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Multi-Stage Procedures to Select the Largest Mean of Two Normal Distributions : The Case when Variances are Unequal and Unknown

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Abstract : In this paper the problem of selecting the largest mean of two normal distributions is considered when variances of two distributions are unequal and unknown. Various multi-stage procedures are proposed and the second-order approximations are obtained.

1. Introduction

In every day life, one selects or decides on the best product, best treatment, best candidate for a position or the best route for a destination etc. among many other decisions. Such type of problems give rise to a new dimension of 'ranking and selection procedures'. The fundamental and pioneering work, done in fifties in this area is due to Bahadur[1], Bahadur and Robbins[2], Bechhofer[3]. For a brief review on 'ranking and selection' problems and inferential procedures to deal with them, one may refer to Bachhofer, Dunnett and Sobel[4], Gibbons, Olkin and Sobel[5], Gupta and Huang[6], Dudewicz and Koo[7], Laan and Verdooren[8] and Mukhopadhyay[9].

Bechhofer[3] considered the problem of selecting the largest mean of $k\ (\geq 2)$ normal populations having common and known variance and provided the fixed sample size solution. Robbins, Sobel and Starr[10] established the failure of the fixed sample size solution to the problem when the common variance is unknown. They proposed a sequential procedure to handle the problem and studied its first-order asymptotic properties. For the case of two populations, Mukhopadhayay[11] proposed 'improved' two-stage and purely sequential procedures. For the sequential procedures, he derived second-order asymptotics. For the sequential procedure of Robbins, Sobel and Starr[10], Mukhopadhyay and Judge[12] obtained the second-order approximations. Dudewicz and Dalal[13] and Rinott[14] proposed two-stage procedures for selecting the largest mean of two normal populations assuming the variances to be unequal and unknown. Under the same set-up, Mukhopadhyay[9] proposed purely sequential procedure and obtained first-order asymptotics.

In the present paper, the problem of selecting the largest mean of two normal populations is revisited. The population variances are assumed to be ubequal and unknown. Failure of the fixed sample size procedure to deal with the problem is established and various multi-stage procedures are developed to deal with the situation. Improving the already obtained results, the second-order approximations are derived.

In Section 2, we give the set-up of the problem and prove failure of the fixed sample size procedure to deal with it. In Section 3, we propose a purely sequential procedure and applying the theory of Woodroofe[15], obtain the second-order approximations. In Section 4, following Hall[16], a three-stage procedure is proposed and the associated second-order asymptotics are derived. In Section 5, as in Hall[16], an 'accelerated' sequential procedure is developed and second-order approximations are obtained. Finally, in Section 6, motivated by the work of Kumar and Chaturvedi[17] and Chaturvedi and Rani[18], a two-stage procedure is adopted and the second-order approximations are obtained.

2. The Set-up of the Problem and the Failure of the Fixed Sample Size Procedure

Let $\{X_{ij}\}, j = 1, 2, \ldots$ be a sequence of independent and identically distributed (iid) randon variables (rv's) from the i^{th} normal population Π_i $(i = 1, 2,)$ with mean $\mu_i \in (-\infty, \infty)$ and variance $\sigma_i^2 \in (0, \infty)$ i.e., Π_i has the probability density function (pdf)

$$f(x_i\,;\mu_i, \sigma_i^2) = \frac{1}{\sigma_i\sqrt{2\pi}}\exp\left\{\frac{(x_i - \mu_{\{i\}})^2}{2\sigma_i^2}\right\};\ -\infty < x_i < \infty. \tag{2.1}$$

All the four parameters μ_i and σ_i^2 are assumed to be unknown. Our problem is to select the population corresponding to, $\mu_{[2]}$,where $\mu_{[1]} \le \mu_{[2]}$ are the ordered means. Let $\delta \in (0, \infty)$, $P^* \in (1/2,\ 1)$, $\underline{\theta} = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$ and

$$\Omega(\delta) = \{\underline{\theta} : (\mu_{[2]} - \mu_{[1]} \ge \delta)\}.$$

The configuration $\mu_{[2]} = \mu_{[1]} + \delta$ is known as the least favourable configuration (LFC). Denoting by 'CS' the 'correct selection', we have to select the better population such that $P(CS) \ge P^*$ for $\underline{\theta} \in \Omega(\delta)$. It has been shown by Mukhopadhyay[11] that under the LFC, to achieve $P(CS) \ge P^*$, the sample size required from Π_i is the smallest positive integer $n_i \ge n_i^*$, where

$$n_i^* = (2a^2 .\sigma_i^2/\delta^2), \quad i = 1, 2. \tag{2.2}$$

Here the constant 'a' is determined from the equation $\Phi(a) = P^*$, where $\Phi(.)$ denotes the cumulative distribution function of a standard normal variate. Since σ_i^2's have been assumed to be unknown, it follows from (2.2) that the fixed sample size procedure fails to meet the goal for all σ_i^2's.

In what follows, we propose and study various multi-stage procedures. Throughout the remaining part of this paper, we denote by

$$s^2_{i(ni)} = (n_i - 1)^{-1} \sum_{j=1}^{n_i} \left(X_{ij} - \overline{X}_{i(n_i)}\right)^2$$

and

$$\overline{X}_{i(ni)} = n_i^{-1} \sum_{j=1}^{n_i} X_{ij}\,.$$

3. The Purely Sequential Procedure

Let us start with a sample of size $m\,(\geq 2)$ from each Π_i $i = 1, 2$. Then the stopping time $N_i \equiv N_i(\delta)$ is defined by

(3.1) $$N_i = \inf\left[n_i \geq m : n_i \geq (2a^2/\delta^2)\, S^2_{i(n_i)} \right].$$

When we stop, we compare $\overline{X}_{1(N_1)}$ and $\overline{X}_{2(N_2)}$ and choose Π_1 (Π_2) as the better population if $\overline{X}_{1(N_1)} > (<)\ \overline{X}_{2(N_2)}$.

It has been shown by Mukhopadhyay[11] that under the LFC, for the sequential procedure (3.1),

(3.2) $$P(CS) = E\left[\Phi\left(\delta\left\{\frac{\sigma_1^2}{N_1} + \frac{\sigma_2^2}{N_2}\right\}^{-\frac{1}{2}}\right)\right].$$

Mukhopadhyay[11] studied the asymptotic properties of the sequential procedure (3.2) and proved that

(i) $\lim_{\delta \to 0} E\,(N_i/n_i^*) = 1$ and (ii) $\lim_{\delta \to 0} P\,(CS) = P^*$.

Thus, only first-order asymptotics for the expected sample size and probability of correct selection associated with the sequential procedure were achieved.

The following theorem provides the second-order approximations for the sequential procedure (3.1).

Theorem 1 : *For the rule* (3.1) *and all* $m \geq 4$, *as* $\delta \to 0$,

(3.3) $$E(N_i) = n_i^* - 1.59 + o(1)$$

and

(3.4) $$P(CS) = P^* + a(-.71 + .0884a^2)\left(\frac{1}{n_1^*} + \frac{1}{n_2^*}\right)\phi\,(a) + o(\delta^2),$$

where ϕ (.) *denotes the pdf of a standard normal variate.*

Proof : It is easy to see that

$$\lim_{\delta \to 0} N_i = \infty \tag{3.5}$$

i.e., letting $\delta \to 0$ is equivalent to studying the asymptotic properties of the sequential procedure (3.1). From (3.1), we notice the inequality

$$\left(\frac{2a^2}{\delta^2}\right) s^2_{i(N_i)} \leq N_i \leq \left(\frac{2a^2}{\delta^2}\right) s^2_{i(N_i)} + (m - 1)$$

or

$$\left(s^2_{i(N_i)}/\sigma^2_i\right) \leq \left(N_i / n^*_i\right) \leq \left(s^2_{i(N_i)}/\sigma^2_i\right) + (m - 1)/n^*_i ,$$

which on using (3.5), the fact that $s^2_{i(N_i)} \xrightarrow{\text{a. s.}} \sigma^2_i$ as $N_i \to \infty$ $\lim_{\delta \to \infty} n^*_i = \infty$, gives that

$$1 \leq \lim_{\delta \to 0} (N_i/n^*_i) \leq \lim_{\delta \to 0} \sup (N_i/n^*_i) \leq 1,$$

or

$$\lim_{\delta \to 0} (N_i/n^*_i) = 1 \text{ a. s.} \tag{3.6}$$

Using the result that

$$(n_i - 1) s^2_{i(n_i)}/\sigma^2_i = \sum_{j=1}^{n_i - 1} Z^2_j , \tag{3.7}$$

where Z_j is a standard normal variate, we can write the stopping rule (3.1) as

$$N_i = \inf\left\{ n_i \geq m : \sum_{j=1}^{n_i - 1} Z^2_j \leq (n_i - 1)(n_i/n^*_i) \right\}. \tag{3.8}$$

Let us define another stopping rule N^*_i as

$$N^*_i = \inf\left\{ n_i \geq m - 1 : \sum_{j=1}^{n_i} Z^2_j \leq n^2_i (1 + n_i^{-1})/n^*_i \right\}. \tag{3.9}$$

Comparing (3.9) with equation (1.1) of Woodroofe[15], we obtain in his notations

$c = (n^*_i)^{-1}$, $\alpha = 2$, $\beta = 1$, $\mu = 1$, $\tau^2 = 2$, $L(n) = 1 + n^{-1}$, $L_0 = 1$ and $\lambda = n^*_i$. Moreover, for generic B (> 0), denoting by $F(x)$, the cumulative distribution function (cdf) of Z^2_j , we have

$$F(x) = B\int_0^x e^{-y/2}\, y^{\frac{1}{2}-1}\, dy \le B x^{1/2},$$

so that, $a = 1/2$. Table 2.1 of Woodroofe[15] gives that $\nu = .41$. Theorem 2.4 of Woodroofe[15] now gives that, for all $m \ge 4$, as $\delta \to 0$,

$$E(N_i^*) = n_i^* - 2.59 + o(1).$$

Since $N_i^* = N_i - 1$, we get (3.3).

We can write (3.2) as

$$P(CS) = E\left[\Psi\left(\frac{N_1}{n_1^*}, \frac{N_2}{n_2^*}\right)\right] \tag{3.10}$$

where

$$\Psi(x, y) = \Phi\left(a\sqrt{2}\left(x^{-1} + y^{-1}\right)^{-1/2}\right). \tag{3.11}$$

Expanding $\Psi(x, y)$ around '$x = 1$, $y = 1$' by Taylor's series for two variables, we obtain for $|u - 1| \le |x - 1|$ and $|v - 1| \le |y - 1|$.

$$\begin{aligned}\Psi(x, y) = \Psi(1, 1) &+ \left[(x-1)\left\{\left.\frac{\partial\Psi(x, y)}{\partial x}\right|_{x=1,\, y=1}\right\} + (y-1)\left\{\left.\frac{\partial\Psi(x, y)}{\partial y}\right|_{x=1,\, y=1}\right\}\right] \\ &+ (1/2)\left[(x-1)^2\left\{\left.\frac{\partial^2\Psi(x, y)}{\partial x^2}\right|_{(u, v)}\right\} + 2(x-1)(y-1)\left\{\left.\frac{\partial^2\Psi(x, y)}{\partial x\,\partial y}\right|_{(u, v)}\right\}\right. \\ &\left. + (y-1)^2\left\{\left.\frac{\partial^2\Psi(x, y)}{\partial y^2}\right|_{(u, v)}\right\}\right].\end{aligned} \tag{3.12}$$

We note that

$$\Psi(1, 1) = \Phi(a) = P^* \tag{3.13}$$

$$\frac{\partial\Psi(x, y)}{\partial x} = \left(\frac{a}{\sqrt{2}\,x^2}\right)\left(x^{-1} + y^{-1}\right)^{-3/2} \phi\left(a\sqrt{2}\left(x^{-1} + y^{-1}\right)^{-1/2}\right), \tag{3.14}$$

$$\frac{\partial\psi(x, y)}{\partial y} = \left(\frac{a}{\sqrt{2}\,y^2}\right)\left(x^{-1} + y^{-1}\right)^{-3/2} \phi\left(a\sqrt{2}\left(x^{-1} + y^{-1}\right)^{-1/2}\right), \tag{3.15}$$

(3.16)
$$\frac{\partial^2\psi(x,y)}{\partial x^2} = \left(\frac{a}{\sqrt{2}\,x^3}\right)\left[-2+\left(\frac{3}{2x}\right)\left(x^{-1}+y^{-1}\right)^{-1} + \left(\frac{a^2}{x}\right)\left(x^{-1}+y^{-1}\right)^{-3/2}\right]$$
$$\times\left(x^{-1}+y^{-1}\right)^{-3/2}\phi\left(a\sqrt{2}\left(x^{-1}+y^{-1}\right)^{-1/2}\right)$$

(3.17)
$$\frac{\partial^2\psi(x,y)}{\partial y^2} = \left(\frac{a}{\sqrt{2}\,y^3}\right)\left[-2+\left(\frac{3}{2y}\right)\left(x^{-1}+y^{-1}\right)^{-1} + \left(\frac{a^2}{y}\right)\left(x^{-1}+y^{-1}\right)^{-3/2}\right]$$
$$\times\left(x^{-1}+y^{-1}\right)^{-3/2}\phi\left(a\sqrt{2}\left(x^{-1}+y^{-1}\right)^{-1/2}\right)$$

and

(3.18)
$$\frac{\partial^2\psi(x,y)}{\partial x\,\partial y} = \left(\frac{a}{2\sqrt{2}\,x^2y^2}\right)\left[3+2a^2\left(x^{-1}+y^{-1}\right)^{-1/2}\right]$$
$$\times\left(x^{-1}+y^{-1}\right)^{-5/2}\phi\left(a\sqrt{2}\left(x^{-1}+y^{-1}\right)^{-1/2}\right).$$

It is easy to check from (3.6) that, as $\delta \to 0$,

(3.19)
$$U \xrightarrow{\text{a. s.}} 1, \quad V \xrightarrow{\text{a. s.}} 1.$$

It follows from Theorem 2.1 of Woodroofe[15] that

$$\left(N_i - n_i^*\right)/\left(n_i^*\right)^{-1/2} \to N(0,\,2)$$

and his Theorem 2.3 gives that $\left(N_i - n_i^*\right)^2/\left(n_i^*\right)$ is uniformly integrable for all $m \geq 4$.

Utilizing (3.3), (3.12) – (3.19) and the independence of N_1 and N_2, it follows from (3.10) that, for all $m \geq 4$, as $\delta \to 0$,

(3.20)
$$P(CS) = P^* + E\left[\left(\frac{N_1}{n_1^*} - 1\right)\left(\frac{a}{4}\right)\phi(a) + \left(\frac{N_2}{n_2^*} - 1\right)\left(\frac{a}{4}\right)\phi(a)\right]$$
$$+\left(\frac{1}{2}\right)E\left[\left(\frac{N_1}{n_1^*} - 1\right)^2\left(\frac{a}{4}\right)\phi(a)\left(-\frac{5}{4}+\frac{a^2}{2^{3/2}}\right)\right.$$

$$+ 2\left(\frac{N_1}{n_1^*} - 1\right)\left(\frac{N_2}{n_2^*} - 1\right)\left(\frac{a}{16}\right)\left(3 + a^2\sqrt{2}\right)\phi(a)$$

$$+ \left(\frac{N_2}{n_2^*} - 1\right)^2\left(\frac{a}{4}\right)\phi(a)\left(-\frac{5}{4} + \frac{a^2}{2^{3/2}}\right)\Bigg]$$

$$= P^* + \left(\frac{a}{4}\right)\phi(a)\left[\frac{1}{n_1^*}\left\{-1.59 + o(1)\right\} + \frac{1}{n_2^*}\left\{-1.59 + o(1)\right\}\right]$$

$$+ \left(\frac{a}{8}\right)\phi(a)\left[\frac{2}{n_1^*}\left\{-1.25 + \frac{a^2}{2^{3/2}}\right\} + \frac{2}{n_2^*}\left\{-1.25 + \frac{a^2}{2^{3/2}}\right\}\right] + o(\delta^2)$$

$$= P^* + a\left(-0.71 + 0.0884a^2\right)\left(\frac{1}{n_1^*} + \frac{1}{n_2^*}\right)\phi(a) + o(\delta^2).$$

Thus, we have (3.4).

4. The Three-Stage Procedure

Let $\eta_i \in (0, 1)$ be specified. We start with the sample of size m (≥ 2) from each of Π_i, $i = 1, 2$, where m is chosen in such a manner that $m = o(\delta^2)$ as $\delta \to 0$ and $\limsup_{\delta \to 0}(m/n_i^*) < 1$. Then we collected $M_i - m$ more observations from Π_i, where

(4.1) $$M_i = \max\left\{m, \left[\frac{2\eta a^2 s_{i(m)}^2}{\delta^2}\right]^+ + 1\right\}.$$

Here, $[y]^+$ denotes the largest positive integer $< y$. Finally, at the third stage, we take $N_i - M_i$ observations from Π_i, where

(4.2) $$N_i = \max\left\{M_i, \left[\frac{2a^2 s_{i(M_i)}^2}{\delta^2}\right]^+ + 1\right\}.$$

We choose Π_1 (Π_2) as the better population if $\overline{X}_{1(N_1)} > (<)\ \overline{X}_{2(N_2)}$

Before proving the main theorem, we establish some lemmas.

Lemma 1 : *For the three-stage procedure* (4.1) – (4.2) *as* $\delta \to 0$.

(4.3) $$E(N_i) = n_i^* - \left(\frac{3}{\eta_i}\right) + \left(\frac{1}{2}\right) + o(1)$$

and

(4.4) $$E(N_i^2) = n_i^{*2} + (1 - 3\eta_i^{-1})n_i^* - \eta_i^{-1}(3 + 5\eta_i^{-1}) + (1/2) + o(\delta^2).$$

Proof : By the definition,

$$E(N_i) = \mathrm{I} + \mathrm{II},$$

where, denoting by I (.), the indicator function,

$$I = E\left[N_i I\left(\{M_i \le m\}\, U \left\{N_i \le \left[\frac{2\eta_i a^2 s_{(m)}^2}{\delta^2}\right]^+ + 1\right\}\right)\right]$$

and

$$II = E\left[N_i I\left(\left[\frac{2a^2 s_{i(M_i)}^2}{\delta^2}\right]^+ + 1 > M_i\right)\right].$$

It follows from Hall[16] that, $\delta \to 0$,

(4.5) $$I = o(1).$$

Now, denoting by
$$T_{M_i} = 1 - \left(\left\{\frac{2a^2 s_{i(M_i)}^2}{\delta^2}\right\} - \left[\frac{2a^2 s_{i(M_i)}^2}{\delta^2}\right]^+\right)$$

we can write

$$II = \left(\frac{2a^2}{\delta^2}\right) E\left(s_{i(M_i)}^2\right) + E\left\{T_{M_i}\right\}.$$

It follows from Hall[16] that, as $\delta \to 0$, T_{M_i} is uniform over (0, 1). Thus, as $\delta \to 0$,

(4.6) $$II = \left(\frac{2a^2}{\delta^2}\right) E\left(s_{i(M_i)}^2\right) + \left(\frac{1}{2}\right).$$

We now evaluate the value of $E(s_{i(M_i)}^2)$. To this end, using (3.7), we write

$$E(s_{i(M_i)}^2) = \sigma_i^2 E\left[\frac{1}{(M_i - 1)} \sum_{j=m}^{M_i - 1} Z_j^2\right]$$

$$= \sigma_i^2 E\left[\left\{\left(M_i - m\right) + (m - 1)\right\}^{-1}\right.$$

$$\times \left\{ \sum_{j=1}^{m-1} Z_j^2 + E\left(\sum_{j=m}^{m_i-1} Z_j^2 \,/\, Z_1, \ldots, Z_{m-1} \right) \right\} \Bigg] \Bigg]$$

(4.7) $$= \sigma_i^2 + \sigma_i^2 E\left[M_i^{-1} \sum_{j=1}^{m-1} Z_j^2 \right] + O\left(M_i^{-1} \right).$$

Expanding M_i^{-1} around $\eta_i \eta_i^*$ by Taylor's series and denoting by R_m^*, the remainder term, we obtain

$$M_i^{-1} = \left(\eta_i n_i^*\right)^{-1} - \left(\eta_i n_i^*\right)^{-2} \left(M_i - \eta_i n_i^*\right) + R_{n_i^*}.$$

(4.8) $$= \left(\eta_i n_i^*\right)^{-1} - \left(\eta_i n_i^*\right)^{-2} \left[\frac{\eta_i n_i^*}{(m-1)} \left(\sum_{j=1}^{m-1} Z_j^2 \right) + T_m - \left(\eta_i n_i^*\right) \right] + R_{n_i^*}.$$

Substituting (4.8) in (4.7), we get

$$E(s_{i(M_i)}^2) = \sigma_i^2 + \sigma_i^2 E\left[2\left(\eta_i n_i^*\right)^{-1} \left(\sum_{j=1}^{m-1} Z_j^2 \right) - \frac{(\eta_i n_i^*)^{-1}}{(m-1)} \left(\sum_{j=1}^{m-1} Z_j^2 \right)^2 \right.$$

(4.9) $$\left. - \left(\eta_i n_i^*\right)^{-2} T_m \left(\sum_{j=1}^{m-1} Z_j^2 \right)^{-1} \right] + R_{n_i^*}.$$

It follows from Cauchy-Schwartz inequality that

$$\mathrm{Cov}^2 \left(T_m, \sum_{j=1}^{m-1} Z_j^2 \right) \le \mathrm{Var}\,(T_m)\,\mathrm{Var} \left(\sum_{j=1}^{m-1} Z_j^2 \right)$$

$$= \frac{(m-1)}{2}$$

(4.10) $$= 0\,(\delta)^2 \text{ by the choice of } m,$$

implying that T_m and $\sum_{j=1}^{m-1} Z_j^2$ are asymptotically uncorrelated. Utilizing this result, we obtain from (4.9) that as $\delta \to 0$,

(4.11) $$E(s_{i(M_i)}^2) = \sigma_i^2 - \frac{3\sigma_i^2}{(\eta_i n_i^*)} + o(\delta^2).$$

Making substitution from (4.11) in (4.6), we get

(4.12) $$II = n_i^* - \frac{3}{\eta_i} + \frac{1}{2} + o(1).$$

Result (4.3) now follows on combining (4.5) and (4.12).

Furthermore, we have

$$E(N_i^2) = E\left\{\frac{2\,a^2 s_{i(M_i)}^2}{\delta^2} + T_{M_i}\right\}^2$$

(4.13) $$= \left(\frac{4a^4}{\delta^4}\right) E\left\{s_{i(M_i)}^4\right\} + \left(\frac{4a^2}{\delta^2}\right) E\left\{s_{i(M_i)}^2\, T_{M_i}\right\} + E\left\{T_{M_i}^2\right\}$$

$$E\left\{s_{i(M_i)}^2\right\} = \sigma_i^4\, E\left\{\frac{1}{(M_i-1)} \sum_{j=1}^{M_i-1} Z_i^2\right\}^2$$

$$= \sigma_i^4\, E\left[\left(M_i - m\right)^{-2} \sum_{1=0}^{2} \binom{2}{1} \left(\sum_{j=1}^{m-1} Z_j^2\right)^1 E\left\{\left(\sum_{j=1}^{M_i-1} Z_j^2\right)^{2-1} \cdot \middle| Z_i, \ldots, Z_{m-1}\right\}\right] + O(M_i^{-2})$$

$$= \sigma_i^4\, E\left[1 + 2M_i^{-1}\left(\sum_{j=1}^{M_i-1} Z_j^2\right) + M_i^{-2}\left(\sum_{j=1}^{m-1} Z_j^2\right)^2\right] + O(M_i^{-1}),$$

which on using (4.8) and Taylor's series expansion of M_i^{-2} gives that

$$E\left\{s_{i(M_i)}^4\right\} = \sigma_i^4\, E\left[1 - \frac{3}{(\eta_i n_i^*)} + \left\{(\eta_i n_i^*)^{-2} - 2(\eta_i n_i^*)^{-3}(M_i - \eta_i n_i^*)\right\}\left\{\sum_{j=1}^{m-1} Z_j^2\right\}^2\right] + o(\delta^2)$$

(4.14) $$= \sigma_i^4\left[1 - \frac{3}{(\eta_i n_i^*)} - \frac{5}{(\eta_i n_i^*)^2}\right] + o(\delta^2).$$

Substituting from (4.11) and (4.14) in (4.13), we get

$$E\left(N_i^2\right) = n_i^* - \frac{3n_i^*}{\eta_i} - \frac{5}{\eta_i^2} + n_i^* - \frac{3}{\eta_i} + \frac{1}{2} + o(\delta^2),$$

and (4.4) holds.

Lemma 2 : *For* $0 < \varepsilon < 1$ *and some positive integer* r, *as* $\delta \to 0$.

$$P\left(N_i \le \varepsilon\, n_i^*\right) = 0(\delta).$$

Proof : It follows from the definition of N_i that

$$P\left(N_i \le \varepsilon n_i^*\right) \le P\left[\frac{2a^2 s_{i(M_i)}^2}{\delta^2} \le \varepsilon n_i^*\right]$$

$$\le P\left[\,\left| s_{i(M_i)}^2 - \sigma_i^2 \right| \ge \sigma_i^2 (1-\varepsilon)\right]$$

$$\le P\left[\max_{m \le M \le \left[\varepsilon n_i^*\right]^2} \left\{\left| s_{i(M_i)}^2 - \sigma_i^2 \right| \ge \sigma_i^2 (1-\varepsilon)\right\}\right]$$

$$= O\left(m^{-1}\right),$$

by Hajeck- Rényi inequality and the result follows.

Now we prove the main result of this section, which provides second-order approximations for the probability of correct selection associated with three-stage procedure.

Theorem 2 : *As* $\delta \to 0$,

$$P(CS) = P^* + \left(\frac{a}{32}\right)\phi(a)\left[\frac{1}{n_1^*}\left\{\sqrt{2}\,a^2\left(1+\frac{3}{\eta_1}\right) - \left(1+\frac{39}{\eta_1}\right)\right\}\right.$$

$$\left. + \frac{1}{n_2^*}\left\{\sqrt{2}\,a^2\left(1+\frac{3}{\eta_2}\right) - \left(1+\frac{39}{\eta_2}\right)\right\}\right] + o\left(\delta^2\right).$$

Proof : It is easy to check that the coverage probability corresponding to three-stage procedure is same as that given at (3.2). Using the expansion (3.12) an applying Lemma 1 , we get as $\delta \to 0$,

$$P(CS) = P^* + \left(\frac{a}{4}\right)\phi(a)E\left[\left(\frac{N_1}{n_i^*}-1\right)+\left(\frac{N_2}{n_2^*}-1\right)\right]+\left(\frac{a}{8}\right)\left(-\frac{5}{4}+\frac{a^2}{2^{3/2}}\right)\phi(a)E\left[\left(\frac{N_1}{n_1^*}-1\right)^2+\left(\frac{N_2}{n_2^*}-1\right)^2\right]$$

$$= P^* + \left(\frac{a}{4}\right)\phi(a)\left[\frac{1}{n_i^*}\left\{-\frac{3}{\eta_1}+\frac{1}{2}+o(1)\right\}+\frac{1}{n_2^*}\left\{-\frac{3}{\eta_2}+\frac{1}{2}+o(1)\right\}\right.$$

$$\left. + \left(\frac{5}{8}+\frac{a^2}{2^{5/2}}\right)\left\{\frac{1}{n_i^*}\left(1+\frac{3}{\eta_2}\right)\frac{1}{n_2^*}\left(1+\frac{3}{\eta_2}\right)\right\}\right] + o\left(\delta^2\right)$$

$$= P^* + \left(\frac{a}{32}\right)\phi(a)\left[\frac{1}{n_i^*}\left\{\sqrt{2}\,a^2\left(1+\frac{3}{\eta_1}\right) - \left(1+\frac{39}{\eta_1}\right)\right\}\right.$$

$$+ \frac{1}{n_2^*} \sqrt{2}\, a^2 \left(1 + \frac{3}{\eta_1}\right) - \left(1 + \frac{39}{\eta_1}\right)\Bigg] + o(\delta^2)$$

and the result follows.

5. An 'Accelerated' Sequential Procedure

Take $m\ (\geq 2)$ to be the initial sample size from each of the two populations, where m is chosen so as to satisfy $m = o(\delta^2)$ as $\delta \to 0$ and $\lim\limits_{\delta \to 0} \sup (m/n_i^*) < 1$. Let $\eta_i \in (0, 1)$ be specified. Start sampling sequentially from the i^{th} population with stopping time M_i defined by

$$M_i = \inf \left\{ n_i \geq m : n_i \geq \frac{2\eta_i a^2 s_{i(M_i)}^2}{\delta^2} \right\}. \tag{5.1}$$

Based on these M_i observations, we compute $s_{i(M_i)}^2$. Then we jump ahead and collect $N_i - M_i$ more observations from the ith population, where

$$N_i = \max \left\{ M_i, \left[\frac{2\, a^2 s_{i(M_i)}^2}{\delta^2} \right] + 1 \right\}. \tag{5.2}$$

We select $\Pi_1\ (\Pi_2)$ as the better population if $\bar{X}_{1(N_1)} > (<)\ \bar{X}_{2(N_2)}$. The probability of correct selection is same as that given at (3.2) with N_i determined by the present rule.

We first prove a Lemma.

Lemma 3 : *For all* $m \geq 4$, *as* $\delta \to 0$,

$$E(N_i) = n_i^* + (1/2 - 2\eta_i^{-1}) + o(1) \tag{5.3}$$

and

$$\mathrm{Var}(N_i) = 2\,\eta_i^{-1} n_i^* + o(\delta^{-2}). \tag{5.4}$$

Proof : Denoting by

$$U_{M_i} = 1 - \left\{ \left(\frac{2\, a^2 s_{i(M_i)}^2}{\delta^2} \right) - \left[\frac{2\, a^2 s_{i(M_i)}^2}{\delta^2} \right] \right\} \tag{5.5}$$

we can write

$$E(N_i) = I + II,$$

where

$$\mathrm{I} = E\left[N_i \;\middle|\; M_i > \left[\frac{2\,a^2 s_{i(M_i)}^2}{\delta^2} \right] + 1 \right] \tag{5.6}$$

and

$$\mathrm{II} = \left(\frac{2\,a^2}{\delta^2} \right) E\,(s_{i(M_i)}^2) + E\,(U_{M_i}). \tag{5.7}$$

It follows from Hall[16] that, as $\delta \to 0$, $I = o(1)$ and U_{M_i} is uniform over (0, 1). Thus, we conclude from (5.5) (5.6) and (5.7) that, as $\delta \to 0$,

$$E\,(N_i) = \left(\frac{2\,a^2}{\delta^2} \right) E\left(s_{i(M_i)}^2 \right) + \frac{1}{2} + o(1). \tag{5.8}$$

Now, we evaluate $E\,(s_{i(M_i)}^2)$. To this end, we write the stopping rule (5.1) as

$$M_i = \inf \left\{ n_i \ge m : \sum_{j=1}^{n_i - 1} Z_j^2 \le (\eta_i n_i^*)^{-1} (n_i - 1) n_i \right\}. \tag{5.9}$$

Define another stopping rule M_i^* , as

$$M_i^* = \inf \left\{ n_i \ge m - 1 : \sum_{j=1}^{n_i} Z_j^2 \le (\eta_i n_i^*)^{-1} - n_i^2 (1 + n_i^{-1}) \right\}. \tag{5.10}$$

Comparing (5.10) with equation (1.1) of Woodroofe[15], we obtain in his notations,

$c = (\eta_i n_i^*)^{-1}$, $\alpha = 2$, $\beta = 1$, $\mu = 1$, $\tau^2 = 2$, $L(n) = 1 + n^{-1}$, $L_0 = 1$,

$\lambda = \eta_i n_i^*$, $a = 1/2$, and $\nu = 0.41$.

It now follows from Theorem 2.4 of Woodroofe[15] that for all $m \ge 4$, as $\delta \to 0$,

$$E\,(M_i^*) = \eta_i n_i^* - 2.59 + o\,(1).$$

Since $M_i^* = M_i - 1$, we get for all $m \ge 4$, as $\delta \to 0$,

$$E\,(M_i) = \eta_i n_i^* - 1.59 + o\,(1). \tag{5.11}$$

Let us consider the difference

$$D_\delta = M_i - \frac{2\,\eta_i a^2 s^2_{i(M_i)}}{\delta^2} \tag{5.12}$$

It follows from Woodroofe[15] that the mean of the asymptotic distribution of D_δ is 0.41. Thus, we obtain from (5.11) and (5.12) that, for all $m \geq 4$, as $\delta \to 0$,

$$\left(\frac{2a^2}{\delta^2}\right) E\left(s^2_{i(M_i)}\right) = \eta_i^{-1}\left[E(M_i) - 0.41\right]$$

$$= n_i^*\left[-2\,\eta_i^{-1} + o(1)\right]. \tag{5.13}$$

Substituting from (5.13) in (5.8), we get

$$E(N_i) = n_i^* + \left(\frac{1}{2} - 2\,\eta_i^{-1}\right) + o(1)$$

and (5.3) holds.

Let

$$h(M_i) = \left|\frac{(M_i - \eta_i n_i^*)}{(\eta_i n_i^*)^{1/2}}\right|.$$

Then it follows that $h_{(M_i)} \xrightarrow{L} N(0,\ 2)$ as $\delta \to 0$. Moreover, from Theorem 2.3 of Woodroofe[15], $h^2_{(M_i)}$ is uniformly integrable for all $m \geq 4$. Hence, for all $m \geq 4$, as $\delta \to 0$,

$$E\left[h^2(M_i)\right] = 2 + o(1).$$

By the definition

$$\mathrm{Var}\ (N_i) = \eta_i^{-2}\ \mathrm{Var}\ (M_i)$$

$$= \eta_i^{-2}\left[(\eta_i^{-2} n_i^*)\ \{\ 2 + o(1)\ \}\right]$$

and (5.4) holds.

The following theorem provides the main result of this section.

Theorem 3 : *For all $m \geq 4$, as $\delta \to 0$,*

$$P(CS) = P^* + \left(\frac{a}{4}\right)\left[\frac{1}{n_1^*}\left(\frac{1}{2} - \frac{13}{4\eta_1} + \frac{0.3535a^2}{\eta_1}\right)\right.$$

$$+ \frac{1}{n_2^*}\left(\frac{1}{2} - \frac{13}{4\eta_2} + \frac{0.3535a^2}{\eta_2}\right)\Bigg]\phi(a) + o(\delta^2).$$

Proof : Applying Lemma 3 to the expansion (3.12), we get for all $m \geq 4$, as $\delta \to 0$,

$$P(CS) = P^* + \left(\frac{a}{4}\right)\phi(a)\,E\left[\left(\frac{N_1}{n_1^*} - 1\right) + \left(\frac{N_2}{n_2^*} - 1\right)\right]$$

$$+ \left(\frac{a}{\delta}\right)\left[-\frac{5}{4} + \frac{a^2}{2^{3/2}}\right]\phi(a)\,E\left[\left(\frac{N_1}{n_1^*} - 1\right)^2 + \left(\frac{N_2}{n_2^*} - 1\right)^2\right]$$

$$= P^* + \left(\frac{a}{4}\right)\phi(a)\left[\frac{1}{n_1^*}\left(\frac{1}{2}2\eta_1^{-1}\right) + \frac{1}{n_2^*}\left(\frac{1}{2}2\eta_2^{-1}\right)\right] + \left(\frac{a}{8}\right)\left(-\frac{5}{4} + \frac{a^2}{2^{3/2}}\right)\phi(a)$$

$$\times\left[\frac{1}{n_1^{*2}}\left\{\mathrm{Var}(N_1) + (E(N_1) - n_2^*)^2\right\} + \frac{1}{n_2^{*2}}\left\{\mathrm{Var}(N_2) + (E(N_2) - n_2^*)^2\right\}\right]$$

$$= P^* + \left(\frac{a}{4}\right)\phi(a)\left[\frac{1}{n_1^*}\left(\frac{1}{2}2\eta_1^{-1}\right) + \frac{1}{n_2^*}\left(\frac{1}{2}2\eta_2^{-1}\right) + \left(\frac{a}{4}\right)\left(-\frac{5}{4} + \frac{a^2}{2^{3/2}}\right)\phi(a)\right]$$

$$\times\left[\frac{1}{(\eta_1 n_1^*)} + \frac{1}{(\eta_2 n_2^*)}\right]$$

$$= P^* + \left(\frac{a}{4}\right)\left[\frac{1}{n_1^*}\left(\frac{1}{2} - \frac{13}{4\eta_1} + \frac{0.3535a^2}{\eta_1}\right)\right.$$

$$\left. + \frac{1}{n_2^*}\left(\frac{1}{2} - \frac{13}{4\eta_2} + \frac{0.3535a^2}{\eta_2}\right)\phi(a) + o(\delta^2)\right].$$

and the result follows.

6. The Two-Stage Procedure

Start with a sample of size $m \geq 2$ from each of the two populations, where, as in Kumar and Chaturvedi[17] and Chaturvedi and Rani[18], m is chosen in such a manner that $m = o(\delta^2)$ as $\delta \to 0$ and $\limsup_{\delta \to 0} (m/n_i^*) < 1$. Based, on these m observations , we compute $s_{i(m)}^2$. Then, the second stage sample size being given by

(6.1) $$N_i = \max\left\{ m, \left[\frac{2h_{(m-1)}s^2_{i(m)}}{\delta^2}\right] + 1 \right\},$$

where

(6.2) $$P\left\{ t^2_{(m-1)} \leq h_{(m-1)} \right\} = P^*$$

and $t_{(m-1)}$ follows Student's t-distribution with ($m - 1$) degrees of freedom. We select Π_1 (Π_2) as the better population if $\bar{X}_{1(N_1)} > (<) \bar{X}_{2(N_2)}$. The probability of correct selection for the two stage procedure (6.1) - (6.2) is the same as that given at (3.2).

The main results are proved in the following theorem.

Theorem 4 : *As* $\delta \to 0$,

(6.3) $$E(N_i) = n_i^* + \frac{1}{2} + o(1)$$

and

(6.4) $$P(CS) \geq P^*.$$

Proof : Denoting by

$$T_m = 1 - \left\{ \left(\frac{2\,h_{(m-1)}s^2_{i(m)}}{\delta^2}\right) - \left[\frac{2\,h_{(m-1)}s^2_{i(m)}}{\delta^2}\right]^+ \right\}$$

we can write

(6.5) $$E(N_i) = \left(\frac{2h_{(m-1)}}{\delta^2}\right) E\left(s^2_{i(m)}\right) + E(T_m).$$

It follows from Hall[16] that T_m is uniform over (0, 1) as $m \to \infty$. Utilizing the unbiasedness of $s^2_{i(m)}$ for σ_i^2 and the fact that

$$h_{(m-1)} \xrightarrow{a.s.} a^2 \text{ as } m \to \infty,$$

we obtain from (6.5) that, as $\delta \to 0$,

$$E(N_i) = n_i^* + \frac{1}{2} + o(1)$$

and (6.3) holds.

Utilizing (6.1), we obtain from (3.2) that

$$P(CS) \geq E\left[\Phi\left(a\sqrt{2}\left(\frac{n_1^* \delta^2}{2h_{(m-1)} s_{1(m)}^2} + \frac{n_2^* \delta^2}{2h_{(m-1)} s_{2(m)}^2}\right)^{-1/2}\right)\right]$$

$$\geq \Phi\left(\left(\frac{2h_{(m-1)} \chi^2_{(m-1)}}{(m-1)}\right)^{-1/2}\right)$$

$$= P\left[t^2_{(m-1)} \leq h_{(m-1)}\right]$$

$$= P^*,$$

and (6.4) follows.

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