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On *C*-Concircularly Flat and **P*-Finsler Spaces

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1. Introduction

H. Izumi[1] discussed a C-concircular transformation in detail and defined a tensor $Z^i_{jk\,h}$ which is invariant under such transformation. He established therein that the tensor $Z^i_{jk\,h}$ vanishes identically in a Finsler space of constant curvature. In his papers[2,3] he gave the concept of *P-Finsler spaces which was the generalization of C^h-recurrent spaces and $P2$-like spaces of Makoto Matsumoto[4,5]. He further defined a C-Concircularly flat Finsler space and tried to correlate it with a *P-Finsler space of constant curvature. He proved therein the following two theorems[2] :

Theorem A : *If a Finsler space of scalar curvature is C-Concircularly flat, then the space is a *P-Finsler space of constant curvature.*

Theorem B : *If a Finsler space of scalar curvature is C-Concircularly flat, the space is a Riemannian space of constant curvature or a *P-Finsler space of constant curvature satisfying* $F^2 K + \lambda_1 y^1 + \lambda^2 = 0$, *where K is the Reimannian curvature.*

We prove the following theorem.

Theorem : *A C- Concircularly flat Finsler space* $F_n(n > 2)$ *is a Finsler space of constant curvature.*

It can be shown that when the condition of our theorem is satisfied, then the space will be one of scalar curvature. Thus in theorems A and B the assumption of the space being of scalar curvature makes the statement of the theorems unnecessarily restricted in appearance.

It will be noticed that we have not used this condition in our proof, whereas, Izumi has used this condition in the course of the proofs of theorem A and theorem B. This is the main advancement which the present paper brings about.

2. *C*-Concircularly Flat and **P*-Finsler Spaces

Let F_n be an n-dimensional Finsler space equipped with a metric function F satisfying the requisite conditions[6], corresponding symmetric metric tensor g and Berwald connection G. Let the components of the metric tensor g and the coefficients of Berwald connection G be g_{ij} and G^i_{jk} respectively. The $(h)hv$-torsion tensor C^h_{ik}, defined by

$$C^h_{ik} = g^{hj} C_{ijk}$$

where

$$C_{ijk} = \frac{1}{2} \dot{\partial}_j g_{ik}, \ \dot{\partial}_j = \partial/\partial y^j \text{ and } g^{hj} g_{jk} = \delta^h_k,$$

is symmetric in its lower indices. The inner product of this tensor with y^i vanishes identically.

H. Izumi[2] defined the tensors ${}^*P^i_{jk}$ and Z^i_{jkh} by

(2.1) $$ {}^*P^i_{jk} = P^i_{jk} - \lambda C^i_{jk}$$

and

(2.2) $$Z^i_{jkh} = H^i_{jkh} - \left(g_{jk} {}^*H^i_h - g_{jh} {}^*H^i_k \right) / (n-1)$$

where H^i_{jkh} are components of Berwald curvature tensor and

(2.3) (a) $ {}^*H^i_h = H^i_{rsh} g^{rs}$, (b) $P^i_{jk} = C^i_{jk|r} y^r$

$C^i_{jk|r}$ being the Cartan's covariant derivative of the tensor C^i_{jk}.

Izumi[2] characterised his *${}^*P$-Finsler space by*

(2.4) $$ {}^*P^i_{jk} = 0,$$

while the C-concircularly flat space by the condition (2.4) and

(2.5) $$Z^i_{jkh} = 0.$$

We know[6] that a Finsler space F_n $(n > 2)$ of scalar curvature is characterised by the condition

$$H^i_h = F^2 K\left(\delta^i_k - l^i l_k \right)$$

where H^i_h are components of Berwald deviation tensor, K is the Riemannian curvature and $l^i = y^i/F$.

A Finsler space of constant curvature is characterized by the condition :

(2.6) $$H^i_{jkh} = K\left(g_{jk} \delta^i_h - g_{jh} \delta^i_k \right),$$

where the Riemannian curvature K of the space is constant.

We must note that every Finsler space of constant Riemannian curvature is a Finsler space of scalar curvature but the converse is not necessarily true.

Now we propose :

Theorem 2.1 : *A non-flat Finsler space F_n $(n > 2)$ satisfying (2.5) is a Finsler space of constant curvature.*

Proof : Let us consider a Finsler space satisfying (2.5). In view of (2.5), (2.2) gives

$$H^i_{jkh} = \left(g_{jk}\ {}^*H^i_h - g_{jh}\ {}^*H^i_k\right)/(n-1). \tag{2.7}$$

Transvecting (2.7) by y^j and using $H^i_{jkh}\, y^j = H^i_{kh}$, we have

$$H^i_{kh} = \left(y_k\ {}^*H^i_h - y_h\ {}^*H^i_k\right)/(n-1). \tag{2.8}$$

Transvecting (2.8) by y_i and using $y_i H^i_{kh} = 0$, we have

$$y_k y_i\ {}^*H^i_h - y_h y_i\ {}^*H^i_k = 0 \tag{2.9}$$

which implies at least one of the conditions

$$\text{(a) } y_i\,{}^*H^i_k = 0, \qquad \text{(b) } y_i\,{}^*H^i_k = \phi\, y_k \tag{2.10}$$

If (2.10) is true, the transvection of (2.7) by y_i yields $y_i H^i_{jk} = 0$; which, in view of the lemma proved by P. N. Pandey[7], implies $H^i_{jkh} = 0$. Therefore for a non-flat Finsler space satisfying (2.5), (2.10a) can not be true. Consequently (2.10b) holds. Transvecting (2.7) by y_i and using (2.10b), we get

$$y_i H^i_{jkh} = \frac{1}{n-1}\left(g_{jk}\, y_h - g_{jh}\, y_k\right). \tag{2.11}$$

Differentiating $y_i H^i_{kh} = 0$ partially with respect to y^j and using $\dot{\partial}_j H^i_{hk} = H^i_{jkh}$ and $g_{ij} = \dot{\partial}_i y_j$, we find

$$g_{ij} H^i_{kh} + y_i H^i_{jkh} = 0. \tag{2.12}$$

From (2.11) and (2.12) we have

$$-\, g_{ij} H^i_{kh} = \phi\left(g_{jk}\, y_h - g_{jh}\, y_k\right)/(n-1) \tag{2.13}$$

Transvecting (2.13) by g^{jm} and using the fact that g^{jm} and g_{ij} are inverse matrices, we have

$$H^m_{kh} = \phi\left(\delta^m_h\, y_k - \delta^m_k\, y_h\right)/(n-1). \tag{2.14}$$

Contracting the indices m and h in (2.14) and using $H^r_{kr} = H_k$, we get $H_k = \phi\, y_k$. In view of $H_k\, y^k = (n-1)H$, the transvection of this equation by y^k yields

(2.15) $$(n-1)H = \phi F^2.$$

Consequently (2.14) becomes

(2.16) $$H^m_{kh} = K\left(\delta^m_h\, y_k - \delta^m_k\, y_h\right).$$

where $K = H/F^2$. In view of Berwald's theorem[8,6], the equation (2.16) implies that the scalar K is a constant. Differentiating (2.16) partially with respect to y^j and using $g_{ij} = \dot{\partial}_i y_j$ and $\dot{\partial}_j H^i_{hk} = H^i_{jkh}$, we have the condition (2.6). Hence the space considered is of constant curvature.

From this theorem and the theorem 4.4 of H. Izumi[1] we have :

Corollary 2.1 : A non-flat Finsler space $F_n (n > 2)$ *is of constant curvature if and only if the tensor* Z^i_{jkh} *vanishes.*

Since the tensor Z^i_{jkh} vanishes in a C-concircularly flat Finsler space, we may conclude that a C-concircularly flat Finsler space is of constant curvature. Again every Finsler space of constant curvature is a Finsler space of scalar curvature[6]. In view of this fact, theorems A and B stated in introduction may be written in the form .

Theorem 2.2 : *A C-concircularly flat Finsler space is a *P-Finsler space of constant curvature.*

Theorem 2.3 : *A C-concircularly flat Finsler space is a Riemannian space of constant curvature or a *P-Finsler space of constant curvature satisfying* $F^2 K + \lambda_l\, y^l + \lambda^2 = 0$.

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