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On Summabilty of Random Fourier - Stieltjes Series

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Abstract: If $X(t,\omega), t \in R$ is stable process of index $\alpha, 1 < \alpha \le 2$ then it is seen that the stochastic integral $\int_a^b f(t)dX(t,\omega)$ is defined in the sense of mean for $f \in L^\alpha$. Denote $a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)dt$ for $f \in L^\alpha, 1 < \alpha \le 2$ and $A_n(\omega) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} dX(t,\omega)$. In this paper it is established that the Random Fourier-Stieltjes (RFS) series $\sum_{n=-\infty}^{\infty} a_n A_n(\omega) e^{int}$ converges to the stochastic integral $\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u,\omega)$ in the sense of mean. Further it is proved that the RFS series $\sum_{n=-\infty}^{\infty} a_n A_n(\omega)e^{int}$ is Abel summable to $\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u,\omega)$.

1. Introduction

Let $X(t,\omega), t \in R$ be a continous stochastic process with independent increments and f be a continuous function in [a,b]. Then the stochastic integral $\int_a^b f(t)dX(t,\omega)$ is defned in the sense of convergence in probability and is a random variable (cf. Lukacs[1], p.148). If $X(t,\omega)$ is a symmetric stable process of index $\alpha, 1 \le \alpha \le 2$ then the stochastic integral $\frac{1}{2\pi}\int_0^{2\pi} f(t)dX(t,\omega)$ is defined in the sense of convergence in probability for $f \in L^p[0,2\pi], p \ge \alpha$ (cf. Nayak, Pattanayak, and Mishra[2]). Further if $X(t,\omega)$ is a stable process of index $\alpha, 1 \le \alpha \le 2$, and $f \in L^{\alpha}[0,2\pi]$ then the stochastic integral $\int_a^b f(t)dX(t\omega)$ is defined in the sense of mean (cf. Kwapien, Woyczynski[3]). Hence in particular

(1.1)
$$A_n(\omega) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}dX(t,\omega)$$

exists for the orthonormal set $\{e^{-int}\}_{n=-\infty}^{\infty}$ and is the Fourier - Stieltjes coefficients of $X(t,\omega)$.

The series $\sum_{n=-\infty}^{\infty} A_n(\omega)e^{int}$ is a Fourier - Stieltjes expansion of $X(t,\omega)$. No doubt $A_n(\omega)$ are not independent random variables. A series of the form

(1.2)
$$\sum_{n=-\infty}^{\infty} a_n A_n(\omega)e^{int}$$

with weights $(a_n)_{n=-\infty}^{\infty}$ is known as Random Fourier - Stieltjes (RFS) series. Convergence of the series (1.2) in the sense of probability were studied extensively by Mishra, Nayak and Pattanayak[4] and Nayak, Pattanayak, and Mishra[2]. They have shown that for a symmetric stable process of index $\alpha, 1 < \alpha < 2$, the RFS series (1.2), where $A_n(\omega)$ are defined as in (1.1), converges in probability to the stochastic intergral $\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u,\omega)$ for some $f \in L^p[0,2\pi], p \ge \alpha$ such that a_n are the Fourier cofficients of f.

If $X(t,\omega)$ is a symmetric stable process of index $\alpha, 1 < \alpha \le 2$, then it can also be shown (see Theorem 3.4) that the seris (1.2) converges in the sence of mean to the stochastic $\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u,\omega)$. If $f \in L^p[0,2\pi], p \ge 1$ and

(1.3)
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}f(t)dt n = \ldots -2,-1,0,1,2..$$

are the Fourier coefficients of f then it is easy to see that for each r with $0 \le r < 1$, the series

(1.4)
$$\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{int}$$

converges uniformaly and represents a continuous function on $[0,2\pi]$. Let us write

(1.5)
$$f_r(t) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{int}.$$

Then the stochastic integral

(1.6) $$\int_0^{2\pi} f_r(t)dX(t,\omega)$$

is defined. Since each $f_r(t)$ is continuous, it belongs to $L^p[0,2\pi]$ for all $p \geq 1$ and $a_n r^{|n|}$, $n = \ldots,-2,-1,0,1,2,\ldots$ are the Fourier coefficients of $f_r(t)$. So the random series

(1.7) $$\sum_{n=-\infty}^{\infty} a_n A_n r^n e^{int}$$

will converge to the stochastic integral

(1.8) $$\frac{1}{2\pi}\int_0^{2\pi} f_r(t-u)dX(u,\omega)$$

in the sense of convergence in mean. Here $f_r(\theta)$ is the harmonic extension of f to the disc $\{z : |z| < 1\}$ given by the Poission integral

(1.9) $$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta - t)+r^2} f(t)dt\ .$$

It can now be shown (see Theorem 3.5) that $\int_0^{2\pi} f_r(t-u)dX(u,\omega)$ converges to $\int_0^{2\pi} f(t-u)dX(u,\omega)$ as $r \to 1^-$ in the sense of mean. This would mean $\sum_{n=-\infty}^{\infty} a_n A_n e^{int}$ is Abel summable to $\int_0^{2\pi} f(t-u)dX(u,\omega)$. In fact a Fourier series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is said to be Abel Summable to s if for $0 \leq r < 1$, $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)r^n$ converges to s. In complex form it would mean- a Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{int}$ is said to be Abel summable to s if for each r with $0 \leq r < 1$, $\lim_{r\to 1}\sum_{n=-\infty}^{\infty} a_n r^n e^{int} = s$.

2. Definitions

Definition 2.1. *A sequence of random variables* $\{X_n\}_{n=1}^{\infty}$ *is said to converge in mean to a random variable X if*

$$\lim_{n\to\infty} E\,|\,X_n - X\,| = 0 .$$

Definition 2.2. *The class of functions f satisfying* $\int_a^b |\,f(x)\,|^p\,dx < \infty$ *is denoted by* $L^p[a,b]$.

Definition 2.3. *A series* $\sum_{n=0}^{\infty} u_n$ *is said to be Abel summable to s if for* $|\,r\,| < 1,\ \sum_{n=0}^{\infty} u_n r^n$ *converges to s as* $r \to 1^-$ *i.e.* $\lim_{r\to 1} \sum_{n=0}^{\infty} u_n r^n = s$.

3. Results

To prove that the series (1.2) converges to the stochastic integral (1.8) in the sense of mean, we require the following result.

Lemma 3.1. *If* $X(t,\omega)$ *is a symmetric stable process of index* α, $1 < \alpha \le 2$ *and* $f \in L^p[a,b], p \ge \alpha$ *then the following inequality holds:*

$$E\left(\,\left|\int_a^b f(t)dX(t,\omega)\right|\,\right) \le \frac{4}{\pi(\alpha-1)}\int_a^b |\,f(t)\,|^{\alpha}\,dt$$
$$+\frac{2}{\pi}\int_{|u|>1} \frac{1-\exp(-|\,u\,|^{\alpha}\int_a^b |\,f(t)\,|^{\alpha}\,dt)}{u^2}\,du.$$

Proof of this Lemma requires the following two results:

Lemma 3.2: *(c.f. Shiryayev[5], p.341) A stable random variable X(t) always satisfies the inequality* $E\,|\,X\,|^r < \infty$ for all $r \in (0,\alpha), 0 < \alpha \le 2$.

Lemma 3.3: *(c.f. chow, Teicher[6]) If* φ *is the characteristic function of a random variable X(t) then*

$$\int_{-\infty}^{\infty} |\,X\,|\,dF(X) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1-Re\varphi(t)}{t^2}\,dt.$$

Proof (of Lemma 3.1): Let the characteristic function of the increment $X(t+\tau,\omega)-X(t,\omega)$ be

$$exp(-|\tau||u|^{\alpha}), 1<\alpha\leq 2.$$

Then the random variable $Y = \int_a^b f(t)dX(t,\omega)$ will have characteristic function

$$\psi(u) = exp\left(-\int_a^b |u|^{\alpha}|f(t)|^{\alpha}\,dt\right), 1<\alpha\leq 2$$

as is established in Lukacs[1][p.150]. Lemma 3.2 implies that E|Y| exists as $1\in(0,\alpha), 1<\alpha\leq 2$. Using Lemma 3.3 we get

$$E|Y| = \int_{-\infty}^{\infty}|Y|\,dF(Y)$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1-\psi(u)}{u^2}du$$

$$= \int_{|u|\leq 1}\frac{1-\psi(u)}{u^2}du + \int_{|u|>1}\frac{1-\psi(u)}{u^2}du.$$

Now

$$\int_{|u|\leq 1}\frac{1-\psi(u)}{u^2}du = \int_{-1}^{1}\frac{1-exp\left(-|u|^{\alpha}\int_a^b |f(t)|^{\alpha}dt\right)}{u^2}du$$

$$\leq \int_{-1}^{1}|u|^{\alpha-2}\int_a^b |f(t)|^{\alpha}\,dt\;du(\because 1-e^{-x}<x \text{ for } x>0)$$

$$= 2\int_0^1 |u|^{\alpha-2}du\int_a^b |f(t)|^{\alpha}\,dt$$

$$= \frac{2}{\alpha-1}\int_a^b |f(t)|^{\alpha}\,dt.$$

So

$$E\left(\left|\int_a^b f(t)dX(t,\omega)\right|\right) \leq \frac{2}{\pi}\frac{2}{\alpha-1}\int_a^b |f(t)|^{\alpha}\,dt$$

$$+\frac{2}{\pi}\int_{|u|>1}\frac{1-\exp\left(-|u|^{\alpha}\int_{a}^{b}|f(t)|^{\alpha}dt\right)}{u^{2}}du$$

which proves the Lemma.

Theorem 3.4: *Let* $X(t,\omega), t\in R$ *be a symmetric stable process of index* α, $1<\alpha\le 2$ *with period* 2π *and*

$$A_n(\omega)=\frac{1}{2\pi}\int_0^{2\pi}e^{-int}dX(t,\omega),\ n\in Z.$$

Then the RFS series

(1.10) $$\sum_{n=-\infty}^{\infty}a_nA_ne^{int}$$

converges in mean to the stochastic integral

$$\frac{1}{2\pi}\int_0^{2\pi}f(t-u)dX(u,\omega)$$

for some $f\in L^p[0,2\pi]$, $p\ge\alpha$ *such that* α_n *are the Fourier coefficients of f.*

Proof: Let $S_n(t)=\sum_{k=-n}^{n}a_kA_ke^{ikt}$ be the partial sum of the RFS series

(1.10) and that of f be $s_n(t)=\sum_{k=-n}^{n}a_ke^{ikt}$.

$$S_n(t)=\sum_{k=-n}^{n}a_kA_ke^{ikt}$$

$$=\sum_{k=-n}^{n}a_k\left(\frac{1}{2\pi}\int_0^{2\pi}e^{-iku}dX(u,\omega)\right)e^{ikt}$$

$$=\sum_{k=-n}^{n}a_k\frac{1}{2\pi}\int_0^{2\pi}e^{-ik(t-u)}dX(u,\omega)$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=-n}^{n}a_ke^{ik(t-u)}dX(u,\omega)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} s_n(t-u)dX(u,\omega)$$

Now

$$E\left(\left|\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u,\omega) - S_n(t)\right|\right)$$

$$= E\left(\left|\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u,\omega) - \frac{1}{2\pi}\int_0^{2\pi} s_n(t-u)dX(u,\omega)\right|\right)$$

$$= E\left(\left|\frac{1}{2\pi}\int_0^{2\pi} f(t-u) - s_n(t-u)dX(u,\omega)\right|\right)$$

$$\le \frac{2}{\pi^2(\alpha-1)}\int_0^{2\pi} |f(t-u) - s_n(t-u)|^\alpha du$$

$$+\frac{1}{\pi^2}\int_{u>1}\frac{1-exp(-|v|^\alpha\int_0^{2\pi}|f(t-u)-s_n(t-u)|^\alpha du)}{v^2}dv \text{ (by Lemma 3.1)}$$

It is known (c.f. Zygmund[7] [p.266] that for $f \in L^p[0,2\pi]$, $p > 1$

$$\int_0^{2\pi} |f(t-u) - s_n(t-u)|^p du = 0.$$

Now if $p \ge \alpha$ then we have

$$\lim_{n\to\infty} E\left(\left|\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u,\omega) - S_n(t)\right|\right) = 0.$$

Hence it is proved that the RFS series (1.10) converges in mean to

$$\int_0^{2\pi} f(t-u)dX(u,\omega).$$

In the next theorem it is established that the RFS series $\sum_{n=-\infty}^{\infty} a_n A_n e^{int}$ is Abel summable to $\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u,\omega)$ in the sense of mean.

Theorem 3.5. *Let $X(t,\omega)$ be a symmetric stable process of index $\alpha, 1 < \alpha \le 2$, and $f(t)$ be in* $L^p[0,2\pi]$, $p \ge \alpha$. If $A_n(\omega) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} dX(t,\omega)$ *and* $a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)dt$, *then the series* $\sum_{n=-\infty}^{\infty} a_n A_n e^{int}$ *is Abel summable to* $\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u,\omega)$ *in the sense of mean.*

Proof: As we know, for each r with $0 \le r < 1$, the series $\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{int}$ converges uniformly and represents a continous function and hence belong to L^p, for all $p > 1$. Denote

$$f_r(t) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{int}, 0 \le r \le 1.$$

Since $f \in L^p$, $p \ge \alpha, a_n$ is its Fourier coefficient and $0 \le r < 1$, each f_r is in $L^p, p \ge \alpha$. So by Theorem 3.5 the random series $\sum_{n=-\infty}^{\infty} a_n A_n r^n e^{int}$ will converge to the stochastic integral $\frac{1}{2\pi}\int_0^{2\pi} f_r(t-u)dX(u\omega)$ in the sense of mean. Since the random Fourier series $\sum_{n=-\infty}^{\infty} a_n A_n e^{int}$ converges to the stochastic integral $\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u\omega)$ in the sense of mean we have

$$E\left|\frac{1}{2\pi}\int_0^{2\pi} f_r(t-u)dX(u\omega) - \frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u\omega)\right|$$

$$= E\left|\frac{1}{2\pi}\int_0^{2\pi}\left(f_r(t-u) - f(t-u)\right)dX(u\omega)\right|$$

$$\le \frac{2}{\pi^2(\alpha-1)}\int_0^{2\pi}|f_r(t-u) - f(t-u)|^\alpha du$$

$$+ \frac{1}{\pi^2}\int_{u>1}\frac{1-\exp(-|v|^\alpha \int_0^{2\pi}|f_r(t-u) - f(t-u)|^\alpha du)}{v^2}dv \text{ (by Lemma 3.1).}$$

We know that the integral

$$\int_0^{2\pi}|f_r(t-u) - f(t-u)|^\alpha du$$

tends to 0 as $r \to 1$ if $f \in L^p, p > 1$ (c.f. Zygmund[7] [p.150]. As $\frac{1}{u^2}$ in the integrand of the second integral is dominated by 1 the second integral also tends to 0. Thus it is proved that the RFS series $\sum_{n=-\infty}^{\infty} a_n A_n e^{int}$ is Abel summable to $\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u,\omega)$.

Note: The Abel summability of the random series $\sum_{n=-\infty}^{\infty} a_n A_n e^{int}$ as defined in (1.10) in the sense of probability can also be proved using the inequality

$$P\left(\left|\int_a^b f(t)dX(t,\omega)\right| > \delta\right) \le \frac{c.2^{\alpha+1}}{(\alpha+1)\delta'^{\alpha}}\int_a^b |f(t)|^{\alpha}\,dt,$$

for all $\delta > 0$, where $\delta' < \delta$ and c is a positive constant and the fact that the random series $\sum_{n=-\infty}^{\infty} a_n A_n e^{int}$ converges in the sense of probability to the stochastic integral $\frac{1}{2\pi}\int_0^{2\pi} f(t-u)dX(u,\omega)$ (cf. Nayak, Pattanayak, and Mishra[2]).

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