Vol 2 (1998) Journal of International Academy of Physical Sciences pp.29-33

On the Infinitesimally Deformed Finsler Space

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(Received December 7, 1997)

Abstract : The infinitesimal transformation in a general form has been introduced and Lie-derivatives of various geometric entities have been obtained[1]. The aim of the present paper is to find the various entities of the deformed Finsler space and with the help of these, certain common characteristics possessed by Finsler space and deformed Finsler space have been derived.

1. Introduction

let F_n be an n-dimensional Finsler space equipped with the symmetric metric tensor

(1.1) $$g_{ij}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2}\dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}), \quad \dot{\partial}_i \equiv \partial/\partial \dot{x}^i.$$

Since the metric function $F(x, \dot{x})$ is assumed to be positively homogeneous of degree one in $\dot{x}^i$'s, the metric tensor is homogeneous function of degree zero in $\dot{x}^i$'s. The contravariant components of the metric tensor are given by

(1.2) $$g^{ij} g_{jh} = \delta^i_h = \begin{cases} 1 & \text{if } h = i, \\ 0 & \text{if } h \neq i. \end{cases}$$

The Cartan's covariant derivative of a tensor $T^i_j(x, \dot{x})$ with respect to x^k is given by[2]

(1.3) $$T^i_{j|k}(x, \dot{x}) = \partial_k T^i_j - \left(\dot{\partial}_l T^i_j\right) G^l_k + T^l_j \Gamma^{*i}_{lk} - T^i_l \Gamma^{*l}_{jk}, \quad \partial_k \equiv \partial/\partial x^k,$$

where

(1.4) $$G^l_k(x, \dot{x}) \overset{\text{def}}{=} \dot{\partial}_k G^l = \Gamma^{*l}_{mk}(x, \dot{x})\, \dot{x}^m.$$

The functions $G^m(x, \dot{x})$ are homogeneous of degree two in $\dot{x}^i$'s and $\Gamma^{*1}_{mk}(x, \dot{x})$ are Cartan's connection coefficients. The completely symmetric part of a geometric object Ω_{ij} is given by

(1.5) $$\Omega_{(ij)} \overset{\text{def}}{=} \frac{1}{2}\left(\Omega_{ij} + \Omega_{ji}\right).$$

The infinitesimal transformation in the general form is given by

(1.6) $$\bar{x}^i = x^i + v^i(x, \dot{x})\, d\tau$$

where $v^i(x, \dot{x})$ are the contravariant components of a vector and dx is an infinitesimal constant.

2. Deformed Finsler Space

The deformed geometric object $\bar{\Omega}(x, \dot{x})$ of any geometric object $\Omega(x, \dot{x})$ under the infinitesimal transformation (1.6) is given by[3]

(2.1) $$\bar{\Omega}(x, \dot{x}) = \Omega(x, \dot{x}) + \underset{L}{D}\,\Omega(x, \dot{x}).$$

Thus, we have

(2.2) $$\bar{S}(x, \dot{x}) = S(x, \dot{x}) + \Big[S_{|k}\, v^k + \Big\{v^k_{|h}\, \dot{x}^h + \left(\dot{\partial}_h v^k\right)\left(\ddot{x}^h + 2G^h\right)\Big\}\dot{\partial}_k S\Big]d\tau$$

(2.3) $$\bar{X}^i(x, \dot{x}) = X^i(x, \dot{x}) + \Big[X^i_{|k}\, v^k - X^k\left(v^i_{|k} + G^h_k\, \dot{\partial}_h v^i\right) + \left(\dot{\partial}_k X^i\right)\Big\{v^k_{|h}\, \dot{x}^h + \left(\dot{\partial}_h v^k\right)\left(\ddot{x}^h + 2G^h\right)\Big\}\Big]d\tau$$

(2.4) $$\bar{g}_{ij}(x, \dot{x}) = g_{ij}(x, \dot{x}) + \Big[2\, g_{m(i}\Big\{v^m_{|j)} + G^r_{j)}\, \dot{\partial}_r v^m\Big\} + \left(\dot{\partial}_m g_{ij}\right)\Big\{v^m_{|r}\, \dot{x}^r + \left(\dot{\partial}_r v^m\right)\left(\ddot{x}^r + 2G^r\right)\Big\}\Big]d\tau$$

and

(2.5) $$\bar{\Gamma}^{*i}_{jk}(x, \dot{x}) = \Gamma^{*i}_{jk}(x, \dot{x}) + \Big[v^i_{|jk} + v^h K^i_{jkh} + \left(G^h_j\, \dot{\partial}_h v^i\right)_{|k} + \left(\dot{\partial}_h \Gamma^{*i}_{jk}\right)\Big\{v^h_{|m}\, \dot{x}^m + \left(\dot{\partial}_m v^h\right)\left(\ddot{x}^m + 2G^m\right)\Big\} + \left(\dot{\partial}_j \dot{\partial}_h v^i + \Gamma^{*i}_{rj}\, \dot{\partial}_n v^r\right) G^h_k\Big]d\tau$$

Definition : *The Finsler space F_n equipped with the above deformed geometric entities is called the deformed space of the Finsler space F_n.*

3. Certain Common Characteristics Possessed by F_n & $\bar{F}_n$

In this section, we have the following theorems :

Theorem 3.1 : *When the Finsler space F_n admits a one parameter group of motions generated by (1.6), a vector of constant magnitude deforms into a vector of the same magnitude.*

Proof : The magnitude of the deformed vector $\bar{X}^i(x, \dot{x})$, say $\bar{X}(x, \dot{x})$ is given by

$$\bar{X}^2(x, \dot{x}) = \bar{g}_{ij}\, \bar{X}^i\, \bar{X}^j . \tag{3.1a}$$

Using (2.3) and (2.4) in (3.1a) and neglecting the terms containing second and higher powers of $d\tau$ we get

$$\bar{X}^2(x, \dot{x}) = X^2(x, \dot{x}) + \Big[\left(X^2\right)_{|k} v^k + \left(\dot{\partial}_k X^2\right)\Big\{v^k_{|h}\, \dot{x}^h + \left(\dot{\partial}_h v^k\right)\left(\ddot{x}^h + 2G^h\right)\Big\}\Big] d\tau \tag{3.1b}$$

where

$$X^2(x, \dot{x}) = g_{ij}\, X^i\, X^j .$$

Since $X^i(x, \dot{x})$ is a vector of constant magnitude, it follows from (3.1b) that

$$\bar{X}^2(x, \dot{x}) = X^2(x, \dot{x}).$$

Theorem 3.2 : *When the space F_n admits a one-parameter group of motions generated by the infinitesimal change (1.6), an orthogonal ennuple in F_n deforms into an orthogonal ennuple.*

Proof : Let $\lambda_{a/}(x, \dot{x})$ $(a = 1, 2, \ldots, n)$ be the unit tangents to n-congruences of an orthogonal ennuple in F_n. The subscript a followed by a solidus simply distinguishes one congruence from the other and has no significance of covariance. The contravariant and covariant components of $\lambda_{a/}$ will be denoted by $\lambda^i_{a/}$ and $\lambda_{a/i}$ respectively. Since n-congruences are mutually orthogonal we have

$$g_{ij}\, \lambda^i_{a/}\, \lambda^j_{b/} = \delta_{ab} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases} \tag{3.2}$$

The deformed vector of $\lambda^i_{a/}(x, \dot{x})$ may be obtained from (2.3) in the form

(3.3) $$\bar{\lambda}^i_{a/}(x,\dot{x}) = \lambda^i_{a/}(x,\dot{x}) + \Big[\lambda^i_{a/|k}\,v^k - \lambda^k_{a/}\left(v^i_{|k} + G^h_k\,\dot{\partial}_h\,v^i\right) + \left(\dot{\partial}_k\,\lambda^i_{a/}\right)\left\{v^k_{|h}\,\dot{x}^h + \left(\dot{\partial}_h\,v^k\right)\left(\ddot{x}^h + 2G^h\right)\right\}\Big]d\tau .$$

Using (2.4) and (3.3) and neglecting the terms containing powers of $d\tau$ higher than one we get

(3.4) $$\bar{g}_{ij}\,\bar{\lambda}^i_{a/}\,\bar{\lambda}^j_{b/} = g_{ij}\,\lambda^i_{a/}\,\lambda^j_{b/} + \Big[\left(g_{ij}\,\lambda^i_{a/}\,\lambda^j_{b/}\right)_{|k}\,v^k + \left\{\dot{\partial}_k\left(g_{ij}\,\lambda^i_{a/}\,\lambda^j_{b/}\right)\right\}\left\{v^k_{|h}\,\dot{x}^h + \left(\dot{\partial}_h\,v^k\right)\left(\ddot{x}^h + 2G^h\right)\right\}\Big]d\tau .$$

From (3.2) and (3.4) it follows that

(3.5) $$\bar{g}_{ij}\,\bar{\lambda}^i_{a/}\,\bar{\lambda}^j_{b/} = g_{ij}\,\lambda^i_{a/}\,\lambda^j_{b/},$$

which proves the proposition.

Theorem 3.3 : *The deformed scalar* $\bar{\gamma}_{abc}(x,\dot{x})$ *of the coefficient of rotation* $\gamma_{abc}(x,\dot{x})$ *in* F_n, given by

(3.6) $$\bar{\gamma}_{abc}(x,\dot{x}) = \gamma_{abc}(x,\dot{x}) + \Big[\gamma_{abc|h}\,v^h + \left(\dot{\partial}_h\,\gamma_{abc}\right)\left\{v^h_{|m}\,\dot{x}^m + \left(\dot{\partial}_m\,v^h\right)\left(\ddot{x}^m + 2G^m\right)\right\}\Big]d\tau$$

is the coefficient of rotation in the deformed space $\bar{F}_n$.

Proof : From previous theorem it follows that the deformed vectors $\bar{\lambda}^i_{a/}(x,\dot{x})$ will also be the unit tangents to the n- congruences of an orthogonal ennuple in the deformed Finsler space. The coefficient of rotation in $\bar{F}_n$ is given by

(3.7) $$\bar{\gamma}_{abc}(x,\dot{x}) = \bar{\lambda}^i_{a/\bar{|}j}\,\bar{\lambda}_{b/i}\,\bar{\lambda}^j_{c/},$$

where $\bar{\lambda}^i_{a/\bar{|}j}(x,\dot{x})$ represents the deformed value of the tensor $\lambda^i_{a/|j}(x,\dot{x})$. It is given by

(3.8) $$\bar{\lambda}^i_{a/\bar{|}j}(x,\dot{x}) = \lambda^i_{a/|j}(x,\dot{x}) + \Big[\lambda^i_{a/|jk}\,v^k - \lambda^k_{a/|j}\left(v^i_{|k} + G^h_k\,\dot{\partial}_h\,v^i\right) + \lambda^i_{a/|k}\left(v^k_{|j} + G^h_j\,\dot{\partial}_h\,v^k\right) + \left(\dot{\partial}_k\,\lambda^i_{a/|j}\right)\left\{v^k_{|h}\,\dot{x}^h + \left(\dot{\partial}_h\,v^k\right)\left(\ddot{x}^h + 2G^h\right)\right\}\Big]d\tau .$$

Also we have

$$\bar{\lambda}_{b/i}(x, \dot{x}) = \lambda_{b/i}(x, \dot{x}) + \Big[\lambda_{b/i|k}\, v^k + \lambda_{b/k}\left(v^k_{|i} + G^h_i\, \dot{\partial}_h\, v^k\right) + \left(\dot{\partial}_k\, \lambda_{b/i}\right)\left\{ v^k_{|h}\, \dot{x}^h + \left(\dot{\partial}_h\, v^k\right)\left(\ddot{x}^h + 2G^h\right)\right\}\Big] d\tau . \tag{3.9}$$

Using the equations (3.3), (3.7), (3.8) and (3.9) and neglecting the terms containing powers of $d\tau$ higher than one we get the equation (3.6).

Theorem 3.4 : *When the space F_n admits a one-parameter group of motions generated by the infinitesimal change* (1.6), *the geodesics of the congruence of an orthogonal ennuple deform into geodesics.*

Proof : If the curves of the congruence of an orthogonal ennuple are geodesics, we have $\gamma_{abb}(x, \dot{x}) = 0$. Putting $c = b$ in (3.6) and using $\gamma_{abb}(x, \dot{x}) = 0$, we get $\bar{\gamma}_{abb}(x, \dot{x}) = 0$. Thus, we have the theorem.

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