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C^v-Reducible Finsler Space

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Abstract : Let C_{ijk} be the (h) hv- torsion tensor of an n-dimensional Finsler space F^n. In view of the well known identity; $C_{ijk}|_l - C_{ijl}|_k = 0$ where we denote by $|$, the v-covariant differentiation; $C_{ijk}|_l$ is symmetric in its all indices. The purpose of the present paper is to study a special Finsler space, the Cartan v-derivative of whose (h) hv-torsion tensor can be written as :

$$L\, C_{ijk}|_l = h_{ijk}\, B_l + h_{jkl} B_i + h_{kli} B_j + h_{lij} B_k$$

where $h_{ijk} = -L^2 L|_i|_j|_k$, L being the fundamental function.

1. Introduction

Let us consider a Finsler space F^n of dimension n, whose Cartan v-derivative of (h) hv-torsion tensor can be written as

(1.1) $$L\, C_{ijk}|_l = A_{ijk}\, B_l + A_{jkl}\, B_i + A_{kli}\, B_j + A_{lij}\, B_k$$

where A_{ijk} is assume to be symmetric but not proportional to C_{ijk} because in such case we can not solve (1.1) for B_i in general. In particular $B_i = l_i$ for which (1.1) is nothing but vanishing of T-tensor which has been studied in detail[1].

Contracting (1.1) by y^l we have :

(1.2a) $$-L\, C_{ijk} = A_{ijk} B_o + A_{jko} B_i + A_{koi} B_j + A_{oij} B_k,$$

where "o" indicate contraction by y^k i.e. $B_o = B_i y^i$. Again contracting (1.2a) by y^k and moreover by y^j and y^i respectively we obtain.

(1.2b) $$0 = 2A_{ijo} B_o + A_{joo}\, B_i + A_{ioo}\, B_j,$$

(1.2c) $$0 = 3A_{ioo}\, B_o + A_{ooo}\, B_i,$$

(1.2d) $$0 = A_{ooo}\, B_o.$$

If $B_o \neq 0$ then (1.2) (d), (c), (b), (a) respectively imply

$$A_{ooo} = 0, \; A_{ioo} = 0, \; A_{ijo} = 0, \; A_{ijk} = - L/B_o \, C_{ijk}$$

which contradicts our assumption that A_{ijk} is not proportional to C_{ijk}. Therefore $B_o = 0$ and hence the equation (1.2c) gives $A_{ooo} = 0$. Further from (1.2b) we have

$$A_{joo} B_i + A_{ioo} B_j = 0$$

which leads to

$$A_{joo} B_i B_k = - A_{ioo} B_k B_j = A_{koo} B_i B_j = - A_{joo} B_i B_k$$

Thus $A_{ioo} = 0$ or $B_i = 0$. If $B_i = 0$ then $L C_{ijk}|_l = 0 = L C_{ijk}|_l y^l = - L C_{ijk}$ leads to $C_{ijk} = 0$. Consequently we have $A_{ioo} = 0$. From (1.2a) we have $A_{ijo} \neq 0$, because F^n is non Riemannian. Thus we have :

Proposition 1 : *If v-derivative of the (h) hv-torsion tensor C_{ijk} of a Finsler space F^n is written in the from* (1.1) *then A_{ijk} and B_i satisfy $A_{ioo} = 0$, $A_{ijo} \neq 0$ and* $B_o = 0$.

Now let us consider a tensor h_{ijk} defined by

$$h_{ijk} = - L^2 L|_i|_j|_k = h_{ij} l_k + h_{jk} l_i + h_{ki} l_j \tag{1.3}$$

where h_{ij} is angular metric tensor and l_i is the unit line element.

It is obvious from the definition of h_{ijk} that

$$h_{ijo} = h_{ijk} y^k = L h_{ij} \neq 0$$

and

$$h_{ioo} = h_{ijk} y^j y^k = L h_{ij} y^j = 0.$$

Hence h_{ijk} satisfies the condition for A_{ijk} as given in proposition.

If $A_{ijk} = h_{ijk}$ in (1.1), then Contracting (1.1) by y^l and using $B_o = 0$ we have

$$-C_{ijk} = h_{jk} B_i + h_{ik} B_j + h_{ij} B_k$$

Again contracting by g^{ki} and using the fact $g^{jk} h_{jk} = n - 1$ and $g^{jk} h_{ik} B_j = B_i$ we have

$$B_i = - C_i / n + 1.$$

2. C^v-Reducible Finsler Space

Definition : *A non Riemannian Finsler space F^n of dimension n is said to be C^v-reducible iff the Cartan v- derivative of (h) hv-torsion tensor can be written as :*

$$(2.1) \qquad L\, C_{ijk}|_l = -1/(n+1)\left[h_{ijk}\, C_l + h_{jkl}\, C_i + h_{kli}\, C_j + h_{lij}\, C_k \right]$$

Proposition 2 : *For $n \geq 3$, every C^v-reducible Finsler space F^n is C-reducible, but converse need not be true. The necessary and sufficient condition for a C-reducible Finsler space to be C^v-reducible is :*

$$(2.2) \qquad L\, C_i|_j + C_i\, l_j + C_j\, l_i = 0.$$

Proof : For *A* C^v-reducible Finsler space contracting (2.1) by y^l, we have,

$$(2.3) \qquad C_{ijk} = 1/(n+1)\left[h_{ij}\, C_k + h_{jk}\, C_i + h_{ki}\, C_j \right]$$

which is (h) hv-torsion tensor of C-reducible Finsler space F^n as defined by *M.* Matsumoto[2]. Since (h) hv-torsion tensor of a C-reducible Finsle spaces can be written in the form (2.3). Hence from proposition (3) of paper[2] there exist a scalar α in C-reducible Finsler space such that

$$(2.4) \qquad L\, C_i|_j + C_i\, l_j + C_j\, l_i = \alpha h_{ij}.$$

Also we have

$$(2.5) \qquad L\, h_{ij}|_k = h_{ij}\, l_k - h_{ijk}.$$

Taking Cartan v-derivative of (2.3) with respect to y^l and using (2.4) and (2.5) we have

$$C_{ijk}|_l = \frac{1}{(n+1)} \cdot \left[\left(h_{ij}|_l\, C_k + h_{ij}\, C_k|_l \right) + \left(h_{jk}|_l\, C_i + h_{jk}\, C_i|_l \right) + \left(h_{ki}|_l\, C_j + h_{ki}\, C_j|_l \right) \right].$$

So

$$L\, C_{ijk}|_l = \frac{1}{(n+1)} \left[\left(h_{ij}\, l_l - h_{ijl} \right) C_k - h_{ij} \left(l_l\, C_k + l_k\, C_l - \alpha h_{kl} \right) + \left(h_{jk}\, l_l - h_{jkl} \right) C_i - h_{jk} \left(l_l\, C_i + l_i\, C_l - \alpha h_{li} \right) + \left(h_{ki}\, l_l - h_{kil} \right) C_j - h_{ki} \left(l_j\, C_l + l_l\, C_j - \alpha h_{jl} \right) \right]$$

$$= \frac{1}{(n+1)}\left[h_{ijk}\, C_l + h_{jkl}\, C_i + h_{kli}\, C_j + h_{lij}\, C_k \right]$$

$$+ \frac{\alpha}{(n+1)}\left[h_{ij}\, h_{kl} + h_{jk}\, h_{li} + h_{ki}\, h_{jl} \right]$$

Thus in order that the space is C^{v}-reducible we must have

$$\frac{\alpha}{(n+1)}\left[h_{ij}\, h_{kl} + h_{jk}\, h_{li} + h_{ki}\, h_{jl} \right] = 0$$

Contracting by g^{ij} and g^{kl} we have $\alpha\,(n-1) = 0$, which implies $\alpha = 0$ for $n \neq 1$.

From (2.4), $\alpha = 0$ is equivalent to

$$L\, C_i|_j + C_i\, l_j + C_j\, l_i = 0$$

which completes the proof of second part.

Proposition 3 : *If a Finsler space F^n is C^{v}-reducible then the T-tensor vanishes identically.*

Proof : T-tensor of a Finsler space F^n can be written as

$$T_{ijkl} = L\, C_{ijk}|_l + C_{ijk}\, l_l + C_{jkl}\, l_i + C_{kli}\, l_j + C_{lij}\, l_k$$

For a C^{v}-reducible Finsler space, $L\, C_{ijk|l}$ is given by (2.1). Putting values of $L\, C_{ijk}|_l$ from (2.1) and C_{ijk} from (2.3) we have

$$T_{ijkl} = -\frac{1}{(n+1)}\left[h_{ijk}\, C_l + h_{jkl}\, C_i + h_{kli}\, C_j + h_{lij}\, C_k \right]$$

$$+ \frac{1}{(n+1)}\Big[\left\{ h_{ij}\, C_k + h_{jk}\, C_i + h_{ki}\, C_j \right\} l_l + \left\{ h_{jk}\, C_l + h_{kl}\, C_j + h_{lj}\, C_k \right\} l_i$$

$$+ \left\{ h_{kl}\, C_i + h_{li}\, C_k + h_{ik}\, C_l \right\} l_j + \left\{ h_{li}\, C_j + h_{ij}\, C_l + h_{jl}\, C_i \right\} l_k \Big]$$

Using (1.3) we get $T_{ijkl} = 0$.

Theorem 1 : *The necessary and sufficient condition for a C^{v}-reducible Finsler space to be Berwald space is* $C_{i|o} = 0$

Proof : Let us consider Ricci identity

$$(2.6)\qquad C_{ijk|l}|_h - C_{ijk}|_{h|_l} = - C_{rjk}\,P^r_{ilh} - C_{irk}\,P^r_{jlh} - C_{ijr}\,P^r_{klh}$$

$$- C_{ijk|r}\,C^r_{lh} - C_{ijk}|_r\,P^r_{lh}\,.$$

Contractig (2.6) by y^l we have

$$(2.7)\qquad C_{ijk|o}|_h - C_{ijk|_h} - C_{ijk}|_{h|o} = 0\,.$$

Since a Berwald space is characterized by $C_{ijk|h} = 0$. Thus from (2.7) for a Berwald space we have

$$(2.8)\qquad C_{ijk}|_{h|o} = 0\,.$$

It is obvious from (2.7) that $C_{ijk}|_{h|o} = 0$ is necessary as well as sufficient condition for a Finsler space to be Berwald space[3].

For a C^v-reducible Finsler space the value of $L\,C_{ijk}|_h$ is of the form

$$L\,C_{ijk}|_h = - \frac{1}{(n+1)}\left[h_{ijk}\,C_h + h_{jkh}\,C_i + h_{khi}\,C_j + h_{hij}\,C_k\right]$$

First differentiating above covariantly with respect to x^l and then contracting by y^l, we have

$$(2.9)\; L\,C_{ijk}|_{h|o} = - \frac{1}{(n+1)}\left[h_{ijk}\,C_{h|o} + h_{jkh}\,C_{i|o} + h_{khi}\,C_{j|o} + h_{hij}\,C_{k|o}\right]$$

For a Berwald space, using (2.8) in (2.9) and contracting by g^{ik} and y^h we have

$$(2.10)\qquad C_{i|o} = 0$$

which is a necessary condition for a C^v-reducible space to be Berwald space. The sufficient part immediately follows from (2.9).

3. C^v-Reducibility of a Two Dimensional Finsler Space

Theorem 2 : *A two dimensional Finsler space is C^v-reducible iff the main scalar I is function of position alone.*

Proof : As is well known in two dimensional case, C_{ijk} is always written in the form where m_i is unit vector orthogonal to the element of support y^l and scalar I is called principal scalar of two dimensional Finsler space. In this case, $h_{ij} = m_i m_j$, hence from (1.3) h_{ijk} for a two dimensional space, can be written as

$$h_{ijk} = m_i m_j l_k + m_j m_k l_i + m_k m_i l_j \tag{3.2}$$

Also we can easily obtain

$$L m_{i|j} = - l_i m_j . \tag{3.3}$$

Differentiating (3.1) covariantly with respect to y^l and using (3.2) and (3.3) we have

$$L C_{ijk|l} = - 1/3 \left[h_{ijk} C_l + h_{jkl} C_i + h_{kli} C_j + h_{lij} C_k \right] + \partial I / \partial y^l m_i m_j m_k . \tag{3.4}$$

From (3.4) It is obvious that in case, two dimensional Finsler space is C^v-reducible, we have

$$\partial I / \partial y^l = 0 \tag{3.5}$$

which implies I is function of position only.

Form (3.4) and (3.5) It is clear that for C^v-reducibility of a two dimensional Finsler space. It is necessary and sufficient that the main scalar is function of position alone.

4. C^v-Reducibility of a Three Dimensional Finsler Space

Theorem 3 : A three dimensional Finsler space is C^v-reducible iff its main scalars are given by

$$H = 3I = \pm 3/\sqrt{2} , J = 0$$

and the v-connection vector vanishes indentically.

Proof : It has already been proved by M. Matsumoto[1] in the (Theorem 30.1) that main scalar J of a three dimensional C-reducible Finsler space is zero. We have shown in Proposition 2 that every C^v-reducible space is C-reducible.

Due to fact, $J = 0$ for any C^v-reducible Finsler space of dimension three.

As shown in the Proposition 3, the T-tensor is identically zero for a C^v-reducible Finsle space. Matsummoto[1] has already shown in the Remark of Theorem [29.5] that vanishing of T-tensor imply $2J^2 + I^2 - HI = - 1$ and v-connection vector vanishes. Hence for a three dimensional C^v-reducible Finsler space we have

$$J = 0, \; 2J^2 + I^2 - HI = - 1, \; v_i = 0 .$$

It is well known that (h) hv-torsion tensor of a three dimensional Finsler space in terms of Moor's frame can be written as

$$LC_{ijk} = Hm_i\, m_j\, m_k - J\pi_{(ijk)}\left(m_i\, m_j\, n_k\right) + I\pi_{(ijk)}\left(m_i\, n_j\, n_k\right) + Jn_i\, n_j\, n_k, \tag{4.1}$$

where H, I, J are main scalars of a three dimensional Finsler space, the scalars H and I are related as $H + I = LC$ and the notation $\pi_{(ijk)}$ denotes the cyclic permutation of indices i, j, k, and summation. Since every C^v-reducible Finsler space is C-reducible. Also due to fact for a three dimensional C^v-reducible Finsler space C_{ijk} can be written as-

$$LC_{ijk} = \frac{L}{4}\left[h_{ij}\, C_k + h_{jk}\, C_i + h_{ki}\, C_j\right] \tag{4.2}$$

as well as from (4.1)

$$LC_{ijk} = \pi_{(ijk)}\left\{\left(H/3\; m_i\, m_j + I\, n_i\, n_j\right)m_k\right\} \tag{4.3}$$

Comparing (4.2) and (4.3) we have $H = 3I$ immediately. Putting $H = 3I$ and $J = 0$ in the relation $2J^2 + I^2 - HI = -1$ we have

$$H = 3I = \pm 3/\sqrt{2}\,.$$

In order to prove sufficient part we put $J = 0$ and $H = 3I = 3\varepsilon/\sqrt{2}$ (where $\varepsilon = \pm 1$) in (4.1), then we have

$$LC_{ijk} = 3\varepsilon/\sqrt{2}\; m_i\, m_j\, m_k + \pi_{(ijk)}\left\{\varepsilon/\sqrt{2}\; m_i\, n_j\, n_k\right\} \tag{4.4}$$

M. Matsumoto[1] introduce two vectors v_i and h_i namely v-and h-connection vectors such that v-and h-Cartan derivative of unit vectors can be written in the form

$$\begin{cases} L\, l^i|_j = \delta^i_j - l^i\, l_j \\ L\, m^i|_j = -\, l^i\, m_j + n^i\, v_j, \\ L\, n^i|_j = -\, l^i\, n_j - m^i\, v_j \end{cases} \tag{4.5a}$$

$$\begin{cases} l^i{}_{|j} = 0 \\ m^i{}_{|j} = n^i\, h_j \\ n^i{}_{|j} = -\, m^i\, h_j \end{cases} \tag{4.5b}$$

and

$$h_{ijk} = \left(m_i\, m_j + n_i\, n_j\right) l_k + \left(m_i\, m_k + n_i\, n_k\right) l_j + \left(m_k\, m_j + n_k\, n_j\right) l_i$$

Differentiating (4.4) and putting values from (4.5a) and (4.5c), we have.

$$L\, C_{ijk}|_l = -1/4\left[h_{ijk}\, C_l + h_{jkl}\, C_i + h_{kli}\, C_j + h_{lij}\, C_k\right].$$

Hence the space is C^v-reducible which completes the proof of sufficient part.

Theorem 4 : *A C^v-reducible three dimensional Finsler space is Landsberg iff h connection vector is perpendicular to line element.*

Proof : The (h) hv-torsion tensor of a three dimensional C^v- reducible Finsler space is of the form (4.4). Differentiating (4.4) h-covariantly and using the fact $L_{|i} = 0$ and (4.5b), we have

$$L\, C_{ijk|l} = \varepsilon/\sqrt{2}\ \pi_{(ijk)}\left(h_{ij}\, n_k\right) h_l \tag{4.6}$$

where $h_{ij} = m_i\, m_j + n_i\, n_j$ for a three dimensional Finsler space. Contracting (4.6) by y^l we have

$$L\, C_{ijk|o} = \varepsilon/\sqrt{2}\ \pi_{(ijk)}\left(h_{ij}\, n_k\right) h_o \tag{4.7}$$

A Landsberg space is characterized by $C_{ijk|o} = 0$. From equation (4.7), for a Landsberg space we have

$$n_i\, h_o = 0 \Rightarrow h_o = 0$$

which completes the necessary part of the Theorem. Sufficient part of the Theorem is obvious from equation (4.7).

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