Common Fixed Point for Uniformly Intimate Pair of Mappings in Semi-Hausdorff Spaces and its Applications

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Abstract: In this paper, we have introduced the concept of "Uniformly intimate" pair of mappings in semi-Hausdorff d-complete topological space. Some common fixed point theorems in semi-Hausdorff d-complete topological space have been proved with above condition. Also, as application, common fixed point theorems for expansion mappings have been given. Our results generalize and improve the results of Fisher[1], Kang and Rhoades[2], Popa[3] and Khan et al[4].

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1. Introduction

In 1976, Jungck[5] generalized the well known Banach fixed point theorem for a pair of commuting mappings. The concept of commuting maps was further generalized by Sessa[6] in 1982 when he introduced pair of weakly commuting maps. Commuting maps are weakly commuting but converse is not true in general. Those are further weakened by the compatibility[7] and the compatibility of type (A)[8].

Hicks and Rhoades[8-9] proved some fixed point theorems in a d-complete topological space (non-metric setting) where the distance function used need not satisfy triangle inequality. Later, on this line Cho, Sharma and Sahu[10] established some existence results under semi-compatibility condition.

In 1965, Murdeshwar and Naimpally[11] introduced the concept of semi-Hausdorff space deriving from topological space. Wherein they have established that the semi-Hausdorff condition is strictly stronger than the T_1-axioms and thus it is between the T_1 and T_2 axioms. For us, this was an interesting space structure in which the existence of fixed point could be studied. With this intention first we have introduced the concept of uniformly intimate condition and then study some existence results in semi-Hausdorff d-complete topological space using uniformly intimate condition wherein the commutativity has been removed by non-commuting pair of mappings.

2. Preliminaries

Let (X, τ) be a topological space and $d : X \times X \to [0, \infty)$ be such that $d(x, y) = 0$ if and only if $x = y$. X is said to be d-complete if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that the sequence $\{x_n\}$ is a Cauchy sequence in X. A mapping $T : X \to X$ is w-continuous at x if $x_n \to x$ implies $Tx_n \to Tx$. The study of d-complete topological spaces have been done by Kasahara[12] and Iseki[13] in detail.

Definition 2.1: A symmetric function on a set X is a real-valued function d on $X \times X$ such that

(i) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$.

Let d be a symmetric function on a set X, and for any $\varepsilon > 0$ and any $x \in X$, let $U_x(\varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$.

In view of Hicks and Rhoades[9], we can define a topology $\tau(d)$ on X by $U \in \tau(d)$ if and only if for each $x \in U$, there exists a $U_x(\varepsilon) \subseteq U$. A symmetric function d is a semi-metric if for each $x \in X$ and each $\varepsilon > 0$, $U_x(\varepsilon)$ is a neighbourhood of x in the topology $\tau(d)$.

A topological space X is said to be symmetrizable or semi-metrizable according as its topology is induced by a symmetric function or a semi-metric function on X.

The d-complete symmetrizable space forms an important class of d-complete topological spaces. Other examples of d-complete topological spaces are given by Hicks and Rhoades[9].

Definition 2.2: A topological space X is said to be semi-Hausdorff iff every sequence in X has atmost one limit[11].

Murdeshwar and Naimpally[11] established the following theorems:

Theorem A: *Every Hausdorff space is semi-Hausdorff, but not conversely.*

Theorem B: *Every semi-Hausdorff space is T_1 but not conversely.*

Theorem C: *Every first countable semi-Hausdorff space is Hausdorff.*

Hicks and Rhoades[9] proved the following interesting theorem:

Theorem D: *Let (X, τ) be a Hausdorff d-complete topological space and f, h be w-continuous self-mappings on X satisfying $d(hx, hy) \leq Q(m(x, y))$ for all x, y in X, where*

$$m(x, y) = \max\{d(fx, fy), d(fx, hx), d(fy, hy)\}$$

and Q be a real-valued function satisfying the following conditions:

(a) $0 < Q(y) < y$ for each $y > 0$ and $Q(0) = 0$,

(b) $g(y) = \dfrac{y}{(y - Q(y))}$ is a non-increasing function on $(0, \infty)$,

(c) $\displaystyle\int_0^{y_1} g(y)\,dy < \infty$ for each $y_1 > 0$,

(d) $Q(y)$ is non-decreasing.

Suppose also that

(e) f and h are commuting,

(f) $h(X) \subseteq f(X)$.

Then f and h have a unique common fixed point in X.

Definition 2.3: Let A and S be self-maps of a semi-Hausdorff d-complete topological space (X, τ). The pair $\{A, S\}$ is said to be Uniformly S-intimate iff

$d(ASu, SSu) \le d(Au, Su)$, for all $u \in X$.

3. Main Results

Theorem 3.1: Let A, B, S and T be self mappings of a semi-Hausdorff d-complete topological space (X, τ) satisfying the following conditions:

(3.1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,

(3.2) $d(Ax, By) \le Q(m(x, y))$,

for all x, y in X, where $m(x, y) = \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By)\}$ and Q satisfying the conditions (a), (b), (c) and (d),

(3.3)   The pairs $\{A, S\}$ and $\{B, T\}$ are Uniformly S and T-intimate respectively,

(3.4) A, B, S and T are w-continuous.

Then A, B, S and T have unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X, there exists a point x_1 in X such that $Ax_0 = Tx_1$ as $A(X) \subseteq T(X)$. For this point x_1, we can choose a point x_2 in X such that $Bx_1 = Sx_2$ as $B(X) \subseteq S(X)$. Thus repeating the foregoing arguments, we define two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(3.5) $\begin{cases} y_{2n} = Ax_{2n} = Tx_{2n+1} \\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \end{cases}$

for all $n = 0, 1, 2, \ldots$.

For simplicity, let $d_n = d(y_n, y_{n+1})$ and applying (3.2), we have

$$d_{2n+1} = d(Ax_{2n+2}, Bx_{2n+1})$$

$$\leq Q(\max\{d(Sx_{2n+2}, Tx_{2n+1}), d(Sx_{2n+2}, Ax_{2n+2}), d(Tx_{2n+1}, Bx_{2n+1})\}),$$

which implies

(3.6) $d_{2n+1} \leq Q(\max\{(d_{2n}, d_{2n+1})\}).$

Suppose that $d_{2n+1} > d_{2n}$ for some n. Then from (3.6), we have

$$d_{2n+1} \leq Q(d_{2n+1}) < d_{2n+1},$$

which is a contradiction. Therefore, from (3.6), we have

$$d_{2n+1} \leq Q(d_{2n}).$$

Similarly, we have

$$d_{2n+2} \leq Q(d_{2n+1}).$$

So, in general, for $d_0 > 0$, we have

(3.7) $d_n \leq Q(d_{n-1}),\ n = 1, 2, \ldots .$

We define a sequence $\{t_n\}$ of positive real numbers such that $t_{n+1} = Q(t_n)$ with $t_1 = d_0 > 0$. By (a), we have $0 < t_{n+1} < t_n < \ldots < t_1$, $n \geq 1$. Moreover, by (b) and (c), the series $\sum t_n$ converges.

We shall now show that $d_n \leq t_{n+1}$, $n \geq 1$. From (3.7), for $n = 1$, we have $d_1 \leq Q(d_0) = Q(t_1) = t_2$ and the desired inequality is valid for $n = 1$. Assume that it is true for some $n > 1$. Again from (3.7), we have

$$d_n \leq Q(d_{n-1}) \leq Q(t_n) = t_{n+1}.$$

Since $\sum t_n$ is convergent, it follows that $\sum d(y_n, y_{n+1})$ is also convergent. Since X is d-complete, $\{y_n\}$ converges to some $z \in X$ and so the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the point z. Let there exist some points u and v in X such that $x_{2n} \to u$ and $x_{2n+1} \to v$ as $n \to \infty$. Then w-continuity of A, B, S and T imply $Ax_{2n} \to Au$, $Sx_{2n} \to Su$, $Bx_{2n+1} \to Bv$ and $Tx_{2n+1} \to Tv$. Since X is semi-Hausdorff, therefore we have

$$z = Au = Su = Bv = Tv.$$

Since the pair $\{A, S\}$ is Uniformly S-intimate, therefore

$$d(ASu, SSu) \leq d(Au, Su).$$

which implies that $Az = Sz$. Now using (3.2), we have

$$d(Az, z) \quad = \quad d(Az, Bv)$$

$$\leq \quad Q(\max\{d(Sz, Tv), d(Sz, Az), d(Tv, Bv)\})$$

$$\leq \quad Q(d(Az, z)) < d(Az, z).$$

which is a contradiction, so $Az = z$.

Similarly when the pair $\{B, T\}$ is Uniformly T-intimate, we obtain $z = Bz = Tz$. Therefore z is a common fixed point of A, B, S and T. Uniqueness easily follows from (3.2).

Theorem 3.2: *Let A, B, S and T be the mappings from a semi-Hausdorff d-complete topological space (X, τ) into itself satisfying the conditions (3.1), (3.3), (3.4) and the following:*

(3.8) $d(Ax, By) \leq k(m(x, y))$ *for all x, y in X,*

where k: $[0, \infty) \to [0, \infty)$ is a function such that k(0) = 0, k(t) < t for all t in $(0, \infty)$ and k is

non-decreasing with $\sum_{n=1}^{\infty} k^n(t) < \infty$ for all $t \in (0, \infty)$ (k is not assumed to be continuous).

Then A, B, S and T have a unique common fixed point in X.

Proof: For an arbitrary point x_0 in X, by (3.1), we can choose a sequence $\{y_n\}$ defined by (3.5) and from (3.5) and (3.8), we obtain

$$d_{2n+1} \leq k(d_{2n}) \text{ and } d_{2n+2} \leq k(d_{2n+1})$$

for all $n = 0, 1, 2, \ldots$.

By induction, for $d_0 > 0$,

$$d_n \leq k d(d_{n-1}) \leq \ldots \leq k^n(d_0)$$

for all $n = 1, 2, \ldots$.

Now $\sum_{n=1}^{\infty} d(y_n, y_{n+1}) < \infty$ implies that $\{y_n\}$ is a Cauchy sequence in X, which means that it converges to some point z in X, since X is semi-Hausdorff d-complete. As in the proof of Theorem 3.1, it can be shown that z is a unique common fixed point of A, B, S and T in X. This completes the proof.

One can also have Theorem 3.1 as a special case of Theorem 3.2, where k forces the w-continuity condition for all A, B, S and T.

In the following theorems, all of A, B, S and T need not be w-continuous.

Let $\Im$ be a family of mappings f from $[0, \infty)^3$ into $[0, \infty)$ such that each f is upper semi-continuous, non-decreasing in each coordinate variable and for any

$$t > 0, \ f(t, t, t) = r(t) < t \text{ and } \sum_{n=1}^{\infty} r^n(t) < \infty,$$

where r be a mapping from $[0, \infty)$ into itself.

Theorem 3.3: *Let A, B, S and T be mappings from a semi-Hausdorff d-complete topological space* (X, τ) *into itself satisfying the conditions (3.1), (3.3) and the followings:*

(3.9)   *there exists an* $f \in \mathfrak{I}$ *such that*

$$d(Ax, By) \leq f(d(Sx, Ty), d(Sx, Ax), d(Ty, By))$$

for all x, y in X,

(3.10)   *one of S and T is w-continuous.*

Then A, B, S and T have a unique common fixed point in X.

Proof: For an arbitrary point $x_0 \in X$, we can choose a sequence $\{y_n\}$ defined by (3.5). From (3.5) and (3.9), we have

$$d_{2n+1} = d(Ax_{2n+2}, Bx_{2n+1})$$

$$\leq f(d(Sx_{2n+2}, Tx_{2n+1}), d(Sx_{2n+2}, Ax_{2n+2}), d(Tx_{2n+1}, Bx_{2n+1})),$$

which implies

(3.11) $d_{2n+1} \leq f(d_{2n}, d_{2n+1}, d_{2n})$.

Suppose $d_{2n+1} > d_{2n}$ for some n. Then from (3.11), we have

$$0 < d_{2n+1} \leq f(d_{2n+1}, d_{2n+1}, d_{2n+1})$$

$$\leq r(d_{2n+1})$$

$$< d_{2n+1},$$

which is a contradiction. Hence from (3.11), we have

$$d_{2n+1} \leq r(d_{2n}) \text{ for all } n = 0, 1, 2, \ldots .$$

Similarly, we have

$$d_{2n+2} \leq r(d_{2n+1}) \text{ for all } n = 0, 1, 2, \ldots .$$

So, in general, for $d_0 > 0$,

$$d_n < r(d_{n-1}) \leq \ldots \leq r^n(d_0)$$

for all $n = 0, 1, 2, \ldots$. Since $\sum_{n=1}^{\infty} r^n(t)$ is convergent for each $t > 0$, it follows that $\sum_{n=1}^{\infty} d(y_n, y_{n+1})$ is convergent. Since X is semi-Hausdorff d-complete, $\{y_n\}$ converges to some point z in X and hence the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the point z.

Now, suppose that T is w-continuous. Since the pair $\{B,T\}$ is Uniformly T-intimate and the subsequences $\{Bx_{2n+1}\}$, $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the point z. Therefore, we have

$$d(BTx_{2n+1}, TTx_{2n+1}) \leq d(Bx_{2n+1}, Tx_{2n+1}).$$

Letting $n \to \infty$, we obtain

$$BTx_{2n+1}, TTx_{2n+1} \to Tz.$$

Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (3.9), we have

(3.12)
$$d(Ax_{2n}, BTx_{2n+1}) \leq f(d(Sx_{2n}, TTx_{2n+1}), d(Sx_{2n}, Ax_{2n}), d(TTx_{2n+1}, BTx_{2n+1})).$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(z, Tz) &\leq f(d(z, Tz), 0, 0) \\ &\leq r(d(z, Tz)) \\ &< d(z, Tz), \end{aligned}$$

which is a contradiction and so $Tz = z$.

Again, replacing x by x_{2n} and y by z in (3.9), respectively, we have

(3.13) $$d(Ax_{2n}, Bz) \leq f(d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz)).$$

As $n \to \infty$, we have

$$\begin{aligned} d(z, Bz) &\leq f(0, 0, d(z, Bz)) \\ &\leq r(d(z, Bz)) \\ &< d(z, Bz), \end{aligned}$$

which means that $Bz = z$. Since $B(X) \subseteq S(X)$, there exists a point u in X such that

$$Bz = Su = z.$$

By (3.9), we have

$$\begin{aligned} d(Au, z) &= d(Au, Bz) \\ &\leq f(d(Su, Tz), d(Su, Au), d(Tz, Bz)) \\ &= f(0, d(z, Au), 0) \\ &< d(z, Au), \end{aligned}$$

which is a contradiction and so $Au = z$. Since the pair $\{A, S\}$ is Uniformly S-intimate and $Au = Su = z$, then we have

$$d(ASu, SSu) \le d(Au, Su) \quad \cdot$$

which implies that $ASu = SSu$ and hence $Az = Sz$.

By using (3.9), we have

$$
\begin{aligned}
d(Az, z) &= d(Az, Bz) \\
&\le f(d(Sz, Tz), d(Sz, Az), d(Tz, Bz)) \\
&= f(d(z, Az), 0, 0) \\
&\le r(d(Az, z)) \\
&< d(Az, z),
\end{aligned}
$$

which is a contradiction, so $Az = z$. Therefore,

$$Az = Bz = Sz = Tz = z,$$

that is, z is a common fixed point of A, B, S and T. The uniqueness of the common fixed point z follows easily from (3.9). Similarly, we can prove the theorem when S is w-continuous. This completes the proof.

Corollary 3.1: *Let A, B, S and T be mappings from a semi-Hausdorff d-complete topological space (X, τ) into itself satisfying the conditions (3.1), (3.3) and (3.10) and there exists an $\alpha \in (0,1)$ such that*

$$(3.14) \qquad d(Ax, By) \le \alpha m(x, y)$$

for all x, y in X. Then A, B, S and T have a unique common fixed point in X.

Proof: If we define a mapping $f : [0, \infty)^3 \to [0, \infty)$ by $f(t_1, t_2, t_3) = \alpha \max\{t_1, t_2, t_3\}$ then $f \in \mathfrak{I}$ and so by Theorem 3.3, this corollary follows.

Let now Φ denote the family of all functions $\phi : [0, \infty) \to [0, \infty)$ which is non-decreasing, upper semi-continuous from the right with $\phi(0) = 0$, $\phi(t) < t$ and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$.

Corollary 3.2: *Let A, B, S and T be mappings from a semi-Hausdorff d-complete topological space (X, τ) into itself satisfying the conditions (3.1), (3.3), (3.10) and*

$$(3.15) \qquad d(Ax, By) \le \phi(m(x, y))$$

for all $x, y \in X$ and $\phi \in \Phi$. Then A, B, S and T have a unique common fixed point in X.

Proof: If we define a mappings $f : [0, \infty)^3 \to [0, \infty)$ by

$$f(t_1, t_2, t_3) = \phi(\max\{t_1, t_2, t_3\}),$$

then $f \in \mathfrak{I}$ and in view of Theorem 3.3, this corollary follows.

Remark 3.1: If we replace the condition (3.3) by $ASx = SAx$ and $BTx = TBx$ for all $x \in X$ in Corollary 3.1, then we get the original theorem of this type proved by Fisher[1] for a complete metric space in this new setting.

Remark 3.2: All corollaries and Theorem 2 of Fisher[1] can be formulated with respect to the stronger condition (3.9) and the uniformly intimate concept in this new setting.

Remark 3.3: It can be verified from the examples of Fisher[1] to see that the condition (3.14), the hypothesis of uniformly intimate pair and the condition that one of mappings S and T is w-continuous can not be omitted from Theorem 3.3.

Remark 3.4: If we let $m(x,y) = d(Sx, Ty)$ in Corollary 3.2, and S and T are surjective, then we obtain a result under the uniformly intimate concept in this new setting. The original theorem of this type was proved by Kang Rhoades[2] under the compatibility concept in a complete metric space.

Example 3.1: Let $X = [0,1]$ with symmetric function d on X defined by $d(x,y) = |x - y|$.

Let $A, B, S, T : X \to X$ such that $Ax = Bx = \dfrac{x^2}{2}$ and $Sx = Tx = x^2$.

It is easy to verify that all the conditions of Theorem 3.1 are satisfied and 0 is unique common fixed point of A, B, S and T.

4. Applications

Fixed points for expansion mappings: Let (X, τ) be a semi-Hausdroff topological space, N the set of all positive integers and Ψ the family of all functions $\psi : [0, \infty)^3 \to [0, \infty)$ satisfying the following properties:

$(\psi - 1)$ ψ is continuous in $([0, \infty))^3$

$(\psi - 2)$ $\psi(1,1,1) = h > 1$ where $h \in [0, \infty)$

$(\psi - 3)$ let $\alpha, \beta \in [0, \infty)$ be such that

$(\psi - 4)$ $\alpha \geq \psi(\beta, \beta, \alpha) = h.\beta$

$(\psi - 5)$ $\alpha \geq \psi(\beta, \alpha, \beta) = h.\beta$

$(\psi - 6)$ $\psi(\alpha, 0, 0) > \alpha$ for all $\alpha \neq 0$.

Proposition 4.1: *Let A, B, S and T be mappings from semi-Hausdorff topological space (X, τ) into itself such that the pairs $\{A, S\}$ and $\{B, T\}$ are uniformly S and T-intimate respectively. Suppose that for all x, y in X and $\psi \in \Psi$ such that:*

(4.1) $d(Sx, Ty) \geq \psi(d(Ax, By), d(Ax, Sx), d(By, Ty))$.

If there exist u, v and z in X such that $Au = Su = Bv = Tv = z$, then $Az = Bz = Sz = Tz = z$.

Proof: Since the pair $\{A,S\}$ is uniformly S-intimate and $Au = Su = z$,

$$d(ASu, SSu) \leq d(Au, Su),$$

which implies that $Az = Sz$. From (4.1), we have

$$d(Sz, z) = d(Sz, Tv)$$

$$\geq \psi(d(Az, Bv), d(Az, Sz), d(Bv, Tv))$$

$$\geq \psi(d(Sz, z), 0, 0)$$

$$> d(Sz, z) \qquad \text{by property } (\psi - 6),$$

which is a contradiction. Therefore $Sz = z$. By symmetry we have $Bz = Tz = z$.

Theorem 4.1: *Let A, B, S and T be mappings from a semi-Hausdorff topological space (X, τ) into itself satisfying the conditions (3.1), (3.3) and (4.1) and S(X) is d-complete. Then A, B, S and T have a unique common fixed point in X.*

Proof: For an arbitrary point x_0 in X, we can choose a sequence $\{y_n\}$ defined by (3.5). Now from (3.5) and (4.1), we have

$$d_{2n} = d(Sx_{2n+2}, Tx_{2n+1})$$

$$\geq \psi(d(Ax_{2n+2}, Bx_{2n+1}), d(Ax_{2n+2}, Sx_{2n+2}), d(Bx_{2n+1}, Tx_{2n+1}))$$

$$= \psi(d_{2n+1}, d_{2n+1}, d_{2n})$$

$$\geq h.d_{2n+1} \qquad \text{by property } (\psi -4).$$

This implies

$$d_{2n+1} \leq 1/h \, d_{2n}.$$

Similarly, we can get

$$d_{2n+2} \leq 1/h \, d_{2n+1}.$$

So, in general, we have for $d_0 > 0$

$$d_n \leq 1/h.d_{n-1} \leq \ldots \leq 1/h^n.d_0$$

for all $n \in N$. Since $h > 1$, this implies that $\lim_{n \to \infty} d_n = 0$. It follows that $\sum_{n=1}^{\infty} d(y_n, y_{n+1})$ is convergent.

Since in addition, $S(X)$ is d-complete, $\{y_n\}$ converges to some z in $S(X)$ and hence the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to z.

Let $Su = z$ for some u in X. Putting $x = u$ and $y = x_{2n+1}$ in (4.1), we obtain

(4.2) $d(Su, Tx_{2n+1}) \geq \psi(d(Au, Bx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1}))$.

Letting $n \to \infty$, we have

$$0 \geq \psi(d(Au, z), d(Au, z), 0)$$

$$\geq h.d(Au, z) \qquad \text{by property } (\psi\text{-}4),$$

which implies $Au = z$.

Since $z = Au \in A(X) \subseteq T(X)$, there exists a point v in X such that $Au = Tv$. Again, replacing x by u and $y = v$ in (4.1), we obtain

$$0 = d(Su, Tv) \geq \psi(d(Au, Bv), d(Au, Su), d(Bv, Tv))$$

$$= \psi(d(z, Bv), 0, d(Bv, z))$$

$$\geq h.d(Bv, z), \qquad \text{by property } (\psi\text{-}5),$$

which implies $Bv = z$. Therefore, we have $Au = Su = Bv = Tv = z$ and hence by Proposition 4.1, it follows that z is a common fixed point of A, B, S and T.

Let us suppose that there exists a second common fixed point w of A, B, S and T. Then from relation (4.1), we have

$$d(z, w) = d(Sz, Tw)$$

$$\geq \psi(d(Az, Bw), d(Az, Sz), d(Bw, Tw))$$

$$= \psi(d(z, w), 0, 0)$$

$$> d(z, w) \qquad \text{by property } (\psi\text{-}6).$$

Thus, there arise a contradiction.

Hence w cannot exists and z is a unique common fixed point of A, B, S and T. This completes the proof.

Remark 4.1: Theorem 4.1 improves and generalizes Theorem 1 of Popa[3] and Theorem 3 of Khan, Khan and Sessa[4], in a semi-Hausdorff d-complete topological space under uniformly intimate condition.

Corollary 4.1: *Let A, B, S and T be mappings from semi-Hausdorff topological space (X, τ) into itself such that, the pairs $\{A, S\}$ and $\{B, T\}$ are uniformly S and T-intimate respectively and satisfying the conditions (3.1), (4.2) and following:*

There exists a, b, $c \in [0, \infty)$ with $a > 1$, $b < 1$, $c < 1$ and $a + b + c > 1$ such that

(4.3) $d^r(Sx, Ty) \geq a.d^r(Ax, By) + b.d^r(Ax, Sx) + c.d^r(By, Ty)$

for all x, y in X, where $r > 0$. Then A, B, S and T have unique common fixed point in X.

Proof: Let us further define the $\psi : [0, \infty)^3 \to [0, \infty)$ as follows:

$$\psi(t_1,t_2,t_3) = (a t_1^r + b t_2^r + c t_3^r)^{1/r}$$

Then $\psi \in \Psi$, and thus by theorem 4.1, this corollary follows.

Remark 4.2: It should be noted that Corollary 4.1 improves and generalizes Theorem 1 of popa[14], in non-metric setting.

If we put $b = c = 0$ in Corollary 1, we obtain the following:

Corollary 4.2: *A, B, S and T be mappings from semi-Hausdorff d-complete topological (X,τ) into itself such that the pairs $\{A,S\}$ and $\{B,T\}$ are uniformly S and T-intimate respectively, satisfying the conditions (3.1), (4.2) and the following:*

There exists a constant $\lambda \in [0,\infty)$ with $\lambda > 1$ such that

$$d(Sx,Ty) \geq \lambda.d(Ax,By)$$

for all x, y in X. Then A, B, S and T have a unique common fixed point in X.

Remark 4.3: If we define ψ as in proof of corollary 4.1, then the result obtained in this new setting, which improves and generalizes Theorem 2.4 of Pathak, Kang and Ryu[15]. The original theorem of this type was proved by these authors in a complete metric space.

Theorem 4.2: *Let A, B, S and T be mappings from semi-Hausdorff space (X,τ) into itself satisfying the conditions (3.1), (3.3), (4.2) and following:*

(4.4) $\phi(d(Sx,Ty)) \geq \max\{d(Ax,By), d(Ax,Sx), d(By,Ty)\}$

for all x, y in X, where $\phi \in \Phi$. Then A, B, S and T have a unique common fixed point in X.

Proof: For an arbitrary point $x_0 \in X$, we can choose a sequence $\{y_n\}$ as defined by (3.5). Then, by (3.5) and (4.4), we have

(4.5) $\phi(d_{2n}) = \phi(d(Sx_{2n+2},Tx_{2n+1}))$

$\geq \max\{d(Ax_{2n+2},Bx_{2n+1}), d(Ax_{2n+2},Sx_{2n+2}), d(Bx_{2n+1},Tx_{2n+1})\}$

$= \max\{d_{2n+1},d_{2n}\}.$

Now, suppose $\max\{d_{2n+1},d_{2n}\} = d_{2n}$, then from (4.5), we have

$$\phi(d_{2n}) \geq d_{2n}$$

which is a contradiction. Hence, we have

$$\max\{d_{2n+1},d_{2n}\} = d_{2n+1}.$$

Therefore from (4.5), we have

$$d_{2n+1} \leq \phi(d_{2n}),$$

and similarly

$$d_{2n+2} \leq \phi(d_{2n+1}),$$

So, in general for $d_0 > 0$, and $n \in N$,

$$d_n \leq \phi(d_{n-1}) \leq \dots \leq \phi^i(d_{n-i}) \leq \dots \leq \phi^n(d_0).$$

Since $\sum\limits_{n=1}^{\infty} \phi^n(t)$ is convergent for each $t > 0$, it follows that $\sum\limits_{n=1}^{\infty} d(y_n, y_{n+1})$ is convergent. In addition since $S(X)$ is d-complete, sequence $\{y_n\}$ converges to some z in $S(X)$ and hence the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to z.

Let $Su = z$ for some u in X. Putting $x = u$ and $y = x_{2n+1}$ in inequality (4.4) and then, letting limits as $n \to \infty$, we obtain:

$$\phi(0) = 0 \geq d(Au, z)$$

which implies $Au = z$. Since in addition $A(X) \subseteq T(X)$, there is a point v in X such that $Au = Tv = z$. Again, replacing x by u and y by v in inequality (4.4), we obtain

$$\phi(0) = 0 \geq d(Su, Tv) \geq d(Bv, z)$$

which means that $Bv = z$. Therefore, $Au = Su = Bv = Tv = z$. Since, the pair $\{A, S\}$ is uniformly S-intimate. Then

$$d(ASu, SSu) \leq d(Au, Su)$$

which implies that $ASu = SSu$, that is $Az = Sz$.

By property (4.4), we have:

$$\phi(d(Sz, z)) = \phi(d(Sz, Tv))$$

$$\geq \max\{d(Az, Bv), d(Az, Sz), d(Bv, Tv)\}$$

$$= d(Sz, z)$$

which is a contradiction, since for each $t > 0$, $\phi(t) < t$. Therefore $Sz = z$, and by symmetry $Bz = Tz = z$, which shows the existence of z as a common fixed point of A, B, S and T. Uniqueness of common fixed point is obvious.

Corollary 4.3: *Let A, B, S and T be mappings from semi-Hausdorff d-complete topological (X, τ) into it self such that the pairs $\{A, S\}$ and $\{B, T\}$ are uniformly S and T-intimate respectively and satisfying the conditions (3.1), (4.2) and at least one of the following conditions:*

For all $x, y \in X$,

(4.6) $\phi(d(Sx, Ty)) \geq d(Ax, By)$,

(4.7) $\phi(d(Sx, Ty)) \geq 1/2 \, [d(Ax, By) + d(Ax, Sx)]$,

(4.8) $\phi(d(Sx, Ty)) \geq 1/2 \, [d(Ax, By) + d(By, Ty)]$,

(4.9) $\phi(d(Sx,Ty)) \geq 1/3 \; [d(Ax,By) + d(Ax,Sx) + d(By,Ty)]$.

Then A, B, S and T have a unique common fixed point in X.

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