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Some Properties of Transversal Hypersurfaces of Trans-Sasakian Manifolds

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Abstract. Transversal hypersurfaces of Trans-Sasakian Manifolds are studied. It is proved that transversal hypersurfaces of almost contact manifold admits an almost complex structure and each transversal hypersurfaces of almost contact metric manifold admits an almost Hermitian structure. The fundamental 2-form on the transversal hypersurfaces of $(\alpha, 0)$ Trans-Sasakian manifolds with (f, g, u, v, λ) structure is closed. The fundamental 2-form on the transversal hypersurfaces of trans-Sasakian manifold with hyperbolic (f, g, u, v, λ)-structure is closed. It is also proved that transversal hypersurfaces of Trans-Sasakian manifold admits a Kaehlerian structure. Some properties of transversal hypersurfaces are proved.

1. Introduction

Tanno[1] classified connected almost contact manifolds whose automorphism group possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say C, he showed that they can be divided into three classes.

1. Homogeneous normal contact Riemann manifold with $C > 0$,
2. Global Riemannian product of a line or a circle with Kaehler manifold of constant holomarphic sectional curvature if $C = 0$ and
3. A warped product space $\mathbb{R} \times_f \mathbb{C}^n$ if $C < 0$

It is known that manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu[2] characterized the differential geometric properties of manifold of class (3); the structure so obtained is now known as Kenmotsu structure. In general these structures are not Sasakian[2]

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In the Gray-Hervella[3] classification of almost Hermitian manifolds, there appears a class, $\mathcal{W}_4$ of Hermitian manifolds. An almost contact metric structure on a manifold $\bar{M}$ is called trans-Sasakian structure[4] if the product manifold $\bar{M} \times R$ belongs to the class $\mathcal{W}_4$. The class $C_6 \oplus C_5$ [4] coincides with the class of trans-Sasakian structure of type (α, β). We note that trans- Sasakian structure of type (0, 0) cosympletic[5], trans- Sasakian structure of type $(0, \beta)$ are β-Kenmotsu[6] and trans-Sasakian structures of type $(\alpha, 0)$ are α-Sasakian[4].

On other hand (f, g, u, v, λ)-structure on manifolds were introduced by Yano and Okumura[7]. It is well known that on a transversal hypersurface of almost contact metric manifold there always exist a (f, g, u, v, λ)- structure. Motivated by this fact, in this paper transversal hypersurface of trans-Sasakian manifold are studied. This paper is organized as follows. Section 2 is devoted to preliminaries.

In Section 3, some properties of of transversal hyper-surfaces are given, it is proved that each transversal hypersurface of an almost contact metric manifold admits an almost Hermitian and almost complex structure. In Section 4, it is shown that 2 form on trans-Sasakian manifold is closed and every transversal hypersurface of trans-Sasakian manifold admits a Kaehlerian structure.

2. Trans-Sasakian Manifold

Let $\bar{M}$ be an almost contact metric manifold[5] with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is a (1, 1) tensor field, ξ is a vector field; η is 1-form and g is a compatible Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(xi) = 1, \quad \phi(\xi) = 0, \quad \eta_0 \phi = 0 \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X) \tag{3}$$

for all $X, Y \in T\bar{M}$.

An almost contact metric structure (ϕ, ξ, η, g), on $\bar{M}$ is called as trans-Sasakian structure (Oubina[3] if $(\bar{M} \times R, J, G)$ belongs to the class $\mathcal{W}_4$ of Gray Hervella classification of almost Hermitian manifolds[2], where J is the almost complex structure on $\bar{M} \times R$ defined by

$$J\left(X, \frac{ad}{dt}\right) = \left(\phi X, -a\xi, \eta(X)\frac{d}{dt}\right)$$

for all the vector fields X on $\bar{M}$ and smooth function a on $\bar{M} \times R$ and G is the product metric on $\bar{M} \times R$ .This may be expressed by the condition (Blair & Oubina)[8]

(4) $$(\overline{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

for smooth functions α and β on $\overline{M}$, and we say that the trans-Sasakian structure is of type (α, β). From the formula (4) it follows that[8]

(5) $$\overline{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi).$$

The class $C_6 \oplus C_5$ [4] coincides with the class of trans-Sasakian structures of type (α, β). We note that the trans-Sasakian structures of type (0, 0) are cosympletic[5], trans-Sasakian structures of type $(0, \beta)$ are β-Kenmotsu[6] and trans-Sasakian structures of type $(\alpha, 0)$ are a-Sasakian[6].

3. Transversal Hypersurface

Let M be a hypersurface of an almost contact manifold $\overline{M}$ equipped with an almost contact structure (ϕ, ξ, η). We assume that the structure vector field ξ never belongs to tangent hyperplane of the hypersurface M,. Such that a hypersurface is called a transversal hypersurface[9] of an almost contact manifold. In this case the structure vector field ξ can be taken as an affine normal to the hypersurface vector field X and ξ are linearly independent, therefore we may write

(6) $$\phi X = JX + \omega(X)\xi,$$

where J is a (1, 1) tensor field and omega is a 1-form on M. Operating by f to the above equation and taking account of equation (1) we get

(7) $$J^2 = -1 \quad \text{and}$$

(8) $$\eta = \omega \; o \; J.$$

Thus, we have

Theorem 3.1. *Each transversal hypersurface of an almost contact manifold admits an almost complex structure.*

From (7) and (8), it follows that

(9) $$\omega = -\eta \; o \; J.$$

Now, we assume that $\overline{M}$ admits an almost contact metric structure (ϕ, ξ, η, g). We denote by g the induced metric on M also. Then for all $X, Y \in TM$, we obtain

(10) $$g(JX, JY) = g(X, Y) - \eta(X)\eta(Y) + \omega(X)\omega(Y).$$

We define a new metric G on the transversal hypersurface given by

(11) $$G(X, Y) = g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

So, $$G(JX, JY) = g(JX, JY) - \eta(JX)\eta(JY)$$

$$= g(X, Y) - \eta(X)\eta(Y) + \omega(X)\omega(Y) - (\eta \; o \; J)(X)(\eta \; o \; J)(Y)$$

$$= g(X, Y) - \eta(X)\eta(Y) + \omega(X)\omega(Y) - \omega(X)\omega(Y)$$

$$= g(X, Y) - \eta(X)\eta(Y) = G(X, Y)$$

Then, we get

(12) $$G(JX, JY) = G(X, Y),$$

where equations (7), (9), (10), and (11) are used.

Then G is Hermitian metric on M. That is (J, G) is an almost Hermitian structure on the transversal hypersurface M of $\bar{M}$.

Thus, we are able to state the following.

Theorem 3.2. *Each transversal hypersurface of an almost contact manifold admits an almost Hermitian structure.*

We now assume that M is orientable and choose a unit vector field N of $\bar{M}$ normal to M. Then Gauss and Weingarten formulae are given respectively by

(13) $$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)N, \quad (X,Y \in TM)$$

(14) $$\bar{\nabla}_X N = -HX$$

where $\bar{\nabla}$ and ∇ are respectively the Riemannian and induced Riamannian connections in $\bar{M}$ and M and h is the second fundamental form related to H by

(15) $$h(X, Y) = g(HX, Y)$$

Defining

(16) $$\phi X = fX + u(X)N$$

(17) $$\phi N = -U$$

(18) $$\xi = V + \lambda N$$

(19) $$\eta(X) = v(X)$$

for $X \in TM$ we get an induced (f, g, u, v, λ)-structure[7] on the transversal hypersurface such that

(20) $$f^2 = -I + u \otimes U + v \otimes V$$

(21) $$fU = -\lambda V, \qquad fV = \lambda U$$

(22) $$u\, o f = \lambda v, \qquad v\, o f = -\lambda U$$

(23) $$u(U) = 1 - \lambda^2, \quad u(V) = v(U) = 0, \quad v(V) = 1 - \lambda^2$$

(24) $$g(fX, fY) = g(X, Y) - u(X)\,u(Y) - v(X)\,v(Y)$$

(25) $$g(X, fY) = -g(fX, Y), \quad g(X, U) = u(X), \quad g(X, V) = v(X)$$

for all $X, Y \in TM$, where

(26) $$\lambda = \eta(N)$$

Thus, we see that every transversal hypersurface of an almost contact metric manifold also admits a (f, g, u, v, λ)-structure. Next we find relation between the induced almost Hermitian structure (J, G) and the induced (f, g, u, v, λ)-structure on the transversal hypersurface of an almost contact metric manifold. In fact, we have the following :

Theorem 3.3. *Let M be a transversal hypersurface of an almost contact metric manifold $\overline{M}$ equipped with almost contact metric structure (ϕ, ξ, η, g).*

Then we have

(27) $$\lambda\,\omega = u$$

(28) $$J = f - \frac{1}{\lambda}\, u \otimes V$$

(29) $$J\,U = -\frac{1}{\lambda}\, V$$

(30) $$u \circ J = u \circ f = \lambda\, v$$

(31) $$JV = fV = \lambda\, U$$

(32) $$U \circ J = -\frac{1}{\lambda}\, u.$$

Proof

$$\phi X = JX + \omega(X)\xi$$

$$\xi = V + \lambda N$$

(a) $$\phi X = JX + \omega(X)V + \lambda\omega(X)N$$

(b) $$\phi X = fX + u(X)N$$

from equation (a) and (b) we have

$$\lambda\omega X = u(X), \quad \omega(X) = \frac{1}{\lambda}\, u(X)$$

$$JX = fX - \omega(X)V$$

$$JX = fX - \frac{1}{\lambda} u(X)V$$

$$J = f - \frac{1}{\lambda} u \otimes v \text{ which is equation (28)}$$

$$(u \circ J)(X) = (u \circ f)(X) - \frac{1}{\lambda} u(X)\, u(V), \qquad u(V) = 0$$

$$u \circ J = u \circ f = \lambda v \text{ which is equation (30)}$$

$$JU = fV - \frac{1}{\lambda} u(v)V$$

$$JU = -\lambda V - \frac{1}{\lambda}(1 - \lambda^2) = -\frac{1}{\lambda} V$$

$$JU = -\frac{1}{\lambda} V \text{ which is equation (29)}$$

$$(u \circ J)(X) = (v \circ f)(X) - \frac{1}{\lambda} u(X)\, u(V) = (v \circ f)(X) - \frac{1}{\lambda} u(X)(1 - \lambda^2)$$

$$= -\lambda u(X) = -\frac{1}{\lambda} u(X) + \lambda u(X)$$

$$= -\frac{1}{\lambda} u(X)$$

$$v \circ J = -\frac{1}{\lambda} u$$

$$JV = fV - \frac{1}{\lambda} u(V)V = fV = \lambda U \text{ which is equation (31)}$$

here equations (21), (22), (23), (24), (25) and (26) are used

4. Kaehlerian Transversal Hypersurface

First, we state

Lemma 4.1. *Let M be a transversal hypersurface (f, g, u, v, λ)-structure of an almost contact metric manifold $\bar{M}$. Then*

(33) $$(\bar{\nabla}_X \phi)Y = ((\nabla_X f)Y - u(Y)HX + h(X, Y)U) + ((\bar{\nabla}_X u)Y + h(X, fY)N)$$

(34) $$\bar{\nabla}_X \xi = (\nabla_X V - \lambda HX + (h(X, V) + X\lambda)N$$

(35) $$(\bar{\nabla}_X \phi)N = (-\nabla_X U + fHX)$$

(36) $$(\bar{\nabla}_X \eta)Y = (\nabla_X u + \lambda h(X, Y)$$

for all $X, Y \in TM$.

The proof is straight forward and hence ommited.

Theorem 4.2. *Let M be a transversal hypersurface (f, g, u, v, λ)-structure of a trans-Sasakian manifold $\bar{M}$. Then*

$$(\nabla_X f)Y = \alpha(g(X, Y)V - v(Y)X) + \beta(g(fX, Y)V - v(Y)fX) + u(Y)HX - h(X, Y)U \tag{37}$$

$$(\nabla_X u)Y = \alpha\lambda\, g(X, Y) + \beta(\lambda g(fX, Y) - u(X)\, v(Y)) - h(X, fY). \tag{38}$$

$$\nabla_X V = \lambda\, HX - \alpha fX + \beta(X - v(X)V). \tag{39}$$

$$h(X,V) = \alpha u(X) - \beta\lambda\, v(X) - X\lambda. \tag{40}$$

$$\nabla_X U = fHX + \alpha\lambda X + \beta(\lambda fX - u(X)V). \tag{41}$$

$$(\nabla_X v) = \lambda h(X, Y) - \alpha\, g(fX, Y) + \beta(g(X, Y) - v(X)\, v(Y)). \tag{42}$$

for all $X, Y \in TM$.

Proof. Using (4), (16) and (18) in (33) we obtain

$$((\nabla_X f)Y - u(Y)HX + h(X, Y)U) + ((\nabla_X u)Y + h(X, fY))N$$
$$= \alpha(g(X, Y)V - v(Y)X) + \beta\,(g\,(fX, Y)V - v(Y)fX) + \alpha\lambda\, g(X, Y) + \beta(\lambda g(fX,Y) - u(X)\, v(Y))$$

Equating tangential and normal parts in the above equation, we get (37) and (38) respectively. Using (5) and (18) in (34), we have

$$(\nabla X\, V - \lambda HX) + (h(X, V) + X\lambda)N$$
$$= -\alpha fX + \beta(X - v(X)V) - (\alpha\, u(X) + \beta\lambda\, v(X))N$$

Equating tangential and normal parts we get (39) and (40) respectively. Using (4), (17)and (18) in (35), and equating tangential we get (41). In the last, (42) follows from (36)

Theorem 4.3. *If M be a transversal hypersurface with (f, g, u, v, λ)-structure of a $(\alpha, 0)$ trans-Sasakian manifold, then the 2-form F on M is given by*

$$F(X, Y) \equiv g(X, fY)$$

is closed

Proof. From (37) we get

$$(\nabla_X F)(Y, Z) = -\alpha(g(X, Y)\, v(Z) - g(X, Z)\, v(Y))$$

$$- \beta(g(fX, Y)\, v(Z) - g(fX, Z)\, v(Y))$$

$$+ h(X, Y)\, u(Z) - h(X, Z)\, u(Y))$$

which gives

$$(\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y)$$

$$= 2\beta(\phi(X, Y)\,\eta(Z) + \phi(Y, Z)\,\eta(X) + \phi(Z, X)\,\eta(Y)..$$

If $\beta = 0$. Then

$$(\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) = 0$$

hence $dF = 0$.

From above theorem.

We have following theorem.

Theorem 4.4. *If M is a transversal hypersurface with hyperbolic (f, g, u, v, λ) structures of a trans-Sasakian manifold $\bar{M}$, then 2-form F given by $F(X, Y) = g(X, fY)$ is closed.*

Theorem 4.5. *If M is a transversal hypersurface with almost Hermitian structure (J, G) of a trans-Sasakian manifold, then the 2-form Ω on M is given by*

$$\Omega(X, Y) = G(X, JY)$$

is closed

Using (37), we calculate the Nijenhuis tensor

$$[J, J] = (\nabla_{JX} J)Y - J(\nabla_{JY} J)X - (\nabla_X J)Y + J(\nabla_Y J)X$$

and find that

$$[J, J] = 0.$$

Therefore, in view of theorem 4.4, we have

Theorem 4.5. *Every transversal hypersurface of a trans-Sasakian manifold, admits a Kehlerian structure.*

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