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# On ξ-Conformally Flat Contact Metric Manifolds

**J. P. Srivastava and Prabha Sharma**
P. G. Department of Mathematics, University of Jammu, Jammu-180 006.

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**Abstract** : Contact Riemannian manifolds satisfying $R(\xi, X)R = 0$ where $\xi$ belongs to the $k$-nullity distribution or a condition similar to it have been studied by various authors[1,2,3]. The aim of this paper is to prove that a $\xi$-conharmonically flat contact metric manifold is locally isometric to a unit sphere.

## 1. *Ak*-Contact Manifold

A $(2n + 1)$ dimensional $C^{\infty}$ manifold $M^{2n+1}$ is said to be a contact manifold if it carries a global *l*-form $\eta$, such that $\eta \Lambda (d\eta)^n \neq 0$. For a given contact form $\eta$ it is well known that there exists a unique vector field $\xi$ (called the characteristic vector field) in $M$ such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$. A Riemannian metric $g$ is said to be an associate metric if there exists a tensor field $\phi$ of type (1, 1) such that $d\eta(X, Y) = g(X, \phi Y)$, $\eta(X) = g(X, \xi)$ and $\phi^2 = -I + \eta \otimes \xi$. The structure $(\Phi, \xi, n, g)$ on $M^{2n+1}$ is called a contact metric structure and $M^{2n+1}$ is called a contact metric manifold.

Given a contact metric structure $(\phi, \xi, \eta, g)$ we define a tensor field $h$ by $h = \frac{1}{2}(L_{\xi}\phi)$ where $L$ denotes the Lie-differentiation. $h$ is a symmetric operator which anticommutes with $\phi$ and hence if $\lambda$ is also an eigenvalue of $h$ with eigen vector $X$, then $-\lambda$ is also an eigenvalue of the eigen vector $\phi X$. Clearly $h\xi = 0$ and it is well known that $\xi$ is a killing vector field with respect to $g$ if $h = 0$.

A contact metric manifold for which $\xi$ is a killing vector field is called a $k$-contact manifold[2, 4], A $k$-contact Riemannian manifold is called Sasakian[5] if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X \tag{1.1}$$

holds, where the operator of convariant differentiation is denoted by $\nabla$.

The $k$-nullity distribution[4] of a Riemannian manifold for a real number $k$ is a distribution.

$$N(k) : x \to N_x(k) = \{Z \in T_x M : R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y) ; X, Y \in T_x M\} \tag{1.2}$$

Thus, if $\xi$ belong to the $k$-nullity distribution then we get

(1.3) $$R(X, Y)\xi = k(g(Y, \xi)X - g(X, \xi)Y)$$

$$= k(\eta(Y)X - \eta(X)Y)$$

From (1.2) it is clear that when $k = 1$, the manifold becomes a Sasakian one.

A Sasakian manifold is $k$-contact but the converse is not true in general. However a 3-dimensional $k$-contact manifold is Sasakian.

## 2. Preliminaries

In a $k$-contact Riemannian manifold, the following relations hold[4,5],

(2.1) $$\phi(\xi) = 0$$

(2..2) $$\eta(\xi) = 1$$

(2.3) $$\phi^2 X = -X + \eta(X)\xi$$

(2.4) $$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

(2.5) $$g(X, \xi) = \eta(X)$$

(2.6) $$g\left(X, \nabla_y \xi\right) + g\left(Y, \nabla_x \xi\right) = \left(L_\xi g\right)(X, Y) = 0$$

(2.7) $$\nabla_X \xi = -\phi X, \nabla_\xi \xi = 0$$

(2.8) $$g(R(\xi, X)Y, \xi) = g(X, Y) - \eta(X)\eta(Y)$$

(2.9) $$\left(\nabla_X \phi\right)(Y) = R(\xi, X)Y$$

(2.10) $$\left(\nabla_X \phi\right)\phi Y + \phi\left(\nabla_X \phi\right)Y = g(X, \phi Y)\xi - \eta(Y)\phi X$$

Thus,

(2.11) $$\phi R(\xi, X)Y + R(X, \xi)\phi Y = g(Y, \phi X)\xi + \eta(Y)\phi X$$

and in particular

(2.12) $$R(X, \xi)\xi = X - \eta(X)\xi$$

and

(2.13) $$g(Q\xi, \xi) = 2\eta$$

where $Q$ is the Ricci operator defined by

(2.14) $$QX = \sum_i R\left(X, e_i\right)e_i$$

for any local orthonormal basis of vector field in $M$, $\{e_i\}_{1 \le i \le 2n+1}$. It should be noted that if we take this local basis in such a way that $e_{2n+1} = \xi$, then $\{\phi e_i, \xi\}_{1 \le i \le 2n}$ is another local orthonormal basis.

## 3. *A* *k*-Contact Manifold with the Characteristic Vector Field ξ Belonging to the *k*-Nullity Distribution

If ξ belong to *k*-nullity distribution, then

(3.1) $$R(X, Y)\xi = k[g(Y, \xi)X - g(X, \xi)Y]$$

$$= k(\eta(Y)X - \eta(X)Y).$$

Putting $X = \xi$ in (3.1) and using (2.2), we get

(3.2) $$R(\xi, Y)\xi = k(\eta(Y)\xi - Y).$$

If possible, let us suppose that $k = 0$, we get

(3.3) $$\phi^2 X = 0$$

which is a contradiction. Thus we have

**Theorem 3.1** : *In a k-contact manifold with real number k for the k-nullity distribution cannot be zero.*

## 4. ξ-Conformally Flat Contact Manifold

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a contact metric manifold, then,

$$\eta(\phi T(M)) = d\eta(\xi, T(M)) = 0$$

Conversely, if $\eta(X) = 0$ then $X = -\phi^2 X \in \phi T(M)$.

The Weyl conformal curvature tensor with respect to the metric $g$ is the tensor field of type (1, 3) defined by

$$(4.1)\quad C(X, Y)Z = R(X, Y)z - \frac{1}{(2n-1)} \times \left\{ g(QY, z)X + g(Y, z)QX - g(QX, z)Y - g(X, z)QY \right\} + \frac{r}{2n(2n-1)}\left\{ g(Y, z)X - g(X, z)Y \right\}.$$

where $X, Y, Z \in T(M)$ and where $Q$ is the symmetric endomorphism of the tangent space at each point. Corresponding to the Ricci tensors[5] i.e.

$$(4.2)\quad g(QX, Y) = S(X, Y)$$

Hence

$$(4.3)\quad \eta(C(X, y)Z) = g(C(X, Y)Z, \xi) = \eta(R(X, Y)Z) + \left\{ \frac{r}{2n(2n-1)} - \frac{2n}{2n-1} \right\}\left[ g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \right] - \frac{1}{(2n-1)}\left\{ S(Y, Z)\eta(X) - S(X, Z)\eta(Y) \right\}$$

Putting $Z = \xi$ in (4.3)

$$(4.4)\quad \eta(C(X, Y)\xi) = 0.$$

Again putting $X = \xi$ in (4.3) we get

$$(4.5)\quad \eta(C(\xi, Y)Z) = \left( \frac{r}{2n} - 1 \right) \frac{1}{(2n-1)} [g(Y, Z) - \eta(Y)\eta(Z)] - \left( \frac{1}{(2n-1)} \right) [S(Y, Z) - 2n\eta(Y)\eta(Z)].$$

On the other hand, the Lie algebra $T(M)$ can be decomposed into a direct sum $T(M) = \phi T(M) \oplus L$ where $L$ is the 1-dimensional distribution on $M$ generated by the structure vector field $\xi$.

**Definition** : A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to $\xi$-conformally flat if the linear operator $C(X, Y)$ is an endomornism of $\phi\, T(M)$ i.e. if

$$C(X, Y)\phi T(M) \subset \phi T(M)$$

Evidently, $\xi$-conformally flat means that the projection of $C(X, Y)\, \phi\, T(M)$ onto $L$ is zero.

We can see that any 3-dimensional contact metric manifold is $\xi$-conformally flat. One can prove that if $C(X, Y)z \in L$ for any $X, Y, z$, then $C = 0$. In this case a $k$- contact metric manifold is locally isometric to the unit sphere. It is easy to prove the following proposition :

**Theorem 4.1** : On a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ the following condition are equivalent

(i) $M$ is $\xi$-conformally flat

(ii) $\eta(C(X, Y)z) = 0$

(iii) $\phi^2 C(X, Y)z = -C(X, Y)z$

(iv) $C(X, Y)\xi = 0$, where $X, Y, z \in T(M)$.

From (iv) in proposition (4.1), we see that a contact metric manifold is $\xi$-conformally flat if and only if

$$R(X, Y)\xi = \frac{1}{(2n-1)}(g(QY, \xi)X + \eta(Y)QX - g(QX, \xi)Y$$

$$- \eta(X)QY) + \frac{r}{2n(2n-1)}(\eta(X)Y - \eta(Y)X).$$

**Theorem 4.2** : If $M^{2n+1}$ be an $\eta$-Einstein Sasakian manifold, then $M^{2n+1}$ is $\xi$-conformally flat.

**Proof** : It is well known that the structure $(\phi, \xi, \eta, g)$ is a Sasakian if and only if the curvature tensor satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

$$S(X, \xi) = g(QX, \xi) = 2n.g(\xi, X)$$

$$Q\xi = 2n.\xi.$$

Since $(\phi, \xi, \eta, g)$ is $\eta$-Einstein, there exists functions $a$ and $b$ such that

$$g(QX, Y) = ag(X, Y) + b\eta(X)\eta(Y). \tag{4.7}$$

Putting $X = \xi$, we get the following :

$$g(Q\xi, Y) = ag(\xi, Y) + b\eta(\xi)\eta(Y)$$

$$S(\xi, Y) = a\eta(Y) + b\eta(Y)$$

$$2n\eta(Y) = (a + b)\eta(Y)$$

$$2n = a + b \tag{4.8}$$

On the other hand, the scalar curvature

$$r = Tr(Q) = (2n + 1)a + b$$

Now, we get

$$C(X, Y)\xi \tag{4.9}$$

$$= R(X, y)\xi - \frac{1}{2n-1}\left(2a + b - \frac{r}{2n}\right)(\eta(Y)X - \eta(Y)X)$$

$$= R(X, Y)\xi - (\eta(Y)X - \eta(X)Y)$$

$$= R(X, Y)\xi - R(X, Y)\xi$$

$$= 0$$

which completes the proof.

Again, using (3.12) and (2.2), we have

$$QX = \left\{(2n - 1) - g(Q\xi, \xi) + \frac{r}{2n}\right\}X - g(QX, \xi) \tag{4.10}$$

$$-\left\{\left(2n - 1 + \frac{r}{2n}\right)\eta(X)\right\}\xi + \eta(X)Q\xi.$$

which is of the form

$$QX = ax + b\eta(X)\xi. \tag{4.11}$$

On substituting in the equation (4.5) we get

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

which means that the manifold is also Sasakian.

**Corollary 4.1** : *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be a* $\xi$ *conformally flat* $k$-contact metric manifold. If there exists function $\lambda$ and $\mu$ on $M^{2n+1}$ such that

(4.12) $$\left(\nabla_X Q\right)Y - \left(\nabla_Y Q\right)X = \lambda X + \mu Y$$

then

$$QX = 2n\,X.$$

**Proof** : From theorem 1, we have $QX = aX + b\xi$ where

$$a = -1 + \frac{r}{2n} \quad \text{and} \quad b = 2n + 1 - \frac{r}{2n}.$$

Thus we have

(4.13) $$\left(\nabla_X Q\right)Y - \left(\nabla_Y Q\right)X = (Xa)Y - (Ya)X + (Xb)\eta(Y)\xi$$
$$- (Yb)\eta(X)\xi - b\{2g(\phi X, Y)\xi$$
$$+ \eta(Y)\phi X - \eta(X)\phi(Y)\}.$$

Replacing $X$ and $Y$ by $\phi X$ and $\phi Y$ in (4.13) we get

(4.14) $$\left(\nabla_{\phi X} Q\right)\phi Y - \left(\nabla_{\phi Y} Q\right)\phi X$$
$$= (\phi X a)\phi Y - (\phi Y a)\phi X - 2bg\left(\phi^2 X, \phi Y\right)\xi.$$

From (4.12) and (4.13) we obtain

$$(\lambda + (\phi Y a))\phi . X + (\mu - (\phi X a)\phi Y) = -2bg\left(\phi^2 X, \phi Y\right)\xi$$

which implies that $-2bg(\phi^2 X, \phi Y) = 0$ but replacing here $X$ by $\phi Y$, we obtain $bg(\phi Y, \phi Y) = 0$ and hence $b = 0$. Therefore $r = 2n(2n + 1)$, which gives $a = 2n$. This completes the proof.

**Corollary 4.2** : *Any conformally flat k-contact metric manifold is locally isometric to unit sphere.*

**Proof** : It is well known that on a conformally flat Riemannian manifold, the following equation holds[6] for $n > 1$

$$\left(\nabla_X Q\right)Y - \left(\nabla_Y Q\right)X = \frac{1}{4n}\left((Xr)Y - (Yr)X\right).$$

The Corollary 1 shows that $QX = 2nX$, therefore $C(X, Y)X = 0$ yield

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

This completes the proof.

**Corollary 4.3** : *Let $M^{2n+1}$ be a $\xi$-conformally flat k-contact metric manifold. If the curvature tensor is harmonic, then $M^{2n+1}$ is $\eta$-Einstein.*

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