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A Note on Horseshoe Lemma for M-Projective and M-Injective Modules

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Abstract : In this paper we have dualized a result of Singh *et. al.*[1] We also generalize one result of same author to get Horseshoe lemma for M-projective and M-injective modules.

Preliminaries

Through out this paper R denotes a ring with unity and all the modules considered are left unitary modules over R.

The following definitions are due to Azumayya *et al.*[2]

Definition 1 : *An R-module U is called M-projective if given a diagram*

$$\begin{array}{ccccc} & & U & & \\ & & \downarrow f & & \\ M & \underset{\alpha}{\rightarrow} & N & \rightarrow & 0 \end{array}$$

of R-modules and R-homomorphisms with exact row there is a R-homomorphism $g : U \rightarrow M$ *such that the resulting diagram is commutative.*

Definition 2 : *An R-module U is called M-injective if given a diagram*

$$\begin{array}{ccccc} O & \rightarrow & N & \overset{\alpha}{\rightarrow} & M \\ & & f\downarrow & & \\ & & U & & \end{array}$$

of R-modules and R-homomorphisms with exact row there is a R-homomorphism $g : M \rightarrow U$ *such that the resulting diagram is cummutative.*

Proposition 1 : *Any R-module U is M-projective if and only if given a diagram*

$$\begin{array}{ccccc} & & U & & \\ & & \downarrow f & & \\ M & \underset{\alpha}{\rightarrow} & Y & \underset{\beta}{\rightarrow} & Z \end{array}$$

of R-modules and R-homomorphisms with exact row and $\beta \circ f = 0$ *there exists a R-homomorphism* $g : U \rightarrow M$ *such that the resulting diagram is commutative.*[1]

Proposition 2 : *An R-module U is M-injective if and only if given a diagram of R-modules and R-homomorphisms of the form*

$$\begin{tikzcd} Z \arrow[r, "\alpha"] & Y \arrow[r, "\beta"] \arrow[d, "f"] & M \\ & U & \end{tikzcd}$$

in which row is exact and $f|_{Im\,\alpha}$ *is monic there exists a R- homomorphism* $g : M \to U$ *such that the resulting diagram is commutative.*

Proof : Let U be M-injective. Since row is exact $f|_{ker\,\beta}$ is also monic so we have a R-homomorphism $f' : Im\beta \to U$ given by $f'(x) = f(y)$ where $y \in Y$ is such that $\beta(y) = x$. Thus if $i : Im\beta \to M$ is the natural injection we have the diagram

$$\begin{tikzcd} 0 \arrow[r] & Im\beta \arrow[r, "i"] \arrow[d, "f"] & M \arrow[dl, "g'"] \\ & U & \end{tikzcd}$$

in which the row is exact. Since U is M-projective there exists a R-homomorphism $g' : M \to U$ such that $g' \circ i = f'$. If $j : Y \to Im\beta$ is defined by $j(y) = \beta(y)$ we have the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\alpha"] & Y \arrow[r, "\beta"] \arrow[d, "j"] & M \arrow[d, equal, "I"] \\ 0 \arrow[r] & Im\beta \arrow[r, "i"] \arrow[d, "f"] & M \arrow[dl, "g'"] \\ & U & \end{tikzcd}$$

where $f' \circ j = f$ and $I : M \to M$ is the identity homomorphism. Converse is easily seen to be true by letting $Z = 0$.

Remark : Here we observe that the proposition 2 above dualizes the proposition 1 of Singh *et. al.*[1] However we can't dualize the proposition 2 of Singh *et. al.*[1]

Definition 3 : *For any module M let* $C_p(M)$ *(respectively* $C_i(M)$*) denotes the class of M-projective (respectively M-injective) modules.*[2]

Proposition 3 : $C_p(M)$ *(respectively* $C_i(M)$*) is closed under direct sums (respectively direct products) and direct summands (respectively direct factors).*[3]

2. Main Result

Proposition 4 : *Consider the diagram of R-modules and R-homomorphisms of the form*

$$
\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & U'_1 \arrow[d, "\alpha_1"'] & & U''_1 \arrow[d, "\beta_1"] & \\
 & U'_0 \arrow[d, "\alpha_0"'] & & U''_0 \arrow[d, "\beta_0"] & \\
0 \arrow[r] & M' \arrow[r, "f"'] \arrow[d] & M \arrow[r, "g"'] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}
$$

in which the row is exact and columns are complexes with each U'_i *and* U''_i *as M-projective,* $i = 0, 1, 2, \ldots$. *Then, there exists a M-projective resolution of M and chain maps so that the columns form an exact sequence of complexes.*

Proof : Since for each i, U'_i and U''_i are M-projective it follows that $U'_i \oplus U''_i$ is also M-projective for each i. Also for each i the sequence

$$0 \to U'_i \xrightarrow{f_i} U'_i \oplus U''_i \xrightarrow{g_i} U''_i \to 0$$

is split exact where f_i is the i-th canonical injection and g_i is the i-th cannonical projection. It now follows from proposition 3 that there exists a map $\gamma_0 : U'_i \oplus U_i \to M$ such that the resulting squares are commutative. The remaining proof now follows by induction using Lemma 6.20 and 3×3 lemma[4].

Proposition 5 : *Consider the diagram of R-modules and R-homomorphisms of the form*

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\alpha_0"'] & M \arrow[r, "g"] & M'' \arrow[r] \arrow[d, "\beta_0"] & 0 \\
 & U'_0 \arrow[d, "\alpha_1"'] & & U''_0 \arrow[d, "\beta_1"] & \\
 & U'_1 \arrow[d] & & U''_1 \arrow[d] & \\
 & \vdots & & \vdots &
\end{tikzcd}
$$

in which the row is exact and columns are complexes with each U'_1 *and* U''_1 *as M-injective,* $i = 0, 1, 2, \ldots$. Then there exists a *M-injective resolution of M and chain maps so that the columns form an exact sequence of complexes.*

Proof : Since

$$\bigoplus_{i \in I} A_i \subset \Pi_{i \in I} A_i \quad \text{and} \quad \bigoplus_{i \in I} A_i = \Pi_{i \in I} A_i$$

if I finite it follows that the proof of proposition 4 can be dualized to prove the result using proposition 3.

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