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Cyclic Purity and Cocyclic Copurity in Module Categories

D. P. Choudhury and Ashok Kumar Pandey

Department of Mathematics, University of Allahabad, Allahabad.

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Abstract : We study relative projectivity and injectivity classes of exact sequences with respect to the classes of cyclic and cocyclic modules respectively. A characterization of cyclic pure exact sequences has been given in terms of exactness of a certain sequence of submodules of the modules appearing in the given sequence. The second half dualizes certain results of Warfield[1] and shows that for certain classes S of modules, every module A can be S-copurely embedded in a direct product of members of A, and from this we obtain other results about S-copure injectives and cocyclic copurity.

1. Introduction

Module-purity plays an important role in the study of R- module categories. The aim of the present paper is to study some aspects of purity relative to a fixed cyclic module R/I for a left ideal I. In the first section of this paper we give a proof of the proposition (2.2) which was stated in[2] without any proof on which our theorem (2.1) depends. In the second section of the paper we dualize certain results of R. B. Warfield[1], which generalizes some results in purity in abellian groups L. Fuchs[3]. In this paper R refers to a ring with identity, which need not be necessarily commutative. Also by an R-module we always mean a left R-module.

2. Cyclic Purity

Definition 2.1 : An exact sequence

$$\begin{tikzcd} & & & M \arrow[ld, dashed, "f'"'] \arrow[d, "f"] & \\ 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

is said to be M-pure if given $f : M \to C$, $\exists f' : M \to B$ such that $\beta o f' = f$.

We shall start with the purity relative to fixed cyclic module R/I for a left ideal I that is the condition that R/I be projective relative to the sequence $0 \to A \to B \to C \to 0$.

The following proposition is stated in D. P. Choudhury[2] without any proof. Since theorem (2.4) uses proposition (2.2), we give a proof of the proposition which does not seem to have appeared anywhere.

Proposition 2.2 : For a left ideal I, *a* submodule K of M is R/I-pure if and only if given $m \in M$ such that $Im \subseteq K$, $\exists\, m' \in M$ such that $Im' = 0$ and $(m - m') \in K$, where R/I is a fixed cyclic left module.

Proof : Given the lower sequence and $f : R/I \to M/K$ we construct the following diagram by projectivity of R.

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\alpha"] \arrow[d] & R \arrow[r, "\beta"] \arrow[d, "g"'] & R/I \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & K \arrow[r, "\alpha'"] & M \arrow[r, "\beta'"] & M/K \arrow[r] & 0 \end{tikzcd}$$

(i)

Let $f(\bar{1}) = m + K\,(\bar{1} \in R/I)$. Since

$$f(\bar{1}) = f(1 + I),\ I(m + K) = If(\bar{1}) = I(f(1 + I)) = \{rf(1 + I) \mid r \in I\}$$

$$= \{f(1 + I) \mid r \in I\} = \{f(0)\} = \{0\}$$

therefore

$$Im \subseteq K \Rightarrow \exists\, m' \in M \text{ with } Im' = 0 \text{ and } (m - m') \in K.$$

We define $h(\bar{r}) = rm'$

Claim : It is well defined. If

$$\bar{r} = \bar{s} \Rightarrow r - s \in I \Rightarrow (r - s)m' = 0 \Rightarrow rm' = sm' \Rightarrow h(\bar{r}) = h(\bar{s})$$

Now

$$(\beta' o h)(\bar{r}) = \beta'(rm') = r\beta'(m') = r(m' + K) = r(m + K) = rf(\bar{1})$$

$$= f(r\bar{1}) = f(\bar{r})\ \forall\, \bar{r} \in R/I$$

therefore the sequence is R/I-pure.

Conversely given that the lower sequence is R/I-pure, given $m \in M$ such that $Im \subseteq K$, we define $f : R/I \to M/K$ by $f(\bar{r}) = rm + K$. This will be well defined as $Im \subseteq K$. The map $h : R/I \to M$ exists such that $\beta' o h = f$. Taking $m' = h(\bar{1})$, $m - m' + K = (m + K) - (m' + K) = f(\bar{1}) - \beta' oh(\bar{1}) = 0$ as $\beta' oh = f$ i.e. $(m - m') \in K$. Also $Im' = Ih(\bar{1}) = h(\bar{I}) = h(0) = 0$.

Definition 2.3 : Let M be a left R-module. For a two sided ideal I *we define* $M[I] = \{m \in M \ s.t. \ Im = 0\}$.

The following theorem is analogue of the corresponding results on purity in abellian groups, Theorem 29.1[3].

Theorem 2.4 : *Suppose K is R/I-pure in M, where R/I is a fixed cyclic left module. Then the following conditions are equivalent* :

(a) $0 \to K \xrightarrow{\alpha} M \xrightarrow{\beta} M/K \to 0$ is an *R/I–pure exact sequence.*

(b) $0 \to K[I] \xrightarrow{\alpha'} M[I] \xrightarrow{\beta'} (M/K)[I] \to 0$ *is exact, where* α' *and* β' *are restrictions of* α *and* β *respectively.*

(c) $0 \to K/K[I] \xrightarrow{\alpha^*} M/M[I] \xrightarrow{\beta^*} (M/K)/(M/K)[I] \to 0$ *is exact, where* α^* *and* β^* *are maps induced by* α *and* β *respectively.*

Proof : $(a) \Rightarrow (b)$.

To show (b) is exact, that is to show that *Image* (α') = *Ker* (β'), let $m \in$ *Image* (α'), therefore $m = \alpha'(k)$, for some $k \in K[I]$, $I(\alpha(k)) = \alpha(Ik) = \alpha \, . \, 0 = 0$ (since $\mathbf{Ik} = 0$) therefore $\alpha(k) \in M[I]$ therefore

(1) $\quad \mathit{Image}(\alpha') \subseteq \mathit{Image}(\alpha) \cap M[I] = \mathit{Ker}(\beta) \cap M[I] = \mathit{Ker}(\beta')$

$$\mathit{Ker}(\beta') = \{m \in M[I] \mid \beta'(m) = 0\} = \{m \in M \mid \beta(m) = 0 \text{ and } Im = 0\}$$

$$= \mathit{Ker}(\beta) \cap M[I] = \mathit{Image}(\alpha) \cap M[I] \, (\text{since } \mathit{Ker}(\beta) = \mathit{Image}(\alpha)\}$$

Take $\alpha(k) \in M[I] \cap$ image (α), therefore, $I\alpha(k) = 0 \Rightarrow \alpha(Ik) = 0$ therefore $Ik = 0$ (since α is injective). Therefore $k \in K[I]$ and $\alpha(k) \in$ Image $(\alpha') \Rightarrow M[I] \cap$ Image (α) = *ker*$(\beta') \subseteq$ Image (α') (2). From these equations we get *Ker*(β') = Image (α'). Now to show that β' is epic, that is to show that for each element $(m + K) \in ((M/K)[I])$ (*i.e.* $I(m + K) = 0$), there exists an element $m' \in M[I]$ such that $\beta'(m') = m' + K = m + K$. Since $I(m + K) = 0 \Rightarrow Im + K = 0 \Rightarrow Im \subseteq K$, therefore from the proposition [2.2], $\exists \ m' \in M$ such that $Im' = 0$ and $(m - m') \in K$. Therefore $\beta'(m') = m' + K \quad = m + K$ (since $(m - m') \in K$) and $Im' = 0 \Rightarrow m' \in M[I]$ therefore β' is epic.

$(b) \Rightarrow (a)$.

Suppose (b) is exact. Then we have to show that (a) is R/I-pure, that is for given

$m \in M$ with $Im \subseteq K$, $\exists\ m' \in M$, such that $Im' = 0$ and $(m - m') \in K$. Since $Im \subseteq K \Rightarrow I(m + K) = 0$ therefore $m + K \in (M/K)[I]$, β' is epic, therefore $(m + K) = \beta'(m')$ (for some $m' \in M[I]) = (m' + K)$. Therefore $Im' = 0$ and $(m - m') \in K$.

Hence $0 \to K \xrightarrow{\alpha} M \xrightarrow{\beta} M/K \to 0$ is R/I-pure.

$(b) \Leftrightarrow (c)$.

We consider

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K[I] \arrow[r, "\alpha"] \arrow[d, "\lambda_1"'] & M[I] \arrow[r, "\beta"] \arrow[d, "\mu_1"] & M/K[I] \arrow[r] \arrow[d, "\gamma_1"] & 0 \\
0 \arrow[r] & K \arrow[r, "\alpha'"] \arrow[d, "\lambda_2"'] & M \arrow[r, "\beta'"] \arrow[d, "\mu_2"'] & M/K \arrow[r] \arrow[d, "\gamma_2"] & 0 \\
0 \arrow[r] & K/K[I] \arrow[r, "\alpha^*"] \arrow[d] & M/M[I] \arrow[r, "\beta^*"] \arrow[d] & \frac{(M/K)}{(M/K)[I]} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(ii)

The proof follows from 3×3 lemma.

3. Cocyclic Copurity

We have called an exact sequence A-copure for a class A of modules if objects of A are injective with respect to the exact sequence, and we call a module A- copure injective or A-copure projective if it is respectively injective or projective relative to A-copure sequences. The following results dualize certain results of R. B. Warfield[1], and at the same time, they generalize certain results in purity in abellian groups (Lemma 30.3 and Theorem 30.4 in Fuchs[3] to module categories.

Definition 3.1 : An R-module M is said to be Cocyclic, if it is a submodule of $E(S)$ for some simple module S. These are nothing but the subdirectly irreducible modules.

Proposition 3.2 : Let S be a class of left R- modules containing the modules $E(S_i)$ (where S_i *are a representative class of all simple modules) such that there is a subclass* S^* which is a set with the property that for any $M \in S$, there is an $N \in S^*$ with $N \simeq M$, then for any module A, there is an S-copure sequence $0 \to A \to C \to C' \to 0$ such that C is a direct product of copies of modules in S.

Proof : Let Λ be the set of pairs (M, f) with $M \in S^*$ and $f \in Hom(A, M)$ and for each $\lambda \in \Lambda$ denote the corresponding M and f by M_λ and f_λ. Let $C = \Pi_{\lambda \in \Lambda} M_\lambda$ and let f : $A \to C$ be the product map of the maps f_λ. Since $p_\lambda o \phi = f_\lambda$, then ϕ is injective because A can be embedded into a direct product of $E(S_i)$'s.

Theorem 3.3 : *A Module P is S-copure injective if and only if it is a direct summand of direct product of copies of modules in S.*

Proof : Suppose P is an S-copure injective module then we have to show that it is a direct summand of direct product of copies of modules in S. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, "\phi"] \arrow[d, "I_P"'] & C \arrow[r] & C/P \arrow[r] & 0 \\ & P & \text{(iii)} & & \end{tikzcd}$$

From the proposition [3.2], there is an S-copure exact sequence (iii) with $C = \Pi_{\lambda \in \Lambda} M_\lambda$ where $M_\lambda \in S$. If we take $M = P$, then the sequence (iii) splits by copure injectivity of P, therefore P is a direct summand of direct product of copies of modules of S.

Conversely, suppose L is a direct summand of direct product of modules of S, then we have to show that L is an S-copure injective module. Let L be a direct summand of a direct product $N = \Pi N_i$ where all N_i are modules of S. Let π_i and ρ_i denote the co-ordinate projections and injections attached to this direct product and let $\pi : N \to L$ and $\rho : L \to N$ be homomorphisms satisfying $\pi o \rho = I_L$.

If

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M_2 \arrow[r, "\beta"] & M_3 \arrow[r] & {} \\ & C & \text{(iv)} & & \end{tikzcd}$$

has S-copure exact row, then every module of S has the injective property relative to this exact sequence $0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0$. That is every R-module $C \in S$, the diagram (iv) can be embedded into a commutative diagram for a suitable choice $\psi : M_2 \to C$/ There exists a map $\psi_i : M_2 \to N_i$ for every i, such that $\pi_i o \rho o \phi = \psi_i o \alpha$.

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M_2 \arrow[r, "\beta"] \arrow[dl, dashed, "\psi"'] \arrow[ddl, dashed, "\psi^*"] \arrow[dd, "\psi_i"] & M_3 \arrow[r] & 0 \\ & L \arrow[d, shift left, "\rho"] & & & \\ & N \arrow[u, shift left, "\pi"] \arrow[r, "\pi_i"] & N_i & & \\ & & \text{(v)} & & \end{tikzcd}$$

Let $\psi^* : M_2 \to N$ be such that $\pi_i o \psi^* = \psi_i$.

Thus $\pi_i \, o \, (\rho o \phi) = \psi_i \, o\alpha = \pi_i \, o(\psi^* \, o\alpha)$ for each i. That is $\rho o \phi = \psi^* o\alpha$ by uniqueness. Hence $\phi = \pi o \, (\, \rho o \phi \,) = \pi o \, (\, \psi^* o\alpha \,)$, and take $\psi = \pi o \psi^*$. Then $\psi o \alpha = \pi o (\psi^* o \alpha) = \phi$.

Corollary 3.4 : Let S be the class of all cocyclic modules, then for any module A there is an S-copure sequence $0 \to A \to N \to N' \to 0$ such that N is the direct product of copies of cocyclic modules in S.

Corollary 3.5 : A left R-module is cocyclic copure injective if and only if it is a direct summand of a direct product of cocyclic modules.

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