}{P}(X)$$

(2.8)b $$\overset{\alpha}{S}(FX) = -\overset{\alpha}{S}(X),$$

(2.8)c $$\overset{x}{A}(FX) = 0$$

Using the equations (2.8)a and (2.8)b in (2.7), we get

(2.9) $$FFX = \{\overset{\alpha}{P}(X)\underset{\alpha}{P} - \overset{\alpha}{S}(X)\underset{\alpha}{S}\}$$

From the equations (2.5)b and (2.9), we get

(2.10) $$FFX = \{X - \overset{x}{A}(X)\underset{x}{T}\}$$

Thus, we see that V_{2m+s} admits a Para Framed structure $\{F, \underset{x}{T}, \overset{x}{A}\}$. Hence the condition is sufficient.

Corollary (2.1). we have

(2.11)a $$\ell = -\overset{\alpha}{P} \otimes \underset{\alpha}{P},$$

(2.11)b $$m = -\overset{\alpha}{S} \otimes \underset{\alpha}{S},$$

(2.11)c $$n = -\overset{x}{A} \otimes \underset{x}{T},$$

Proof. From the equations (2.1)a and (2.1)b, we have

(2.12) $$F = -(\ell - m).$$

Since 1 and -1 are eigen values of F, corresponding eigen vactors being $\underset{\alpha}{P}$ and $\underset{\alpha}{S}$. we have from the (2.6)a

(2.13) $$F = \{\overset{\alpha}{P} \otimes \underset{\alpha}{P} - \overset{\alpha}{S} \otimes \underset{\alpha}{S}\}.$$

Comparing the equations (2.12) and (2.13), we get the equations (2.11)a and (2.11)b, Equations (2.1) yield

(2.14) $$\ell + m + n = -I_n.$$

Equation (2.11)c is obtained from the equations (2.5)a, (2.11)a, (2.11)b and (2.14).

Corollary (2.2). We have

(2.15) $$\ell m = m \ell = n \ell = mn = nm = 0,$$

(2.16)a $$\ell^2 = -\ell$$

(2.16)b $$m^2 = -m$$

(2.16)c $$n^2 = -n$$

Proof. From the equations (2.11)b and (2.2)b, we have

$$\ell m = -\overset{\alpha}{S} \otimes \ell \underset{\alpha}{S} = 0.$$

Other equations of (2.15) follow the same pattern. Equations (2.11)a and (2.2)a yield

$$\ell^2 = -\overset{\alpha}{P} \otimes \ell \underset{\alpha}{P} = \overset{\alpha}{P} \otimes \underset{\alpha}{P} = -\ell .$$

Equations (2.16)b and (2.16)c follow the same pattern.

Lemma (2.1). We have

(2.17)a $$(d\ell)(nX, nY) = 0,$$

(2.17)b $$(d\overset{\alpha}{P})(nX, nY) = 0.$$

Proof. Using the equations (2.11)a and (2.11)c in

$(d\ell)(nX, nY) = -\ell[nX, nY]$, we obtain

$$(d\ell)(nX, nY) = -(\overset{\alpha}{P} \otimes \underset{\alpha}{P})[\overset{x}{A}(X)\underset{x}{T}, \overset{x}{A}(Y)\underset{x}{T}]$$

$$= \overset{x}{A}(X)\overset{x}{A}(Y)\overset{\alpha}{P}([\underset{x}{T}, \underset{x}{T}])\underset{\alpha}{P}$$

$$= 0$$

The proof of the equation. (2.17)b follows the same pattern.

Lemma(2.2). We have

(2.18)a $$2(dm)(\ell X, \ell Y) = F^2[\ell X, \ell Y] - F[\ell X, \ell Y],$$

(2.18)b $$8(dm)(\ell X, \ell Y) = F^2 N[FX, FY] - FN[FX, FY].$$

Proof. In the consequence of the equations (2.1) and (2.15), we have

$$2(dm)(\ell X, \ell Y) = -2m[\ell X, \ell Y]$$

$$= -\{-F^2[\ell X, \ell Y] - F[\ell X, \ell Y]\}$$

$$= F^2[\ell X, \ell Y] - F[\ell X, \ell Y]\}.$$

Also we have

(2.19) $$8(dm)(\ell X, \ell Y) = -2m[2\ell X, 2\ell Y]$$

$$= F^2[F^2X, F^2Y] + F^2[F^2X, F^2Y]$$

$$+ F^2[F^2X, F^2Y] + F^2[F^2X, F^2Y]$$

$$- F[F^2X, F^2Y] - F[F^2X, FY]$$

$$- F[FX, F^2Y] - F[FX, FY]$$

Now, using the equations (1.1)a, (1.1)b and (1.2) in (2.19), we get the equation (2.18).

Lemma (2.3). We have

(2.20) $$(dn)(\ell X, \ell Y) = \overset{x}{A}[\ell X, \ell Y]\underset{x}{T}$$

(2.21) $$4\,(dn)\,(\ell X, \ell Y) = \overset{x}{A}\,(N(X,\; Y))\,\underset{x}{T} + \overset{x}{A}\,(N(FX,\; FY))\,\underset{x}{T}$$

$$= \{\,\overset{x}{A}(N(FX,\; Y)\,\underset{x}{T} + \overset{x}{A}\,(N(X,\; FY)\,\underset{x}{T}\,\}$$

Proof. As in lemma (2.2)

$$(dn)(\ell X, \ell Y) = -n[\ell X, \ell Y].$$

Using the equation (2.1)c in this equation, we get the equation (2.20). Now

(2.22) $$4(\ell X, \ell Y) = [-F^2X - FX, -F^2Y - FY]$$

$$= (F^2X, F^2Y) + [F^2X, FY) + [FX, F^2Y] + [FX, FY]$$

Now, applying n in the equation (2.22) and using the equations (2.1)c, (1.1)d and (1.2), we get the equation (2.21).

Theorem (2.1). *The distribution of* π_s *is integrable*

Proof. From the equation (2.14), π_s is given by

(2.23)a $\ell = 0,$ (2.23)b $m = 0,$ (2.23)c $n = I_n$

In order that π_s is integrable, it is necessary and sufficient that

ℓ =0 and m = 0 be integrable, that is

$$(d\ell)(X,Y) = 0 \text{ and } (dm)(X,Y) = 0$$

be satisfied for any vector satisfying.

(2.23)b or X = nX.

Thus, we have

$$d\ell(nX,nY) = 0.$$

But , from lemma (2.1), we see that see that these equations are identically satisfied. Hence, we have the theorem.

Referemcces

1. S.I.Goldbeg and K. Yano: On normal Globally Framed F-manifold, *Tonhoku Math. J.*, 22(1970) 362-370.
2. S.I.Goldbeg: Globally Framed F-manifolds, *Illinois J. Math*. 15 (1971) 456-474.
3. S.I.Goldbeg: Framed manifold, differential geometry, In honour of Kentao Yano, Kinokunight, Tokyo (1972) 121-132.
4. C.J.Hsu: Note on the integrability of a certain structure on differentiable manifolds, *Tohoku Math. J.* 12 (1960) 349-360.
5. R.S. Mishra: Structures on a Diffeetiable Manifold and their applications, Chandrama Prakashan, 50-A, Balrampur House, Allahabad, 1948.
6. R.S. Mishra: *A course in tensors with applications to Riemannian Geometry*, Pothishala Pvt. Ltd., Allahabad, 1995.