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Hyperbolic R_π-Structure Manifold

S.B. Pandey and M. Pant
Department of Mathematics, Kumaun University, S.S.J. Campus, Almora (U.A.)

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Abstract: In this paper, we have defined Hyperbolic R_π-structure manifold and some of its geometric properties on the pattern of R_π-structure manifold. Integrability conditions are discussed in this manifold. Furthermore, Nijenhuis tensor and its associated are defined and some of their properties have been discussed.

1. Hyperbolic π and Hyperbolic R_π-Structure Manifold

AHGF - structure is defined as:

(1.1)a $F^2 = -\lambda^2 I_n$ or

(1.1)b $$\overline{\overline{X}} = -\lambda^2 X,$$ for complex number 'λ'.

It is hypebolic π-structure if $\lambda \neq 0$.

Definition (1.1). *If on a Hyperbolic π-structure manifold V_n, there exists a symmetric metric tensor a of rank $n_l<n$, such that*

(1.2)a $a(X,\overline{Y}) = \lambda a(X,Y) \Leftrightarrow$ (1.2)b $a(\overline{X},\overline{Y}) = \lambda^2 a(X,Y)$.

Then we say that a is compatible with the Hyperbolic π-structure. Then {F, a} is called Hyperbolic R_π-structure and V_n is said to be Hyperbolic R_π-structure manifold.

A bilinear function A in Hyperbolic R_π-structure manifold is called pure if

(1.3)a $A(\overline{X},\overline{Y}) + \lambda^2 A(X,Y) = 0$

and hybrid, if

(1.3)b $$A(\overline{X},\overline{Y}) - \lambda^2 A(X,Y) = 0$$

Nijenhuis tensor with respect to F is a vector-valued bilinear function N, given by

(1.4) $$N(X,Y) \stackrel{def}{=} [\overline{X},\overline{Y}] + \overline{\overline{[X,Y]}} - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]}.$$

In hyperbolic π-structure manifold, equation (1.4) assumes the form

(1.5) $$N(X,Y) = [\bar{X},\bar{Y}] - \lambda^2 [X,Y] - \overline{[\bar{X},Y]} - \overline{[X,\bar{Y}]}.$$

Let us put

(1.6) $$'N(X,Y,Z) \overset{def}{=} a(N(X,Y),Z).$$

Then $'N$ is called associated Nijenhuis tensor in Hyperbolic R_π-structure manifold.

2. Integrability Conditions

Theorem (2.1). *The necessary and sufficient condition that V_n be a Hyperbolic π-structure manifold is that it contains a distribution $\pi_{n/2}$ of complex dimension n/2 and a distribution $\tilde{\pi}_{n/2}$ complex conjugate to $\pi_{n/2}$, such that $\pi_{n/2}$ and $\tilde{\pi}_{n/2}$ have no direction in common and span together a linear manifold of dimension n, projections on $\pi_{n/2}$ and $\tilde{\pi}_{n/2}$ being L and M given by*

(2.1a) $$2\lambda L(X) = \lambda X - iFX,$$

(2.1b) $$2\lambda M(X) = \lambda X + iFX.$$

Proof. Let V_n be a Hyperbolic π-structure manifold corresponding to the eigen values λi and $-\lambda i$. Let there be n/2 linearly independent eigen vectors $\underset{x}{T}, x = 1,2,\ldots\ldots, n/2$ and n/2 linearly independent complex conjugate eigen vectors $\underset{x}{S}$. Then we have

(2.2a) $$\overset{x}{b}\underset{x}{T} = 0 \Rightarrow \overset{x}{b} = 0 \forall x.$$

(2.2b) $$\overset{x}{c}\underset{x}{S} = 0 \Rightarrow \overset{x}{c} = 0 \forall x.$$

Now,

$$\overset{x}{c}\underset{x}{T} + \overset{y}{d}\underset{y}{S} = 0 \Rightarrow \overset{x}{c}\underset{x}{\bar{T}} + \overset{y}{d}\underset{y}{\bar{S}} = 0 \Rightarrow \lambda i(\overset{x}{c}\underset{x}{T} - \overset{y}{d}\underset{y}{S}) = 0.$$

These equations imply

$$\overset{x}{c}\underset{x}{T} = 0,\ \overset{y}{d}\underset{y}{S} = 0 \Rightarrow \overset{x}{c} = \overset{y}{d} = 0, \text{ for x,y} \Rightarrow \{\underset{x}{T}, \underset{y}{S}\}$$

is a linearly independent set.

Let us put

(2.3a) $$2L(X) = X - \lambda i \overline{X},$$

(2.3b) $$2M(X) = X + \lambda i \overline{X}.$$

Then we have

$$2L(\underset{x}{T}) = (1+\lambda^2)\underset{x}{T},$$

$$2L(\underset{x}{S}) = (1-\lambda^2)\underset{x}{S},$$

$$2M(\underset{x}{T}) = (1-\lambda^2)\underset{x}{T},$$

$$2M(\underset{x}{S}) = (1+\lambda^2)\underset{x}{S}.$$

Thus we have proved that there is a distribution $\pi_{n/2}$ of complex dimension n/2 and there is a complex conjugate distribution $\tilde{\pi}_{n/2}$ of dimension n/2 which has no common direction with $\pi_{n/2}$ and spans with $\pi_{n/2}$ a linear manifold of dimension n, projection of $\pi_{n/2}$ and $\tilde{\pi}_{n/2}$ being L and M.

Conversely, we suppose that there is a distribution $\pi_{n/2}$ of complex dimension n/2 and a distribution $\tilde{\pi}_{n/2}$ complex conjugate of $\pi_{n/2}$ having no common direction with $\pi_{n/2}$ and they span together a linear manifold of dimension n.

Let $\underset{x}{T}$ be n/2 linearly independent vectors in $\pi_{n/2}$ and $\underset{x}{S}$ be n/2 linearly independent vectors in $\tilde{\pi}_{n/2}$. Let $\{\underset{x}{T},\underset{x}{S}\}$ span a linear manifold of dimension n. Then $\{\underset{x}{T},\underset{x}{S}\}$ is a linearly independent set. Let us define the inverse set $\{\overset{x}{T},\overset{x}{S}\}$, such that

(2.4a) $$X = \overset{x}{T}(X)\underset{x}{T} + \overset{x}{S}(X)\underset{x}{S},$$

which yields

(2.4b) $$\overset{x}{T}(\underset{y}{T}) = \overset{x}{S}(\underset{y}{S}) = \overset{x}{\underset{y}{\delta}},$$

(2.4c) $$\overset{x}{T}(\underset{x}{S}) = \overset{x}{S}(\underset{x}{T}) = 0$$

Let us put

(2.5) $$F(X) = \overline{X} = \lambda i\{\overset{x}{T}(X)\underset{x}{T} - \overset{x}{S}(X)\underset{x}{S}\},$$

Then

$$\overline{\overline{X}} = \lambda i\{\overset{x}{T}(FX)\underset{x}{T} - \overset{x}{S}(FX)\underset{x}{S}\}.$$

But from the equatios (2.4) and (2.5)

$$\overset{x}{T}(\overline{X}) = \lambda i\overset{x}{T}(X), \qquad \overset{x}{S}(\overline{X}) = -\lambda i\overset{x}{S}(X).$$

Therefore

$$\overline{\overline{X}} = -\lambda^2\{\overset{x}{T}(X)\underset{x}{T} + \overset{x}{S}(X)\underset{x}{S}\} = -\lambda^2 X.$$

Thus, we have proved that the manifold admits a Hyperbolic π-struture for $\lambda \neq 0$.

Corollary (2.1). *In the manifold V_n, we have*

(2.6a) $$L^2(X) = L(X),$$

(2.6b) $$M^2(X) = M(X)$$

(2.6c) $$L(M(X)) = M(L(X)) = 0.$$

Proof. Using the equations (2.1)a and (1.1)a, we get

$$4\lambda^2 L^2(X) = 2\lambda(\lambda X - iFX) = 4\lambda^2 L(X)$$

$$\Rightarrow L^2(X) = L(X)$$

which is the equation (2.6)a

Similarly from the equations (2.1)b and (1.1)a, we have

$$4\lambda^2 M^2(X) = 2\lambda(\lambda X - iFX) = 4\lambda^2 M(X),$$

$$\Rightarrow M^2(X) = M(X)$$

which is the equation (2.6)b.

Now, from the equations (2.1)a and (2.1)b

(2.7) $$(L+M)X = X.$$

Pre-multiplying (2.7) by L and M, using the equation (2.6)a and (2.6)b, we get (2.6)c.

Corollary (3.2). We have

(2.8a) $$L(X) = \overset{x}{T}(X)\underset{x}{T},$$

(2.8b) $$M(X) = \overset{x}{S}(X)\underset{x}{S}.$$

Proof. From the equations (2.1)a and (2.1)b, we get

(2.9) $$i\lambda(L-M)(X) = F(X),$$

Now, comparing the equatios (2.9) and (2.5), we have the equations (2.8)a and (2.8)b.

Theorem (2.2). *A necessay and sufficient condition for L(X) = 0=M(X) to be integrable is that*

(2.10a) $$dL(M(X),\ M(Y)) = 0,$$

(2.10b) $$dM(L(X),\ L(Y)) = 0.$$

for the structure {F} of class C^∞.

Proof. We know that

L(X) = 0 be integrable, iff

(dL) (X,Y) = 0 holds and the equation (2.7) reduces to

$$M(X) = X$$

Thus, we have

$$(dL)\ (MX,\ MY) = 0$$

Similarly, we can show that the necessary and sufficient condition fo MX = 0 to be integrable is

$$dM\ (LX,\ LY) = 0.$$

Theorem (2.3). *The necessary and sufficient condition for F of class C^∞ to be integrable is that Nijenhuis tensor vanishes (Calabi and Spencer, 1951; Eckmann and FrÖlicher, 1951, Hodge, 1951; Yano, 1954, Schouten and Yano, 1955).*

Proof. In consequence of the equation (2.6)c, equation (2.10)a assumes the from

(2.11) $$L[MX, MY] = 0,$$

(2.12) $$\Rightarrow 2\lambda L[2\lambda MX, 2\lambda MY] = 0.$$

Using the equation (2.1)a in (2.12), we obtain

(2.13) $$\lambda[2\lambda MX, 2\lambda MY] - iF[2\lambda MX, 2\lambda MY] = 0.$$

Now, using the equation (2.1)b in (2.13), we get

(2.14) $$\lambda\{[FX,FY] - \lambda^2[X,Y] - F[X,FY] - F(FX,Y]\}$$
$$+ i\{F[FX,FY] - \lambda^2 F[X,Y] + \lambda^2 F(FX,Y] + \lambda^2[X,FY]\} = 0$$

From the equations (1.5) and (2.14), we have

(2.15) $$-\lambda N(X,Y) + iFN(X,Y) = 0$$

Also complex conjugate of the equation (2.15) satisfies the equation

(2.16) $$-\lambda(N(X,Y)) - iFN(X,Y) = 0$$

Adding (2.15) and (2.16), we get

$$-\lambda N(X,Y) = 0$$

$$\Rightarrow N = 0$$

Hence the necessary and sufficient condition for F to be integrable is that Nijenhuis tensor vanishes.

3. Nijenhuis and Associated Nijenhuis Tensors

Theorem (3.1). *In the Hyperbolic π-structure manifold, we have*

(3.1) $$N(X,Y) = -N(Y,X) = [\bar{X},\bar{Y}] - \lambda^2[X,Y] - [\bar{X},Y] - [X,\bar{Y}],$$

i.e. Nijenhuis tensor is skew-symmetric in X and Y.

(3.2) $$N(\bar{X},\bar{Y}) = -\lambda^2 N(X,Y) = -\lambda^2\{[\bar{X},\bar{Y}] - \lambda^2[X,Y] - \overline{[\bar{\bar{X}},Y]} - \overline{[X,\bar{\bar{Y}}]}\},$$

i.e. N is pure in X and Y.

(3.3) $$N(\bar{X},Y) = N(X,\bar{Y}) = -\bar{N}(X,Y)\cdot$$
$$= -\lambda^2[X,\bar{Y}] - \lambda^2[\bar{X},Y] + \lambda^2[X,Y] - [\bar{\bar{X}},\bar{Y}],$$

(3.4) $$N(\bar{X},Y)=N(X,\bar{Y})=\lambda^2 N(X,Y)=-\lambda^2\{[X,\bar{Y}]+[\bar{X},Y]+\lambda^2[X,Y]-[\bar{X},\bar{Y}]\},$$

(3.5) $$N(\bar{X},\bar{Y})=-\lambda^2 N(X,Y)=-\lambda^2\{[\bar{X},\bar{Y})-\lambda^2[X,Y]+\lambda^2[\bar{X},Y]+\lambda^2[X,\bar{Y}]\}.$$

Proof. Interchanging the vectors X and Y in the equation (1.5) and using the fact that [X,Y] = -[Y,X], we obtain the equation (3.1). Barring the vectors X and Y in the equation (1.5), using (1.1) and (1.5), we get the equation (3.2). Barring X and Y separately in the equation (1.5), using (1.1) and (1.5) and then equating the resulting equations, we have the equation (3.3). Proof of the eqautions (3.4) and (3.5) follows the same pattern.

Theorem (3.2). *Let us put*

(3.6) $$P(X,Y)\overset{def}{=}[\bar{X},\bar{Y}]-[\bar{X},Y].$$

Then

(3.7a) $$P(X,\bar{Y})=\lambda^2 P(X,Y)=-\lambda^2[\bar{X},Y)+\lambda^2[\bar{X},\bar{Y}],$$

(3.7b) $$P(X,\bar{Y})=-\overline{P(X,Y)}=-\lambda^2(\bar{X},Y)-[\overline{\bar{X},\bar{Y}}],$$

(3.7c) $$P(\bar{X},\bar{Y})=-P(\bar{X},Y)=\lambda^4[X,Y]+\lambda^2[X,\bar{Y}],$$

(3.7d) $$P(\bar{X},\bar{Y})=\lambda^2 P[\bar{X},Y]=\lambda^4[X,Y)-\lambda^4[X,\bar{Y}].$$

Consequently

(3.8a) $$P(\bar{X},\bar{Y})-\lambda^2 P(X,Y)=N[\bar{X},\bar{Y}]$$

(3.8b) $$P(\bar{X},Y)+P(X,\bar{\bar{Y}})=N(\bar{X},Y),$$

(3.8c) $$P(X,Y)-P(\bar{X},Y)=N(X,Y).$$

Proof. Barring the equation (3.6) throughout and different vectors in it, using the equations (1.1) and (1.5), we get the equation from (3.7)a to (3.7)d.

Now, subtracting the equations (3.7)a from (3.7)d, using (3.2), we have the eqaution (3.8)a. Proof of the equations (3.8)b and (3.8)c follows similarly.

Theorem (3.3). *Let us put*

(3.9) $$Q(X,Y) \stackrel{def}{=} -\lambda^2[X,Y] - [X,\bar{Y}]$$

Then

(3.10a) $$\overline{Q(X,Y,)} = -Q(X,\bar{Y}),$$

(3.10b) $$\overline{Q(X,\bar{Y},)} = \lambda^2 Q(X,Y),$$

(3.10c) $$\overline{Q(\bar{X},\bar{Y},)} = \lambda^2 Q(\bar{X},Y),$$

(3.10d) $$Q(\bar{X},\bar{Y}) = -\overline{Q(\bar{X},Y)},$$

Consequently,

(3.11a) $$Q(\bar{X},\bar{Y}) - \lambda^2 Q(X,Y) = N(\bar{X},\bar{Y}) = -\lambda^2 N(X,Y),$$

(3.11b) $$\overline{Q(\bar{X},\bar{Y})} - \lambda^2 \overline{Q(X,Y)} = \overline{N(\bar{X},\bar{Y})},$$

(3.11c) $$Q(\bar{X},Y) + Q(X,\bar{Y}) = N[\bar{X},Y],$$

(3.11d) $$\overline{Q(\bar{X},Y)} + \overline{Q(X,\bar{Y})} = \overline{N(X,\bar{Y})} = \overline{N(\bar{X},Y)}.$$

Proof. Barring the eqation (3.9) throughout or different vectors in it, using the equations (1.1) and (1.5), we get the equations from (3.10)a to (3.10)d.

Now, subtracting the equation (3.10)b from (3.10)d, we get

(3.12) $$Q(\bar{X},\bar{Y}) - \lambda^2 Q(X,Y) = -\lambda^2[\bar{X},\bar{Y}] + \lambda^2[\bar{\bar{X}},Y] + \lambda^4[X,Y] + \lambda^2[X,\bar{Y}]$$

Using the equation (3.2) in (3.12) and using the fact that Nijenhuis tensor is pure in X and Y, we get the equation (3.11)a. Similarly, we can obtain the other equations.

Corollary (3.1). In the Hyperbolic π-structure manifold, we have

(3.13a) $$P(\bar{X},Y) = Q(X,\bar{Y})$$

(3.13b) $$P(X,\bar{Y}) = Q(\bar{X},Y)$$

(3.13c) $$\lambda^2 P(X,Y) = -Q(\bar{X},\bar{Y})$$

(3.13d) $$P(\overline{X},\overline{Y}) = -\lambda^2 Q(X,Y)$$

(3.14e) $$P(X,Y) = -Q(\overline{X},Y)$$

Proof. Proof of this corollary is obvious from the theorems (3.2) and (3.3).

Corollary (3.2). *In the Hyperbolic π-structure manifold, we have*

(3.14a) $$P(X,Y)+Q(X,Y) = N(X,Y)$$

(3.14b) $$P(\overline{X},Y)+Q(\overline{X},Y) = -\overline{N(X,Y)} = N(\overline{X},Y)$$

(3.14c) $$P(X,\overline{Y})+Q(X,\overline{Y}) = -\overline{N(X,Y)} = N(X,\overline{Y})$$

(3.14d) $$P(\overline{X},\overline{Y})+Q(\overline{X},\overline{Y}) = N(\overline{X},\overline{Y}) = \lambda^2 N(X,Y).$$

Proof. Proof of this corollary is evident from the theorems (3.1), (3.2) and (3.3).

Theorem (3.3). *Let us put*

(3.15) $$V(X,Y) \overset{def}{=} [\overline{X},Y] + [X,\overline{Y}]$$

Then

(3.16a) $$V(X,Y) = -V(Y,X) = -\{[X,\overline{Y}]+[\overline{X},Y]\}$$

that is V is skew-symmetic in X and Y.

(3.16b) $$V(\overline{X},\overline{Y}) = -\lambda^2 V(X,Y) = -\lambda^2\{[X,\overline{Y}]+[\overline{X},Y]\},$$

(3.16c) $$V(\overline{X},Y) = V(X,\overline{Y}) = -\lambda^2\{[X,Y]+[\overline{X},\overline{Y}]\},$$

(3.16d) $$V(\overline{X},Y) = V(X,\overline{Y}) = \lambda^4[X,Y]-\lambda^2[\overline{X},\overline{Y}],$$

(3.16e) $$V(\overline{X},\overline{Y}) = -\lambda^2 V[X,Y] = \lambda^4[X,\overline{Y}]+\lambda^4[\overline{X},Y],$$

Consequently,

(3.17a) $$V(\overline{X},Y)+V[X,Y] = N[X,Y],$$

(3.17b) $$V(X,\overline{Y})+V[X,Y]=N[X,Y],$$

(3.17c) $$V(\overline{X},\overline{Y})-V(X,\overline{Y})=N[X,\overline{Y}],$$

(3.17d) $$V(\overline{X},\overline{Y})-V(\overline{X},Y)=N[\overline{X},Y],$$

(3.17e) $$V(\overline{X},\overline{Y})+\lambda^2 V[X,\overline{Y}]=N(\overline{X},\overline{Y}),$$

Proof. The proof follows the pattern of the proof of the theorem (3.2).

Theorem (3.4). *Let us put*

(3.18) $$R(X,Y)\overset{def}{=}[\overline{X},Y]-\lambda^2[X,Y]$$

Then

(3.19a) $$R(Y,X)=-R(X,Y)=-[\overline{X},\overline{Y}]+\lambda^2[X,Y]$$

i.e. R is skew-symmetric in X and Y.

(3.19b) $$R(\overline{X},\overline{Y})=-\lambda^2 R(X,Y)=-\lambda^2\{[\overline{X},\overline{Y}]-\lambda^2[X,Y]\},$$

(3.19c) $$R(\overline{X},Y)=R(X,\overline{Y})=-\lambda^2[X,\overline{Y}]-\lambda^2[\overline{X},Y],$$

(3.19d) $$R(\overline{X},Y)=R(X,\overline{Y})=-\lambda^2[X,\overline{Y}]-\lambda^2[\overline{X},Y].$$

Consequently

(3.20a) $$R(\overline{X},\overline{Y})+\lambda^2 R(\overline{X},Y)=N(\overline{X},\overline{Y}),$$

(3.20b) $$R(X,Y)-R(\overline{X},Y)=N(X,Y),$$

(3.20c) $$R(\overline{X},Y)-R(X,\overline{Y})=N(\overline{X},Y)=N(X,\overline{Y}).$$

Proof. The proof follows the pattern of the proof of the theorem (3.2).

Corollary (3.3). *In the Hyperbolic π-structure manifold, we have*

(3.21a) $$V(\bar{X},Y) = R(X,Y),$$

(3.21b) $$V(\bar{X},\bar{Y}) = -\lambda^2 R(X,\bar{Y}),$$

(3.21c) $$V(\bar{X},\bar{Y}) = R(\bar{\bar{X}},Y),$$

(3.21d) $$V(X,Y) = R(\bar{X},Y).$$

Consequently

(3.22a) $$R(\bar{X},Y) - V(\bar{X},Y) = N(\bar{X},Y),$$

(3.22b) $$R(\bar{X},\bar{Y}) - V(\bar{X},\bar{Y}) = N(\bar{X},\bar{Y}),$$

(3.22c) $$R(X,Y) - V(X,Y) = N(X,Y),$$

(3.22d) $$R(\bar{\bar{X}},Y) - V(\bar{X},Y) = N(\bar{X},Y),$$

(3.22e) $$R(X,Y) - V(X,Y) = N(X,Y),$$

Proof. Proof is obvious from the Theorems (3.1), (3.3) and (3.4).

Theorem (3.5). *In Hyperbolic R_π-structure manifold, we have*

(3.23a) $${}'N(X,Y,Z) = -{}'N(Y,X,Z) = -a(N(Y,X),Z),$$

(3.23b) $$a(X,Y)'N(\bar{X},Y,Z) = -a(X,\bar{Y})'N(X,Y,Z),$$

(3.23c) $$a(X,Y)'N(X,\bar{Y},Z) = -a(X,\bar{Y})'N(X,Y,Z),$$

(3.23d) $$a(X,Y)'N(\bar{X},\bar{Y},Z) = -a(\bar{X},\bar{Y})'N(X,Y,Z),$$

(3.23e) $$a(X,Y)'N(X,Y,\bar{\bar{Z}}) = a(X,\bar{Y})'N(X,Y,Z),$$

(3.23f) $$a(X,Y)'N(\bar{X},Y,\bar{\bar{Z}}) = a(X,\bar{Y})'N(X,\bar{Y},Z),$$

(3.23g) $$a(X,Y)a(X,Y)'N(\bar{X},\bar{Y},\bar{\bar{Z}}) = -a(\bar{X},\bar{Y})a(X,\bar{Y})'N(X,Y,Z),$$

(3.23h) $$a(X,Y)'N(X,\bar{Y},\bar{\bar{Z}}) = a(X,\bar{Y})'N(\bar{X},Y,Z),$$

(3.23i) $$a(X,Y)'N(X,\overline{Y},\overline{Z}) = -a(\overline{X},\overline{Y})'N(X,Y,Z),$$

(3.23j) $$a(X,Y)'N(\overline{X},Y,\overline{Z}) = -a(\overline{X},\overline{Y})'N(X,Y,Z),$$

Proof. Using the equation (3.1) in (1.7), we have the equation (3.23)a. In view of the equations (3.3), (1.3)a and (1.7), we have

(3.24) $$'N(\overline{X},Y,Z) = -\lambda'N(X,Y,Z).$$

Now, putting the value of λ in (3.24) from (1.3)a, we get the equation (3.23)b. Equation (3.23)c can be obtained on the same pattern.

In view of the equations (3.2) and (1.7), we get

(3.25) $$'N(\overline{X},\overline{Y},Z) = -\lambda^2\,'N(X,Y,Z).$$

Putting the value of λ^2 in (3.25) from the equation (1.3)b, we get the eqation (3.23)d. Proof of the remaining equations follow the same pattern.

Theorem (3.6). *In the Hyperbolic R_π-structure manifold, we have*

(3.26) $$a(X,Y)\{'N(\overline{X},Y,Z)+'N(X,\overline{Y},Z)+'N(X,Y,\overline{Z})\}=-a(X,\overline{Y})'N(X,Y,Z).$$

(3.27) $$a(X,Y)\{'N(\overline{X},\overline{Y},Z)+'N(X,\overline{Y},\overline{Z})+'N(\overline{X},Y,\overline{Z})\}=-3a(\overline{X},\overline{Y})'N(X,Y,Z).$$

Proof. Adding the equation (3.23)b, (3.23)c and (3.23)e, we get the eqaution (3.26) and adding the equations (3.23)d, (3.23)i, and (3.23)j, we get (3.27)

Theorem (3.7). In the Hyperbolic R_π-structure manifold, we have

(3.28) $$a(X,Y)P(\overline{X},\overline{Y})-a(\overline{X},\overline{Y})P(X,Y) = a(X,Y)N(\overline{X},\overline{Y}),$$

(3.29) $$a(X,Y)Q(\overline{X},\overline{Y})-a(\overline{X},\overline{Y})Q(X,Y) = a(X,Y)N(\overline{X},\overline{Y}),$$

(3.30) $$a(X,Y)\overline{Q(\overline{X},\overline{Y})}-a(\overline{X},\overline{Y})\overline{Q(X,Y)} = a(X,Y)\overline{N(\overline{X},\overline{Y})},$$

(3.31) $$a(\overline{X},\overline{Y})P(X,Y)=-a(X,Y)Q(\overline{X},\overline{Y}),$$

(3.32) $$a(X,Y)P(\overline{X},\overline{Y})=-a(\overline{X},\overline{Y})Q(X,Y),$$

(3.33) $$a(X,Y)P(\overline{X},\overline{Y})+a(X,Y)Q(\overline{X},\overline{Y})= -a(\overline{X},\overline{Y})N(X,Y),$$

(3.34) $$a(X,Y)\overline{V(\overline{X},\overline{Y})}+a(\overline{X},\overline{Y})V(X,\overline{Y})= -a(X,Y)\overline{N(\overline{X},\overline{Y})},$$

(3.35) $$a(X,Y)R(\overline{X},\overline{Y})+a(\overline{X},\overline{Y})R(\overline{X},Y)=a(X,Y)N(\overline{X},\overline{Y}),$$

(3.36) $$a(X,Y)V(\overline{X},\overline{Y})=-a(\overline{X},\overline{Y})R(X,\overline{Y})$$

where P, Q, V and R are defined by the equations (3.6), (3.9), (3.15) and (3.18) respectively.

Proof. Putting the value of λ^2 from the equation (1.3)b in the equation (3.8)a, (3.11)a, (3.11)b, (3.13)c, (3.13)d, (3.14)d, (3.17)e, (3.20)a and (3.21)b, we have the equations from (3.28) to (3.37).

Reference

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