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A Class of Shrinkage Estimators of a Shape Parameter of Generalized Burr Distribution

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Abstract: In this paper, a class shrinkage estimators has been proposed for a shape parameter of the Generalized Burr Distribution by using the maximum likelihood estimator in the kernel. The proposed class of the estimators are compared with the maximum likelihood estimators in terms of the mean squared error and their effective intervals of dominance are obtained.

1. Introduction

As a member of Burr[1] family of distributions which includes twelve type of cumulative distributions with a variety of density shapes. The two parameter generalized Burr (type XII) distributions has pdf of the form.

(1.1) $$f(x;c,k) = ckx^{c-1}(1 + x^{c})^{-(k+1)} \; ; \; (c,k) > 0, \; x > 0$$

and its cdf is

(1.2) $$F(x;c,\theta) = 1 - (1+ x^{c})^{-k} \; ; \; (c,k) > 0, \; x > 0$$

where c and k are shape parameters.

The Burr (c,k) distribution was proposed as a life time model by Dubey[2,3]. The Burr distribution is a unimodel distribution as shown by Burr and Cislak[4] Rodriguez[5]and Tadikamalla[6] show that the Burr distribution covers the curve shape caracteristics for the Normal, Logistic and exponential (Pearson typeX) distribution as well as a significant portion of the curve shape characteristic for Pearson type I (beta), II, III (gamma), V, VII, IX and XII distributions. Lewis[7] noted that the Weibull and exponential distribution are special limiting cases of the parameter values of the Burr distribution. Wingo[8,9] has described the method for fitting the Burr distribution to life test data for complete and type II censored samples. Inferences based on Burr (c,k) distribution and some of its testing measures were made by Popadopoulos1[10]. Evans and Ragab[11], Lingappaiah[12], Al-Hussaini et al.[13]. In 1997 Anwar Hossain and Shyamal[14] studies the estimation of the parameters in the presence of outliers for the Burr XII distribution.

In this paper, a class shrinkage estiamators the of a shape parameter of Generalized Burr distribution have been proposed. It has been shown that the MLE is also the MVB estimator. Properties of these estimators have been studied with the help of mean squared errors.

We reparameterize the c.d.f. (1.2) to get the c.d.f. of generalized Burr distribution in the following form

(1.3) $$F(x) = 1-(1+x^c)^{-\frac{1}{\theta}} ; (c,\theta) > 0, 0 < x < \infty.$$

and its probability density function (pdf) comes out to be

(1.4) $$f(x;c,\theta) = \frac{c}{\theta} x^{c-1}(1+x^c)^{-\left(\frac{1}{\theta}+1\right)} ; c,\theta > 0, x > 0$$

where c and θ are the shape parametes. This reparameterization leads to mathematical tractibility in calculation.

Statistical properties:

The probability density function of the form (1.4) of GBD is unimodel with mode

(1.5) $$x_{mode} = \left[\frac{c-1}{(c/\theta)+1}\right]^{1/c}$$, if $c > 1$ and L-shaped if $c \leq 1$.

The rth moment of generalized Burr distribution is given by

(1.6) $$\mu_r' = E\left[X^r\right] = \frac{1}{\theta} B\left[\frac{1}{\theta} - \frac{r}{c}, \left(1+\frac{r}{c}\right)\right]$$

so that the fourth moment is finite if $\frac{c}{\theta} > r$ or $\left(\frac{c}{\theta} > 4\right)$. Therefor, the mean and variance are given by

(1.7) $$\text{Mean}(\mu_1') = \frac{1}{\theta} B\left[\frac{1}{\theta} - \frac{1}{c}, \left(1+\frac{1}{c}\right)\right]$$

(1.8) and $$\text{Variance}(\mu_2) = \frac{1}{\theta} B\left[\frac{1}{\theta} - \frac{2}{c}, \left(1+\frac{2}{c}\right)\right] - \left\{\frac{1}{\theta} B\left[\frac{1}{\theta} - \frac{1}{c}, \left(1+\frac{1}{c}\right)\right]\right\}^2$$

exists for $\frac{c}{\theta} > 4$

2. Maximum likelihood estimator

Let us consider a random sample n, $\underline{x} = (x_1, \ldots\ldots\ldots, x_n)$ from the p.d.f. (1.4) when c is known. The MLE is given by

(2.1)
$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} log(1+x_i^c)$$

We obtained the pdf of $\hat{\theta}$ as

(2.2)
$$f(\hat{\theta}) = \frac{\left(\frac{n}{\theta}\right)^n}{\Gamma(n)}(\hat{\theta})^{n-1} e^{-n\hat{\theta}\,\theta} \quad ;\hat{\theta} > 0$$

The log likelihood function may be written as

(2.3)
$$log\, f(\underline{x}\,|\,\theta) = n\, log\left(\frac{c}{\theta}\right) + log\left(\prod_{i=1}^{m} x_i^{c-1}\right) - \left(\frac{1}{\theta}+1\right)\sum_{i=1}^{n} log(1+x_i^c)$$

Diffentiating with respect to θ, we get

(2.4)
$$\frac{d}{d\theta} log\, f(\underline{x}\,|\,\theta) = \frac{n}{\theta^2}\left[\frac{1}{n}\sum_{i=1}^{n} log(1+x_i^c) - \theta\right]$$

Now, it is easy to verify that the regularity condition of Rao-Cramer inequality are satisfied by the p.d.f. (1.4) when the parameter c is known

Thus the estimator

(2.5)
$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} log(1+x_i^c)$$

is MVB (minimum variance bound) estimator and with the variance

(2.6)
$$\text{Var}(\hat{\theta}) = \frac{\theta^2}{n}$$

it is very easy to varify that $E[\hat{\theta}] = \theta$

We mnow that if MVB estimator exists, it exists for one and only one specific function of θ (Kendall and Stuart[15]). This consideration has led us to the repameterization of the p.d.f. in the form (1.4).

The idea of shrinkage estimator using the point guess value of the parameter was introduced by Thompson[16]. He suggests that a procedure, known as shrinkage technique and proposed an estimator T_1 of the parameter μ as

(2.7)
$$T_1 = k\hat{\mu} + (1-k)\mu_0 \qquad ;0 \le k \le 1$$

which is better than the uiformly minimum variance unbaised estimator (UMVUE) under squared error loss criterion in the neighbourhood of the guess value μ_0. Here k is known as shrinking factor, specified by the experimenter according to his belief in μ_0. The value of k near zero imply strong belief in μ_0. The optimum value of k, say $\hat{k}$, is obtained by minimizing $MSE(T_1)$ with respect of k and substituting the usual estimator of the parameter in resulting expression for k. Thompson[16] considered the estimation problem of mean of normal, binomial, Poisson and Gamma distributions. Mehta and Srinivasan[17] proposed a more general class of estimators by shrinking the maximum likelihood estimator $\hat{\mu}$ towards μ_0, as

$$T_2 = \hat{\mu}\left[1 - exp\left\{\frac{b(\hat{\mu}-\mu_0)^2}{Var(\hat{\mu})}\right\}\right] \tag{2.8}$$

where a and b are positive constants to be suitably chosen such that $0 < a < 1$ and $b > 0$ and showed that the MSE of these estimators are bounded and smaller than Thompson type estimators T_1 in the wider effective interval of the parametic space. Pandey[18,19] applied the shrinkage technique in estimation of normal variance and scale parameter of exponential distributins. Pandey and Singh[20], and Pandey and Srivastava[21,22] proposed shrinkage estimators of the scale in exponential distribution from censored sample. Pandey et. al[23] considered the problem of estimation of the shape parameter of Weibull distribution from type II censored sample. Jani[24] proposed a class of shrinkage estimators by taking the uniformly minimum vaiance unbaised estimator (UMVUE) $\hat{\mu}$ in the 'kernel', for the scale parameter of exponential distribution as

$$T_{3(b)} = \mu_0\left[1 - k\left\{\frac{\mu_0}{\hat{\mu}}\right\}^b\right] \quad ; 0 \le k \le 1 \tag{2.9}$$

where b is a non-zero real number. The class of estimators includes the estimators porposed in Pandey and Srivastava[21], as special cases and gives other better estimates for wider range of parametric space. Srivastava and Kumar[25] proposed a class of shrinkage estimator. over an interval. Kotani[26] proposed the best shrinkage predictor of a preassigned dominance level for a future order statistic of an exponential distribution under type II censoring assuming a prior estimator of the scale having some distribution.

We have considered the estimation porblems of the shape parameter of generalized Burr distribution using shrinkage technique. We have proposed a class of shrinkage estimators with MLE in the 'kernel' of the proposed estimator by shrinking towards the prior estimate or guess value for the shape parameter of the Generalized Burr distribution. The proposed class of estimators are compared with the maximim likelihood estimator in terms of mean square error (MSE) and their effective intervals of dominance are obtained.

3. Shrinkage Estimator

Let us consider the class of shrinkage estimator of θ for the generalized Burr distribution (GBD) with p.d.f. (1.4) as

(3.1) $$T_{(b)} = \theta_0\left[1 + k\left\{\frac{\theta_0}{\hat{\theta}}\right\}^b\right] \quad ; 0 \le k \le 1$$

where b is a non-zero real number and $\hat{\theta}$ is the MLE of θ. This estimator gives rise to a class of shrinkage estimators for different choice of b. Now the MSE of $T_{(b)}$ is given by

(3.2) $$MSE\left[T_{(b)}\right] = E\left[T_{(b)} - \theta\right]^2 ;$$

$$= (\theta_0 - \theta)^2 + k^2\theta_0^{2(b+1)}\, E\left[\hat{\theta}^{-2b}\right] + 2k(\theta_0 - \theta)\theta_0^{b+1} E\left[\hat{\theta}^{-b}\right]$$

where k is chosen such that MSE $[T_{(b)}]$ is minimum,. differentiating (3.2) with respect to k and equating it to zero, i.e.

$$\frac{d}{dk} MSE\left[T_{(b)}\right] = 0$$

$$\Rightarrow \frac{d}{dk} MSE\left[T_{(b)}\right] = 2k\theta_0^{2(b+1)}\, E\left[\hat{\theta}^{-2b}\right] + 2k(\theta_0 - \theta)\theta_0^{b+1} E\left[\hat{\theta}^{-b}\right] = 0$$

(3.3) $$\Rightarrow k = \frac{-(\theta_0 - \theta)E(\hat{\theta}^{-b})}{\theta_0^{(b+1)} E(\hat{\theta}^{-2b})}$$

Since $$\frac{d^2}{dk^2} MSE\left[T_{(b)}\right] > 0$$

k as given in (3.3) leads to minimum value of MSE $[T_{(b)}]$. Now, for any non-zero real number j, we have.

(3.4) $$E\left[\hat{\theta}^{-jb}\right] = \int_0^\infty \hat{\theta}^{-jb} f(\hat{\theta})\, d\hat{\theta}$$

which on simplification leads to

(3.5) $$E\left[\hat{\theta}^{-jb}\right] = \frac{\left[n\theta^{-1}\right]^{jb} \Gamma(n - jb)}{\Gamma(n)}$$

Therefore, substituting the value of $E\left[\hat{\theta}^{-jb}\right]$ for j=1, 2 from (3.5), we have

(3.6) $$E\left[\bar{\theta}^{-b}\right]=\frac{\left[n\theta^{-1}\right]^{b}\Gamma(n-b)}{\Gamma(n)}$$

(3.7) $$E\left[\bar{\theta}^{-2b}\right]=\frac{\left[n\theta^{-1}\right]^{2b}\Gamma(n-2b)}{\Gamma(n)}$$

substituting the value of $E\left[\bar{\theta}^{-jb}\right]$ from (3.6), and (3.7) in (3.3), we get

(3.8) $$k=-\left(\frac{\theta_0}{\theta}-1\right)\left(\frac{\theta_0}{\theta}\right)^{-(b+1)}\frac{\Gamma(n-b)}{n^{b}\Gamma(n-2b)}$$

Since k depends on θ, we replace it by its MLE $\hat{\theta}$ to get

(3.9) $$\bar{k}=-\left(\frac{\theta_0}{\bar{\theta}}-1\right)\left(\frac{\theta_0}{\bar{\theta}}\right)^{-(b+1)}\frac{\Gamma(n-b)}{n^{b}\Gamma(n-2b)}$$

Consequently, the proposed estimator $T_{(b)}$ defined in (3.1) is

(3.10) $$T'_{(b)}=\theta_0+k_1\left(\hat{\theta}-\theta_0\right)$$

where

(3.11) $$k_1=\frac{\Gamma(n-b)}{n^{b}\Gamma(n-2b)}$$

The MSE of estimator $T'_{(b)}$ is

(3.12) $$MSE\left[T'_{(b)}\right]=E\left[\theta_0+k_1\left(\hat{\theta}-\theta_0\right)-\theta\right]^2$$

$$MSE\left[T'_{(b)}\right]=\left[(1-k_1)\theta_0-\theta\right]^2+k_1^2E(\hat{\theta}^2)+2k_1\left[(1-k_1)\theta_0-\theta\right]E(\hat{\theta})$$

Substituting the value of $E[\bar{\theta}^{-b}]$ and $E[\bar{\theta}^{-2b}]$ for j = -1, -2 from (3.5), we get

(3.13) $$E(\hat{\theta})=0 \text{ and}$$

$$E(\bar{\theta}^2)=\theta^2\left(1+\frac{1}{n}\right)$$

Substituting the value of $E[\hat{\theta}]$ and $E[\hat{\theta}^2]$ from (3.14) in (3.12), after simplification we get

(3.14) $$MSE[T'_{(b)}] = \theta^2\left[(1-k_1)^2(\delta-1)^2 + \frac{k_1^2}{n}\right]$$

where $$\delta = \frac{\theta_0}{\theta}$$

Comparisons:

Let us define the relative efficiency of T'_b with respect to MLE $\hat{\theta}$ as

$$Ref\left[T'_{(b)}/\hat{\theta}\right] = \frac{MSE[\hat{\theta}]}{MSE[T'_{(b)}]},$$

(3.15) $$= \frac{1}{n\left[(1-k_1)^2(\delta-1)^2 + \frac{k_1^2}{n}\right]}$$

The porposed class of shrinkage estiamtor $T'_{(b)}$ will be better than MLE $\hat{\theta}$, if

$$Ref\left[\frac{T'_{(b)}}{\hat{\theta}}\right] > 1$$

i.e.

$$MSE\left[T'_{(b)}\right] - MSE\left[\hat{\theta}\right] \le 0$$

or, $$1-\sqrt{\alpha} < \delta < 1+\sqrt{\alpha}$$

where

(3.17) $$\alpha = \frac{1}{n}\frac{(1+k_1)}{(1+k_1)} \quad and \quad k_1 = \frac{\Gamma(n-b)}{n^b\Gamma(n-2b)}$$

Table 1.s The relative efficiencies of T_b with respet to MLE $\hat{\theta}$ for different values of b,δ and sample size n=5

b \ δ	0.10	0.20	0.40	0.80	1.25	1.50	2.00	3.00
-2	0.6943	0.8475	1.3317	3.8375	3.3891	1.7172	0.5775	0.1580
-1	1.2392	1.2766	1.3433	1.4286	1.4222	1.3714	1.2000	0.8000
1	0.9921	1.1468	1.5432	2.5510	2.4390	1.7857	0.8621	0.2809
2	0.2912	0.3683	0.6536	5.6922	3.6914	0.9395	0.2359	0.0591

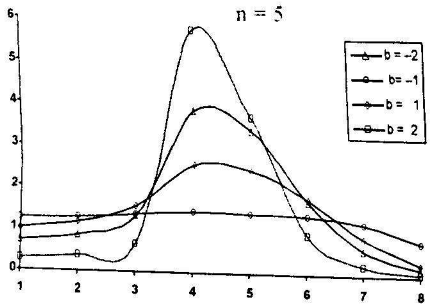

Table 2. The relative efficiencies of T_b with respet to MLE $\hat{\theta}$ for different values of b, δ and sample size n=10

b \ δ	0.10	0.20	0.40	0.80	1.25	1.50	2.00	3.00
-2	0.6874	0.8093	1.1431	2.1624	2.0348	1.3641	0.5884	0.1797
-1	1.1193	1.1372	1.1680	1.2052	1.2025	1.1805	1.1000	0.8643
1	1.0373	1.1161	1.2755	1.5244	1.5038	1.3514	0.9615	0.4464
2	0.3447	0.4293	0.7208	3.2158	2.5863	0.9829	0.2825	0.0734

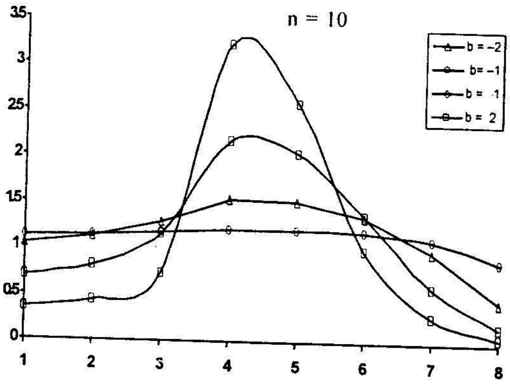

Table 3. The relative efficiencies of T_b with respet to MLE $\hat{\theta}$ for different values of b, δ and sample size n=15

b \ δ	0.10	0.20	0.40	0.80	1.25	1.50	2.00	3.00
-2	0.7184	0.8242	1.0881	1.7162	1.6492	1.2447	0.6283	0.2108
-1	1.0795	1.0912	1.1111	1.1348	1.1331	1.1191	1.0667	0.8982
1	1.0340	1.0849	1.1805	1.3127	1.3025	1.2228	0.9825	0.5501
2	0.4132	0.5040	0.7894	2.2387	1.9828	1.0154	0.3440	0.0944

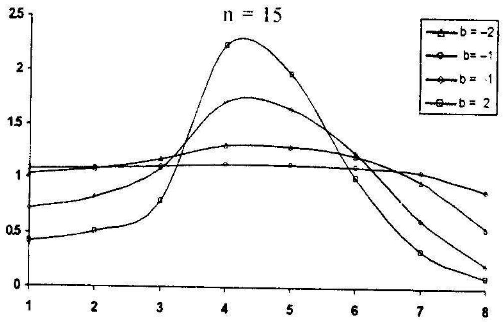

Table 4. The relative efficiencies of T_b with respet to MLE $\hat{\theta}$ for different values of b,δ and sample size n=20

b \ δ	0.10	0.20	0.40	0.80	1.25	1.50	2.00	3.00
-2	0.7486	0.8427	1.0628	1.5151	1.4711	1.1844	0.6655	0.2418
-1	1.0596	1.0683	1.0830	1.1003	1.0991	1.0889	1.0500	0.9188
1	1.0288	1.0661	1.1338	1.2225	1.2158	1.1628	0.9901	0.6211
2	0.4714	0.5640	0.8336	1.8372	1.6938	1.0263	0.3983	0.1155

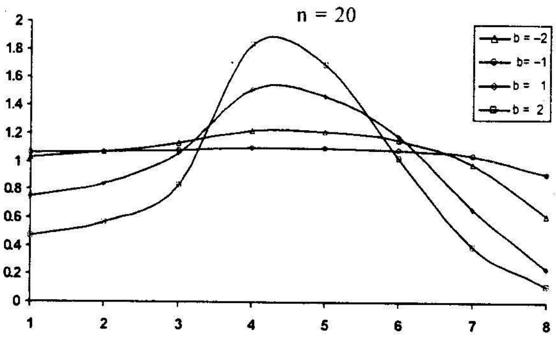

Table 5. The relative efficiencies of T_b with respet to MLE $\hat{\theta}$ for different values of b,δ and sample size n=25

b \ δ	0.10	0.20	0.40	0.80	1.25	1.50	2.00	3.00
-2	0.7743	0.8592	1.0485	1.4016	1.3691	1.1479	0.6973	0.2713
-1	1.0477	1.0546	1.0662	1.0799	1.0789	1.0709	1.0400	0.9324
1	1.0246	1.0540	1.1062	1.1726	1.1677	1.1282	0.9936	0.6728
2	0.5198	0.6117	0.8631	1.6275	1.5321	1.0293	0.4451	0.1361

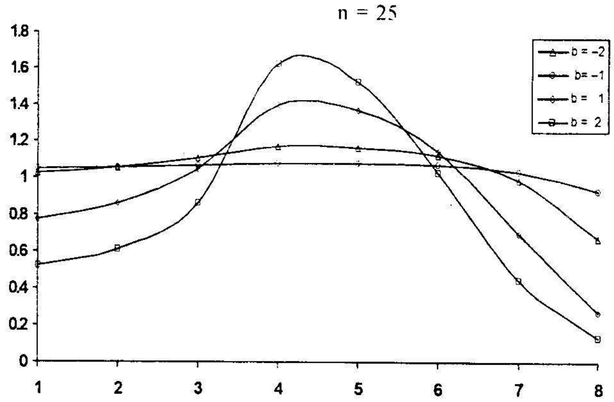

Table 6. The ranges of δ for which $T_{(b)}$ is better than MLE $\hat{\theta}$ for different value of b and sample size n.

b \ δ	0.10	0.20	0.40	0.80	1.25
-2	0.2771~1.7229	0.3239~1.6761	0.3389~1.6611	0.3463~1.6537	0.3506~1.6494
-1	-0.4832~2.4832	-0.4491~2.4491	-0.4376~2.4376	-0.4318~2.4318	-0.4283~2.4283
1	0.1056~1.8944	0.0513~1.9487	0.0339~1.9661	0.0253~1.9747	0.0202~1.97798
2	0.5155~1.4845	0.5052~1.4948	0.4941~1.5059	0.4877~1.5123	0.5320~1.4680

The Tables (1) to (5) show the relative efficiencies $T'_{(b)}$ with respect to the MLE $\hat{\theta}$ for different choices of b and sample size n and for different values of δ. It is evident from the table that the relative efficiencies are more than one for almost all samle sizes and δ when b = 1, -1. For b = 2, -2 the relative efficiencies are more than one for a narrow range of δ. Thus estimators with b = 1, -1 perform better. Figures (1) to (5) show the same picture on graph. From these curves we can find the values of δ at which relative efficiencies are equal to one.

The Table (6) shown the ranges of δ for which $T'_{(b)}$ is better than $\hat{\theta}$. It is clear from this table that for each choice b and n the ranges are fairly wide.

Thus we can choose a suitable shrinkage estimator according to the situation at hand.

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