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Invariant Submanifolds of a Pseudo Normal Nearly Co-symplectic Manifold

Bhagwat Prasad
Department of Mathematics, M. M. T. D. College, Ballia (U.P.), India

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Abstract : Tanno[1], Yano and Ishihara[2] proved that any invariant submanifold of a Sasakian manifold is Sasakian and minimal. Further Kon[3] and Endo[4] proved that an invariant submanifold of a K-contact manifold is K-contact and minimal. The author[5] generalized this result to the case of a pseudo normal nearly co-symplectic manifold. Kon[6] proved that the ϕ-sectional curvature K of an invariant submanifold M of a normal contact metric manifold $\overline{M}$ with $\overline{\phi}$ - sectional curvature $\overline{K}$ is less than or equal to $\overline{K}$. The equality holds if and only if M is totally geodesic. The purpose of this paper is to prove similar results if $\overline{M}$ is a pseudo-normal nearly co-symplectic manifold in place of a normal contact metric manifold.

1. Preliminaries

Let $\overline{M}$ be a $(2n + 1)$ dimensional contact Riemannian manifold with structure tensors $(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$. Then they satisfy

(1.1) $$\overline{\phi}\,\overline{\xi} = 0,\ \overline{\eta}(\overline{\xi}) = 0,\ \overline{\phi}^2 = -I + \overline{\eta} \otimes \overline{\xi},$$

(1.2) $$\overline{g}(\overline{\phi}\overline{X}, \overline{\phi}\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \overline{\eta}(\overline{X})\,\overline{\eta}(\overline{Y})$$

and

(1.3) $$\overline{g}(\overline{\phi}\overline{X}, \overline{Y}) = d\,\overline{\eta}(\overline{X}, \overline{Y}),\ \overline{\eta}(\overline{X}) = \overline{g}(\overline{\xi}, \overline{X})$$

for any vector fields $\overline{X}$ and $\overline{Y}$ on $\overline{M}$.[7] On such a manifold we can always define a 2-form $\overline{\Phi}$ by $\overline{\Phi}(\overline{X}, \overline{Y}) = \overline{g}(\overline{\phi}\overline{X}, \overline{Y})$. $\overline{M}$ is called a pseudo normal nearly co-symplectic manifold if[8]

(1.4) $$(\overline{D}_X\,\overline{\phi})\,\overline{\phi}\,\overline{Y} + (\overline{D}_X\,\overline{\phi})\,\overline{Y} = \overline{\eta}(\overline{Y})(\overline{D}_{\phi X}\,\overline{\xi}) - \overline{\eta}(\overline{X})(\overline{D}_{\phi Y}\,\overline{\xi}).$$

In a pseudo normal nearly co-symplectic manifold, the following formulae are satisfied[5]

(1.5) $$\left(\bar{D}_X \bar{\eta}\right)(\bar{Y}) + \left(\bar{D}_{\bar{\phi} X} \bar{\eta}\right)(\bar{\phi}\, \bar{Y}) = 0\,,$$

(1.6) $$\text{(a)} \quad \bar{D}_{\bar{\xi}}\, \bar{\xi} = 0, \qquad \text{(b)} \quad \bar{D}_{\bar{\xi}}\, \bar{\eta} = 0$$

where $\bar{D}$ is the covariant differentiation with respect to g.

Let M be a $(2m + 1)$-dimensional submanifold of $\bar{M}$. Applying $\bar{\phi}$ to a tangent vector field X to M, we obtain the vector field $\bar{\phi}X$ which can be represented as a sum of its tangential and normal parts i.e.,

$$\bar{\phi}\bar{X} = \phi X + \sum_A V_A(X) N_A\,,$$

where N_A $(A = 1, 2, \ldots, 2(n - m))$ are locally mutually orthogonal unit normal vector fields to M, and ϕ and V_A define respectively a (1, 1)-type tensor and a 1-form on M[9]. Moreover, we can put $\bar{\xi} = \xi + \sum u_A N_A$, where ξ is a vector field on M and u_A is a function on M. Now we define 1-form by $\eta(X) = \bar{\eta}(X)$ for any vector field X on M.

Let us assume that D_X denotes the Riemannian connection on M determined by the induced metric g, the Gauss formula and Weingarten formula can be written as

(1.7) $$\bar{D}_X Y = D_X Y + \sum_A h_A(X, Y) N_A\,,$$

$$\bar{D}_X N_A = -H_A X + \sum_B L_{BA}(X) N_B\,,$$

and Gauss equation is given by

(1.8) $$\bar{g}(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W)$$

$$-\sum_B g\left(H_B Y, Z\right) g\left(H_B X, W\right) + \sum_B g\left(H_B X, Z\right) g\left(H_B Y, W\right),$$

for any vector field X, Y, Z and W on M, where $\bar{R}$ is the Riemannian curvature tensor of $\bar{M}$, R is the Riemannian curvature tensor of M, L_{BA} are the third fundamental forms and h_A and H_A are the second fundamental forms. h_A and H_A satisfy

$$h_A(X, Y) = g\left(H_A X, Y\right) = g\left(X, H_A Y\right) = h_A(X, Y)\,,$$

M is said to be invariant if $\bar{\phi}X$ is tangent to M and $\bar{\xi}$ is always tangent to M.

If there exists a unit vector $\bar{X}$ in $T_X(\bar{M})$ (where $T_X(\bar{M})$ denotes tangent space at the point x on $\bar{M}$) orthogonal to ξ such that $\{\bar{X}, \bar{\phi}\bar{X}\}$ is an orthonormal basis of the plane section, then the sectional curvature

$$\bar{K}(\bar{X}, \bar{\phi}\bar{X}) = g(\bar{R}(\bar{X}, \bar{\phi}\bar{X})\bar{\phi}\bar{X}, \bar{X})$$

is called a $\bar{\phi}$-sectional curvature. In the same way $K(X, \phi X)$ is defined at a point on M.

2. Invariant Submanifolds of a Pseudo Normal Nearly Co-symplectic Manifold

Lemma 2.1 : *For an invariant submanifold M of a pseudo normal nearly co-symplectic manifold $\bar{M}$, we have*

$$g\left(H_A\xi, \xi\right) = 0.$$

Proof : From (1.7), we have

$$\bar{D}_\xi \xi = D_\xi \xi + \sum_A h_A(\xi, \xi) N_A .$$

By taking the normal parts, we get the result.

Lemma 2.2 : *For an invariant submanifold M of a pseudo normal nearly co-symplectic manifold $\bar{M}$, we get*

$$g\left(H_A\phi X, \phi X\right) + g\left(H_A X, X\right) = 0, \tag{2.1}$$

for a vector field X on M orthogonal to ξ.

Proof : By virtue of (1.7); we get

$$\bar{D}_X(\phi Y) = D_X(\phi Y) + \sum_A h_A(X, \phi Y) N_A$$

$$= \left(D_X\phi\right)(Y) + \phi\left(D_X Y\right) + \sum_A h_A(X, \phi Y) N_A .$$

Moreover,

$$\bar{D}_X(\phi Y) = \bar{D}_X(\bar{\phi}Y) = \left(\bar{D}_X\bar{\phi}\right)Y + \bar{\phi}\left(\bar{D}_X Y\right)$$

$$= \left(\bar{D}_X\bar{\phi}\right)Y + \bar{\phi}\left(D_X Y + \sum_B h_B(X, Y) N_B\right).$$

$$= \left(\bar{D}_X\bar{\phi}\right)Y + \bar{\phi}\left(D_X Y\right) + \sum_B h_B(X, Y)\bar{\phi} N_B .$$

Thus, we have

(2.2)
$$\left(D_X \phi\right)(Y) + \sum_A g\left(H_A X, \phi Y\right) N_A$$
$$= \left(\overline{D}_X \overline{\phi}\right) Y + \sum_B g\left(H_B X, Y\right) \phi N_B .$$

Putting $Y = \phi Y$ and $X = \phi X$ in the above equation, we find

(2.3)
$$\left(D_{\phi X} \phi\right)\phi Y + \sum_A g\left(H_A \phi X, \phi^2 Y\right) N_A$$
$$= \left(\overline{D}_{\phi X} \overline{\phi}\right)\phi Y + \sum_B g\left(H_B \phi X, \phi Y\right)\overline{\phi} N_B .$$

Combining (2.2) and (2.3), we get

(2.4)
$$\left(D_X \phi\right) Y + \left(D_{\phi X} \phi\right)\phi Y + \sum_A \left(g\left(H_A X, \phi Y\right)\right.$$
$$\left. + g\left(H_A \phi X, \phi^2 Y\right)\right) N_A = \left(\overline{D}_X \overline{\phi}\right) Y + \left(\overline{D}_{\phi X} \overline{\phi}\right)\phi Y$$
$$+ \sum_B \left(g\left(H_B X, Y\right) + g\left(H_B \phi X, \phi Y\right)\right)\overline{\phi}_{N_B}$$

In consequence of (1.4) and (2.4), we obtain

(2.5)
$$\eta(Y)\left(D_{\phi X} \xi\right) - \eta(X)\left(D_{\phi Y} \xi\right) + \sum_A \left(g\left(H_A X, \phi Y\right)\right.$$
$$\left. + g\left(H_A \phi X, \phi^2 Y\right)\right) N_A = \overline{\eta}(Y)\left(\overline{D}_{\phi X} \overline{\xi}\right) - \overline{\eta}(X)\left(\overline{D}_{\phi Y} \overline{\xi}\right)$$
$$+ \sum_B \left(g\left(H_B X, Y\right) + g\left(H_B \phi X, \phi Y\right)\right)\overline{\phi} N_B .$$

Setting $Y = X$ in (2.5) and using the fact that a unit vector field X orthogonal to $\overline{\xi} = \xi$, we get

(2.6)
$$\sum_A \left(g\left(H_A X, \phi X\right) - g\left(H_A \phi X, X\right)\right) N_A$$
$$= \sum_B \left(g\left(H_B X, X\right) + g\left(H_B \phi X, \phi X\right)\right)\overline{\phi}_{N_B} .$$

Thus we have

$$g\left(H_A\,\phi X,\ \phi X\right) = -\,g\left(H_A X,\ X\right).$$

Hence we have the Lemma 2.2.

Theorem 2.1 : *Let M be an invariant submanifold of a pseudo-normal nearly co-symplectic manifold $\bar{M}$ with $\bar{\phi}$-sectional curvature $\bar{K}$. If M has ϕ-sectional curvature K, then $K \le \bar{K}$. The equality holds if and only if M is totally geodesic.*

Proof : Taking a unit vector field X orthogonal to $\bar{\xi} = \xi$ and using (1.8), we get

$$(2.7)\qquad \bar{g}\left(\bar{R}(X,\ \phi X)\ \phi X,\ X\right) = g\left(R(X,\ \phi X)\ \phi X,\ X\right)$$

$$-\sum_B g\left(H_B\ \phi X,\ \phi X\right) g\left(H_B X,\ X\right) + \sum_B g\left(H_B\ X,\ \phi X\right) g\left(H_B\ \phi X,\ X\right).$$

By the assumption, we have

$$(2.8)\qquad \bar{K} = K - \sum_B g\left(H_B\ \phi X,\ \phi X\right) g\left(H_B X,\ X\right)$$

$$+ \sum_B g\left(H_B\ X,\ \phi X\right) g\left(H_B\ \phi X,\ X\right).$$

By virtue of (2.2), (2.8) yields

$$(2.9)\qquad \bar{K} = K + \sum_B \left(g\left(H_B X,\ X\right)\right)^2 + \sum_B \left(g\left(H_B\ \phi X,\ X\right)\right)^2.$$

Hence, we get $K \le \bar{K}$. Here if M is totally geodesic, we have $K = \bar{K}$. Conversely, if we have $K = \bar{K}$, we have $g(H_A X,\ X) = 0$ for a unit vector field X orthogonal of ξ. Therefore we get $g(H_A\ \phi X,\ \phi X) = 0$. Moreover, we have $g(H_A\,\xi,\ \xi) = 0$. However, any vector X is expressed by the linear combination of a ϕ-basis $(e_1, \ldots, e_m,\ \phi e_1, \ldots,\ \phi e_m,\ \xi)$ in $T_x(M)$. Therefore, if we use the polarization identity, we obtain $g(H_A X,\ Y) = 0$, that is, M is totally geodesic.

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