On *P- and P- reducible Finsler Spaces of Recurrent Curvature

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Abstract. In this paper we observe that a Finsler space of recurrent Berwald's curvature is either a Landsberg space or $\det((H^i_h)) = 0$. A*P-Finsler spaces with recurrent Berwald's curvature is either Riemannian or $\det((H^i_h)) = 0$, while a P-reducible Finsler space with recurrent Berwald's curvature is necessarily a Landsberg space. The associate tensors of the torsion tensor and deviation tensor of a Finsler space of recurrent curvature are recurrent but the associate tensor of curvature tensor H^i_{jkh} is, in general, not recurrent. However, it is recurrent if $\det((H^i_h)) \neq 0$.

1. Introduction

A three dimensional Riemannian space of recurrent curvature was introduced by H. S. Ruse[1] for the first time. A. G. Walker[2] extended the theory of Ruse to n-dimensional Riemannian space. This theory was further enriched by the extensive contributions of eminent differential geometers like I. Mogi[3], E. M. Patterson[4], Y. C. Wong[5, 6], Y. C. Wong and K. Yano[7]. The credit for extending this concept to Finsler spaces goes to A. Moór.[8, 9] Since then a large number of differential geometers including R. S. Mishra and H. D. Pande[10], R. N. Sen[11], R. B. Misra[12, 13] and P. N. Pandey[14-16] contributed significantly to the theory of Finsler spaces of recurrent curvature. Makoto Matsumoto[17,18] introduced a C^h-recurrent space while H. Izumi[19] considered a *P-Finsler space. The aim of this paper is to discuss *P- and P-reducible Finsler spaces of recurrent curvature. The notations used in this paper are based on the book of H. Rund.[20]

2. Preliminaries

Let F_n be an n-dimensional Finsler space equipped with metric function $F(x^i, y^i)$ satisfying the requisite conditions, the symmetric metric tensor g with components g_{ij} and Berwald connection coefficients G^i_{jk}. The Berwald covariant derivative of an arbitrary tensor T^i_j is given by

$$T^i_{j(k)} = \partial_k T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk} \tag{2.1}$$

where

$$G^r_k = G^r_{sk}\, y^s, \quad \partial_k \equiv \frac{\partial}{\partial x^k} \quad \text{and} \quad \dot{\partial}_k \equiv \frac{\partial}{\partial y^k}.$$

Unlike to Cartan's connection, Berwald's connection is not metrical, i.e. the Berwald covariant derivative of the metric tensor g_{ij} does not vanish in general. In fact, it is given by

(2.2) $$g_{ij(k)} = -2C_{ijk(l)}\, y^l = -2C_{ijk|l}\, y^l$$

where $C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{jk}$ and $C_{ijk|l}$ stands for Cartan's covariant derivative of C_{ijk} with respect to Cartan's connection Γ^{*i}_{jk}. The tensor $C_{ijk|l}\, y^l$ is often denoted by P_{ijk}. Berwald curvature tensor H^i_{jkh}, torsion tensor H^i_{kh} and deviation tensor H^i_h are connected by

(2.3) (a) $H^i_{jkh}\, y^j = H^i_{kh}$, (b) $\dot{\partial}_j H^i_{kh} = H^i_{jkh}$,

(c) $H^i_{kh}\, y^k = H^i_h$, (d) $H^i_{hk} = \frac{1}{3}\left(\dot{\partial}_h H^i_k - \dot{\partial}_k H^i_h\right)$.

The torsion tensor H^i_{kh} also satisfies[16]

(2.4) $$y_i H^i_{kh} = 0.$$

Berwald covariant differentiation and directional differentiation commute according to

(2.5) $$\dot{\partial}_j\left(T^i_{h(k)}\right) - \left(\dot{\partial}_j T^i_h\right)_{(k)} = T^r_h G^i_{jkr} - T^i_r G^r_{jkh}$$

where $G^i_{jkh} = \dot{\partial}_j G^i_{kh}$ and T^i_h are components of an arbitrary tensor.

3. A Finsler Space of Recurrent Berwald Curvature Tensor

Let us consider a Finsler space whose Berwald curvature tensor H^i_{jkh} is recurrent with respect to Berwald's connection i.e.

(3.1) $$H^i_{jkh(m)} = \lambda_m H^i_{jkh}, \quad H^i_{jkh} \neq 0$$

where λ_m is a non-null covariant vector field called recurrence vector.

Transvecting (3.1) by y^j and using (2.3a), we get

(3.2) $$H^i_{kh(m)} = \lambda_m H^i_{kh}.$$

Differentiating (3.2) partially with respect to y^j and using the commutation formula exhibited by (2.5), we have

(3.3) $$H^i_{jkh(m)} + H^r_{kh}\, G^i_{jmr} - H^i_{rh}\, G^r_{jmk} - H^i_{kr}\, G^r_{jmh}$$

$$= \left(\dot{\partial}_j \lambda_m\right) H^i_{kh} + \lambda_m H^i_{jkh}$$

which, in view of (3.1) and the fact that the recurrence vector of a Finsler space with recurrent curvature and non-zero curvature scalar ($H \neq 0$) is independent of y^i, reduces to

(3.4) $$H^r_{kh}\, G^i_{jmr} - H^i_{rh}\, G^r_{jmk} - H^i_{kr}\, G^r_{jmh} = \left(\dot{\partial}_j \lambda_m\right) H^i_{kh}.$$

Transvecting (3.4) by y_i and using (2.4), we get

(3.5) $$H^r_{kh}\, y_i\, G^i_{jmr} = 0.$$

We know that the associate vector y_i of y^i is a covariant constant, i.e. $y_{i(k)} = 0$. Differentiating this equation partially with respect to y^j and using the commutation formula (2.5), we get $g_{ij(k)} = y_r\, G^r_{ijk}$. Using this and (2.2) in (3.5), we find

(3.6) $$H^r_{kh}\, C_{jmr|l}\, y^l = 0.$$

Transvecting the equation (3.6) by y^k and using (2.3c) we have

(3.7) $$H^r_h\, C_{jmr|l}\, y^l = 0.$$

If the det $((H^i_j)) \neq 0$, (3.7) implies

(3.8) $$C_{jmr|l}\, y^l = 0.$$

But this condition characterizes a Landsberg space. Therefore, we may conclude :

Theorem 3.1 : *A Finsler space with recurrent Berwald's curvature is either a Landsberg space or the* det $((H^i_h)) = 0$.

If a Finsler space with recurrent Berwald's curvature is a *P-Finsler space characterized by

(3.9) $$C_{ijk|l}\, y^l = \lambda C_{ijk},$$

the identity (3.7) reduces to

(3.10) $$H^r_h\, C_{jmr} = 0.$$

If the det $((H^i_k)) \neq 0$, (3.10) implies $C_{jmr} = 0$. This shows that the space is Riemannian. Thus, we conclude :

Theorem 3.2 : *A *P-Finsler space with recurrent Berwald's curvature is either Riemannian or* det $((H^i_k)) = 0$.

Consider a P-reducible Finsler space of recurrent Berwald's curvature[18]. We know that a P-reducible Finsler space is characterized by

$$P_{jkh} = \frac{1}{n+1}\left(P_j h_{kh} + P_k h_{hj} + P_h h_{jk}\right) \tag{3.11}$$

where

$$P_{jkh} = C_{jkh|l}\, y^l. \tag{3.12}$$

From (3.7), (3.11) and (3.12), we have

$$H^h_m P_j g_{kh} + H^h_m P_k g_{hj} + H^h_m P_h h_{jk} = 0. \tag{3.13}$$

Transvecting (3.13) by P^k, we get

$$H^h_m P_j P_h + H^h_m P^k P_k g_{hj} + H^h_m P^k P_h h_{jk} = 0. \tag{3.14}$$

But $P^k h_{jk} = P_j - l_j P^k l_k = P_j$ for $P^k l_k = 0$. Therefore

$$g_{hj} H^h_m P_k P^k = -2P_j P_h H^h_m. \tag{3.15}$$

Transvecting (3.15) by P^j, we have

$$P_k P^k P_h H^h_m = 0,$$

which implies at least one of the following two conditions :

$$\text{(a)} \quad P_k P^k = 0, \qquad \text{(b)} \quad P_h H^h_m = 0. \tag{3.16}$$

In view of (3.16b), (3.15) gives $P_k P^k g_{hj} H^h_m = 0$. This implies at least one of the conditions : (3.16a) and $g_{hj} H^h_m = 0$. But $g_{hj} H^h_m = 0$ implies $H^h_m = 0$, which in view of (2.3d) and (2.3b) gives $H^h_{jkm} = 0$, a contradiction. Therefore we have (3.16a). Thus, (3.16b) implies (3.16a). Since the metric of our space is positive definite, the condition (3.16a) gives $P_k = 0$. Substituting $P_k = 0$ in (3.11) we have $P_{jkh} = 0$, which shows that the space is a Landsberg space. Thus, we have :

Theorem 3.3 : *A P-reducible Finsler space with recurrent Berwald's curvature is necessarily a Landsberg space.*

4. Recurrence of Associate Tensors

In a Riemannian space the recurrence of curvature tensor R^i_{jkh} implies and is implied by the recurrence of the associate curvature tensor R_{jkhl}. This is due to the covariant constant character of the metric tensor. The metric tensor of a Finsler space is covariant constant with respect to Cartan's connection, but it is not a covariant constant with respect to Berwald's connection. Therefore, the recurrence of Cartan's curvature tensor implies and is implied by the recurrence of the associate curvature tensor but the situation is quite different in case of Berwald's curvature tensor.

In this section we try to find out a condition sufficient for the equivalence of recurrence of Berwald's curvature tensor and its associate curvature tensor.

Transvecting (3.1) by g_{il} *we get*

(4.1) $$H_{jlkh(m)} - H^r_{jkh}\, g_{rl(m)} = \lambda_m\, H_{jlkh}$$

where $H_{jlkh} = g_{il}\, H^i_{jkh}$. Thus in a space with recurrent Berwald's curvature we have (4.1). This shows that in a space with recurrent Berwald's curvature the associate curvature tensor H_{jlkh} is recurrent iff $H^r_{jkh}\, g_{rl(m)} = 0$. Since $g_{rl(m)} = -2\, C_{mrl|s}\, y^s$, the condition $H^r_{jkh}\, g_{rl(m)} = 0$ *becomes*

(4.2) $$H^r_{jkh}\, C_{mrl|s}\, y^s = 0.$$

In view of Theorem 3.1, if the det $((H^i_j)) \neq 0$, the space is a Landsberg space and the condition $H^r_{jkh}\, C_{mrl|s}\, y^s = 0$ is fully satisfied. Therefore if det $((H^i_j)) \neq 0$, the associate curvature tensor H_{jlkh} of a Finsler space of recurrent Berwald's curvature is recurrent. Thus, we conclude :

Theorem 4.1 : *The necessary and sufficient condition for the recurrence of associate curvature tensor H_{jlkh} of a Finsler space with recurrent Berwald's curvature tensor is $H^r_{jkh}\, C_{lmr|s}\, y^s = 0$. If the det $((H^i_j)) \neq 0$, the associate curvature tensor is necessarily recurrent.*

Transvections of the equation (3.1) by directional arguments show that

(4.3) (a) $H^i_{kh(m)} = \lambda_m\, H^i_{kh}$, (b) $H^i_{h(m)} = \lambda_m\, H^i_h$

hold in a space with recurrent Berwald's curvature tensor. Let us check whether the associate of tensor H^i_{kh} and H^i_h are recurrent. Transvecting (4.3) by g_{il} we have

(4.4) (a) $H_{olkh(m)} - H^i_{kh}\, g_{il(m)} = \lambda_m H_{olkh}$,

(b) $H_{oloh(m)} - H^i_h\, g_{il(m)} = \lambda_m H_{oloh}$

where $g_{il} H^i_{kh} = H_{olkh}$ and $g_{il} H^i_h = H_{oloh}$. Using (2.2) in (4.4), we get

(4.5) (a) $H_{olkh(m)} + 2H^i_{kh}\, C_{mil|s}\, y^s = \lambda_m H_{olkh}$,

(b) $H_{oloh(m)} + 2H^i_h\, C_{mil|s}\, y^s = \lambda_m H_{oloh}$.

In view of (3.6) and (3.7), the equations (4.5a) and (4.5b) reduce to

(4.6) (a) $B_m H_{olkh} = \lambda_m H_{olkh}$, (b) $B_m H_{oloh} = \lambda_m H_{oloh}$.

Thus, we have :

Theorem 4.2 : *The associate of the tensors H^r_{kh} and H^r_h are recurrent in a space with recurrent Berwald's curvature tensor.*

References

1. H. S. Ruse, Three dimensional spaces of recurrent curvature, *Proc. Lond. Math. Soc.*, **50** (1949) 438-446.
2. A. G. Walker, On Ruse's space of recurrent curvature, *Proc. Lond. Math. Soc.*, **52** (1950) 36-64.
3. I. Mogi, A remark on recurrent curvature spaces, *Kodai Math. Sem. Rep.*, (1950) 73-74.
4. E. M. Patterson, Some theorems on Ricci recurrent spaces, *J. Lond. Math. Soc.*, **27** (1952) 285-295.
5. Y. C. Wong, Recurrent tensor on a linearly connected differentiable manifold, *Trans. Amer, Math. Soc.*, 99 (1961) 325-341.
6. Y. C. Wong, Linear connections with zero torsion and recurrent curvature, *Trans. Amer. Math. Soc.*, **102** (1962) 471-506.
7. Y. C. Wong and K. Yano, Projectively flat spaces with recurrent curvature, *Comment, Math. Helv.*, **35** (1961) 223-232.
8. A. Moo'r, Untersuchungen Über Finslerräume von Rekurrenter Krümmung, *Tensor N.S.*, **13** (1963) 1–18.
9. A. Moo'r, Unterräume von rekurrenter krümmung in Finslerräumen, *Tensor N. S.*, **24** (1972) 261-265.
10. R. S. Mishra and H. D. Pande, Recurrent Finsler spaces, *J. Ind. Math. Soc.*, **32** (1968) 17-22.
11. R. N. Sen, Finsler spaces of recurrent curvature, *Tensor N. S.*, **19** (1968) 291-299.
12. R. B. Misra, On a recurrent Finsler space, *Rev, Roumaine Math. Pures Appl.*, **18** (1973) 701-712.
13. R. B. Misra, A turning point in the theory of recurrent Finsler manifolds, *J. South Gujarat Univ.*, **6** (1977) 72-96.
14. P. N. Pandey, A recurrent Finsler manifold with a concircular vector field, *Acta. Math. Acad. Sci. Hungar.*, **35**, (1980) 465-466.

15. P. N. Pandey, A note on Recurrence Vector, *Proc. Nat. Acad. Sci. (India)*, **51A** (1981) 6-8.

16. P. N. Pandey, On decomposability of curvature tensor of a Finsler manifold, *Acta. Math. Acad. Sci. Hungar.*, **38** (1981) 109-116.

17. Makoto Matsumoto, On h-isotropic and C^h- recurrent Finsler spaces, *J. Math. Kyoto Univ.*, **11** (1971) 1-9.

18. Makoto Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Otsu, Japan, 1986.

19. H. Izumi, On *P-Finsler spaces, I, II, *Memo, Defense Acad.*, **16** (1976) 133-138; **17** (1977) 1-9.

20. H. Rund, *The differential geometry of Finsler spaces*, Springer verlag, Berlin-Göttingen- Heidelberg, 1959.