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# An Alternative Proof of a Theorem of Mohanty on the Absolute Riesz Summability of Fourier Series

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**Abstract** : In the present paper a theorem of Mohanty on the absolute Riesz summability of Fourier series at a point is proved by means of an alternative, but equivalent, definition of the method, which has distinct advantages for problems of absolute Riesz summability of certain 'types'.

## 1. Definitions and Notations

Let $\sum a_n$ by an infinite series of real terms, and $\{\lambda_n\}$ a positive, monotonic increasing sequence, diverging to $\infty$. Let

$$A_\lambda(\omega) = A_\lambda^o(\omega) = \sum_{\lambda_n \leq \omega} a_n,$$

$$A_\lambda^r(\omega) = \sum_{\lambda_n \leq \omega} \left(\omega - \lambda_n\right)^r a_n, \qquad r > 0.$$

$$C_\lambda^r(\omega) = A_\lambda^r(\omega)/\omega^r, \; r \geq 0.$$

**Definition I** : The series $\sum a_n$ is said to be absolutely summable by Riesz means of '*type*' $\lambda_n$ and '*order*' $r$, $r \geq 0$, or summable $|R, \lambda_n, r|$, $r \geq 0$, if

$$C_\lambda^r(\omega) \in BV(h, \infty),$$

for some $h \geq 0$ ; Obrechkoff[3,4]. By '$f(\omega) \in BV(a, b)$' we mean that $f$ is a function of bounded variation over $(a, b)$.

An equivalent definition is given below.

**Definition II** : Let $\lambda(\omega)$ be a positive, monotonic increasing, differentiable function of $\omega$, diverging to $\infty$, defined over $(A, \infty)$, where $A$ is some positive constant. We write

$$\tilde{A}_\lambda(\omega) = \tilde{A}_\lambda^o(\omega) = \sum_{n \leq \omega} a_n,$$

$$\tilde{A}^{r}_{\lambda}(\omega) \sum_{n \leq \omega} (\lambda(\omega) - \lambda(n))^{r} a_n, \quad r > 0,$$

$$\tilde{C}^{r}_{\lambda}(\omega) = \tilde{A}^{r}_{\lambda}(\omega) / \{\lambda(\omega)\}^{r}, \quad r \geq 0.$$

$\sum a_n$ is said to be summable $|R, \lambda(\omega), r|$, $r \geq 0$, iff

$$\tilde{C}^{r}_{\lambda}(\omega) \in BV(A, \infty),$$

where $A$ is some positive constant; Mohanty[2].

Evidently summability $|R, \lambda_n, 0|$ and summability $|R, \lambda(\omega), 0|$ are equivalent to each other, each being equivalent to absolute convergence. It is easily seen that for $r > 0$ also $|R, \lambda_n, r|$ and $|R, \lambda(\omega), r|$ are equivalent methods of absolute summability. The 'First Theorem of Consistency' for absolute Riesz summability states that if any infinite series $\sum a_n$ is summable $|R, \lambda_n, r|$, $r \geq 0$, then it is summable $|R, \lambda_n, r'|$ for every $r' > r$. This is due to Obrechkoff[3,4]. It has been demonstrated by Mohanty that summability $|R, e^{\omega}, 1|$ (which is equivalent to summability $|R, e^{n}, 1|$ ) is equivalent to absolute convergence; Mohanty[1,2].

Let $f$ be a periodic, real-valued function, with period $2\pi$, integrable $(L)$ over $(-\pi, \pi)$. Then the Fourier series of $f$ is given by

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right),$$

where

$$\begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \begin{matrix} \cos nt \\ \sin nt \end{matrix} \, dt.$$

We write

$$\varphi(t) = \frac{1}{2} \{ f(x+t) + f(x-t) \}.$$

Evidently the Fourier series of $f$ at $t = x$ is the same as the Fourier series of $\varphi$ at $t = 0$.

## 2. Introduction

Considering the example of the even periodic function $f$ defined as :

$$f(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq \pi/2, \\ 1 & \text{for} \quad \pi/2 < t \leq \pi, \end{cases}$$

and taken as periodic, with period $2\pi$, outside $(-\pi, \pi)$, Mohanty observed that the condition

$$t^{-\delta}\varphi(t) \in BV(0,\ \pi),\ \delta > 0,$$

does not ensure the absolute convergence of the Fourier series of $f$ at $t = x$; Mohanty[2].

However, as proved by Mohanty, the following theorem is true.

**Theorem :** Mohanty[2]. *If* $t^{-\delta}\varphi(t) \in BV\ (0,\ \pi),\ \delta > 0$, *then the Fourier series of* $f$, *at* $t = x$, *is summable* $|\, R\,,\ e^{\omega/(\log\omega)^{1+1/\delta}}\,,\ 1\,|$.

For proving this theorem, Mohanty uses the Definition I of summability $|\, R\,,\ \lambda_n\,,\ r\,|\,,\ r \geq 0$. The purpose of this paper is to give an alternative proof of the theorem by means of the equivalent Definition II. It turns out that our analysis is more convenient. This also corroborates Mohanty's own observations (Mohanty[2]), that Definition II has 'distinct advantages' over Definition I, when we deal with such problems.

For $r > 0,\ m < \omega < m + 1$, we have

$$\frac{d}{d\omega}\left\{\tilde{C}^{\,r}_{\lambda}(\omega)\right\} = \frac{r\,\lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}} \sum_{n\leq\omega} \{\lambda(\omega) - \lambda(n)\}^{r-1}\,\lambda(n)\,a_n\,.$$

Hence summability $|\, R\,,\ \lambda(\omega),\ r\,|\,,\ r > 0$, is equivalent to :

$$\int_A^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}} \left| \sum_{n\leq\omega} \{\lambda(\omega) - \lambda(n)\}^{r-1}\,\lambda(n)\,a_n \right| d\omega < \infty. \tag{2.1}$$

In particular, when $r = 1$, (2.1) reduces to

$$\int_A^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^{2}} \left| \sum_{n\leq\omega} \lambda(n)\ a_n \right| d\omega < \infty. \tag{2.2}$$

## 3. The Lemmas

We shall need the following lemmas.

**Lemma 1**. The case $: 0 < \delta < 1$ is stated in Mohanty[2]. *The Fourier series of the even periodic function defined in* $(-\pi,\ \pi)$ *as* $|\,t\,|^{\delta}\ (\delta > 0)$ *and elsewhere by periodicity, with period* $2\pi$, *is absolutely convergent at* $t = 0$.

**Proof** : Let the Fourier series of the function be

$$\frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos nt\,.$$

Then, for $0 < \delta < 1$,

$$
\begin{aligned}
c_n &= \frac{2}{\pi} \int_0^{\pi} t^{\delta} \cos nt \, dt \\
&= \frac{2}{\pi} \int_0^{1/n} t^{\delta} \cos nt \, dt + \frac{2}{\pi} \int_{1/n}^{\pi} t^{\delta} \cos nt \, dt \\
&= I_1 + I_2 \text{, say.}
\end{aligned}
$$

Now

$$
I_1 = O\left( \frac{1}{n^{\delta}} \left| \int_{\xi}^{1/n} \cos nt \, dt \right| \right) \qquad (0 \le \xi \le 1/n)
$$

by the Second Mean Value Theorem. Hence

$$
I_1 = O\left( \frac{1}{n^{\delta}} \left| \left[ \frac{\sin nt}{n} \right]_{\xi}^{1/n} \right| \right) = O\left( \frac{1}{n^{1+\delta}} \right).
$$

Next

$$
\begin{aligned}
I_2 &= \frac{2}{\pi} \int_{1/n}^{\pi} t^{\delta} \cos nt \, dt \\
&= \frac{2}{\pi} \left[ t^{\delta} \frac{\sin nt}{n} \right]_{1/n}^{\pi} - \frac{2}{\pi} \frac{\delta}{n} \int_{1/n}^{\pi} t^{\delta - 1} \sin nt \, dt \\
&= O\left( \frac{1}{n^{1+\delta}} \right) + O\left( \frac{1}{n} \frac{1}{n^{\delta - 1}} \left| \int_{1/n}^{\xi} \sin nt \, dt \right| \right) \qquad (1/n \le \xi \le \pi)
\end{aligned}
$$

by the Second Mean Value Theorem. Hence

$$
\begin{aligned}
I_2 &= O\left( \frac{1}{n^{1+\delta}} \right) + O\left( \frac{1}{n^{\delta}} \left| \left[ \frac{-\cos nt}{n} \right]_{1/n}^{\xi} \right| \right) \\
&= O\left( \frac{1}{n^{1+\delta}} \right).
\end{aligned}
$$

If $\delta \ge 1$, then

$$
\begin{aligned}
c_n &= \frac{2}{\pi} \int_0^{\pi} t^{\delta} \cos nt \, dt \\
&= \frac{2}{\pi} \int_0^{1/n} t^{\delta} \cos nt \, dt + \frac{2}{\pi} \int_{1/n}^{\pi} t^{\delta} \cos nt \, dt \\
&= J_1 + J_2 \text{, say.}
\end{aligned}
$$

We have

$$J_1 = \frac{2}{\pi}\,\frac{1}{n^{\delta}}\int_{\xi}^{1/n} \cos nt\,dt = O\left(\frac{1}{n^{\delta}}\left|\frac{\sin nt}{n}\right]_{\xi}^{1/n}\right|\right) \qquad (0 \le \xi \le 1/n)$$

$$= O\left(\frac{1}{n^{1+\delta}}\right) = O\left(\frac{1}{n^{2}}\right).$$

Also

$$J_2 = \frac{2}{\pi}\, t^{\delta}\,\frac{\sin nt}{n}\Bigg]_{1/n}^{\pi} - \frac{2}{\pi}\,\frac{\delta}{n}\int_{1/n}^{\pi} t^{\delta-1}\sin nt\,dt$$

$$= O\left(\frac{1}{n^{1+\delta}}\right) + O\left(\frac{1}{n}\left|\int_{\xi}^{\pi}\sin\ nt\,dt\right|\right) \qquad (1/n \le \xi \le \pi)$$

$$= O\left(\frac{1}{n^{1+\delta}}\right) + O\left(\frac{1}{n^{2}}\right)$$

$$= O\left(\frac{1}{n^{2}}\right).$$

Thus, if $0 < \delta < 1$,

$$c_n = O\left(\frac{1}{n^{1+\delta}}\right),$$

and, if $\delta \ge 1$,

$$c_n = O\left(\frac{1}{n^{2}}\right).$$

In either case

$$\sum |c_n| < \infty.$$

**Lemma 2.** *Uniformly in* $0 < t < \pi$, *for* $\delta > 0$,

$$g(\omega,\ t) = \int_0^t u^{\delta}\left(\sum_{n\le\omega}\lambda(n)\ \cos nu\right)du \tag{3.1}$$

$$= O\left(t^{\delta}\,\frac{\lambda(\omega)}{\omega}\,(\log\omega)^{\Delta}\right),$$

*where*

$$\lambda(x) = e^{x/(\log x)^{\Delta}},\ \Delta = 1 + 1/\delta\,.$$

**Proof** : We have

$$g(\omega, t) = \int_0^t u^{\delta}\left(\sum_{n \le \omega} \lambda(n) \cos nu\right) du$$

$$= \sum_{n \le \omega} \lambda(n) \int_0^t u^{\delta} \cos nu\, du$$

$$= t^{\delta} \sum_{n \le \omega} \lambda(n) \int_{\xi}^t \cos nu\, du \qquad (0 \le \xi \le t)$$

$$= t^{\delta} \sum_{n \le \omega} \lambda(n) \frac{\sin nt - \sin n\xi}{n}$$

$$= t^{\delta} \sum_{n \le \omega} \frac{\lambda(n)}{n} \sin nt - t^{\delta} \sum_{n \le \omega} \frac{\lambda(n)}{n} \sin n\xi$$

$$= O\left(t^{\delta} \sum_{n \le \omega} \frac{\lambda(n)}{n}\right). \tag{3.1.1}$$

Now, for $m \le \omega < m + 1$,

$$\begin{cases} \displaystyle\sum_{n \le \omega} \frac{\lambda(n)}{n} = \lambda_1 + \sum_{n=2}^{m} \frac{\lambda(n)}{n} < \lambda_1 + \int_2^m \frac{\lambda(x)}{x} dx + \frac{\lambda(m)}{m}; \\ \displaystyle\int_2^m \frac{\lambda(x)}{x} dx = \int_2^M + \int_M^m \frac{\lambda(x)}{x} dx \qquad (M \text{ defined below}). \end{cases} \tag{3.1.2}$$

Also

$$\lambda'(x) = \left(1 - \frac{\Delta}{\log x}\right) \lambda(x) / (\log x)^{\Delta}$$

and hence

$$\int_M^m \frac{\lambda(x)}{x} dx = \int_M^m \frac{\lambda'(x) (\log x)^{\Delta}}{1 - \dfrac{\Delta}{\log x}} \frac{1}{x} dx \left(x \ge M > e^{\Delta} \Rightarrow \log x \ge \log M > \Delta\right)$$

$$< K \int_M^m \frac{\lambda'(x)}{x} (\log x)^{\Delta} dx$$

(here and in the sequel, $K$ ia an absolute positive constant)

$$= K\left[\frac{\lambda(x)\,(\log x)^{\Delta}}{x}\right]_M^m - K\int_M^m \lambda(x)\,\frac{d}{dx}\left(\frac{(\log x)^{\Delta}}{x}\right)dx$$

$$= O\left(\frac{\lambda(m)\,(\log m)^{\Delta}}{m}\right) + \int_M^m \frac{\lambda(x)}{x}\,F(x)\,dx, \text{ say.}$$

Hence

$$\int_M^m \frac{\lambda(x)}{x}\,dx = O\left(\frac{\lambda(\omega)\,(\log\omega)^{\Delta}}{\omega}\right) + \int_M^m \frac{\lambda(x)}{x}\,F(x)\,dx. \tag{3.1.3}$$

Given $\varepsilon > 0$, $0 < \varepsilon < 1$, we can choose $M$ such that, for $x > M$, $|F(x)| < \varepsilon$, i.e. $-\varepsilon < F(x) < \varepsilon$.

Then

$$(1-\varepsilon)\int_M^m \frac{\lambda(x)}{x}\,dx < \int_M^m \frac{\lambda(x)}{x}\,(1 - F(x))\,dx$$

$$= O\left(\frac{\lambda(\omega)}{\omega}\,(\log\omega)^{\Delta}\right)$$

by (3.1.3), so that

$$\int_M^m \frac{\lambda(x)}{x}\,dx = O\left(\frac{\lambda(\omega)}{\omega}\,(\log\omega)^{\Delta}\right).$$

Hence from (3.1.2), keeping $M$ fixed,

$$\sum_{n\le\omega}\frac{\lambda(n)}{n} = O(1) + O\left(\frac{\lambda(\omega)}{\omega}\,(\log\omega)^{\Delta}\right) + O\left(\frac{\lambda(\omega)}{\omega}\right)$$

$$= O\left(\frac{\lambda(\omega)}{\omega}\,(\log\omega)^{\Delta}\right).$$

Therefore, from (3.1.1),

$$g(\omega, t) = O\left(t^{\delta}\,\frac{\lambda(\omega)}{\omega}\,(\log\omega)^{\Delta}\right).$$

**Lemma 3.** *Uniformly in* $0 < t < \pi$, $\delta > 0$,

$$h(\omega, t) = \int_t^{\pi} u^{\delta}\left(\sum_{n\le\omega}\lambda(n)\cos nu\right)du \tag{3.2}$$

$$= O\left(t^{-1}\,\frac{\lambda(\omega)}{\omega}\right),$$

where

$$\lambda(x) = e^{x/(\log x)^{\Delta}},\quad \Delta = 1 + 1/\delta.$$

**Proof** : We have

$$h(\omega,\, t) = \sum_{n \le \omega} \lambda(n) \int_t^{\pi} u^{\delta} \cos nu\, du$$

$$= \sum_{n \le \omega} \lambda(n)\, \pi^{\delta} \int_{\xi}^{\pi} \cos nu\, du \qquad (t \le \xi \le \pi)$$

$$= -\, \pi^{\delta} \sum_{n \le \omega} \lambda(n)\, \frac{\sin n\xi}{n}$$

$$= O\left( \left| \sum_{n \le \omega} \frac{\lambda(n)}{n} \sin n\xi \right| \right)$$

$$= O\left( \xi^{-1}\, \frac{\lambda(\omega)}{\omega} \right)$$

$$= O\left( t^{-1}\, \frac{\lambda(\omega)}{\omega} \right)$$

by Abel's lemma, since $\lambda(n)/n$ is ultimately monotonic increasing.

## 4. Proof of the Theorem

It is enough to show that

$$\int_A^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \left| \sum_{n \le \omega} \lambda(n)\, A_n(x) \right| d\omega < \infty.$$

Now

$$\sum_{n \le \omega} \lambda(n)\, A_n(x) = \frac{2}{\pi} \int_0^{\pi} \varphi(t) \sum_{n \le \omega} \lambda(n) \cos nt\, dt$$

$$= \left[ \frac{2}{\pi}\, t^{-\delta} \varphi(t) \int_0^t u^{\delta} \left( \sum_{n \le \omega} \lambda(n) \cos nu \right) du \right]_0^{\pi}$$

$$- \frac{2}{\pi} \int_0^{\pi} \left( \int_0^t u^{\delta} \left( \sum_{n \le \omega} \lambda(n) \cos nu \right) du \right) d\left( t^{-\delta} \varphi(t) \right)$$

$$= K g(\omega,\, \pi) - \frac{2}{\pi} \int_0^{\pi} g(\omega,\, t)\, d\left( t^{-\delta} \varphi(t) \right).$$

Thus it is enough to show :

$$\int_A^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \, |g(\omega, \pi)| \, d\omega < \infty \tag{4.1}$$

and

$$I = \int_A^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \int_0^{\pi} |g(\omega, t)| \left| d\left(t^{-\delta} \varphi(t)\right) \right| d\omega < \infty .$$

Since $t^{-\delta} \varphi(t) \in BV(0, \pi)$, $I < \infty$, if, uniformly in $0 < t < \pi$,

$$\int_A^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \, |g(\omega, t)| \, d\omega = O(1). \tag{4.2}$$

**Proof of (4.1)** : This is the same as the summability $|R, \lambda(\omega), 1|$ of the Fourier series of the even function : $|t|^{\delta}$, $\delta > 0$, in $(-\pi, \pi)$ and defined by periodicity outside $(-\pi, \pi)$. This is a consequence of Lemma 1 and the First Theorem of Consistency for absolute Riesz summability.

**Proof of (4.2)** : Let

$$\tau = e^{1/t^{\delta}}, \ \delta > 0.$$

We have

$$\begin{aligned}
\int_A^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \, |g(\omega, t)| \, d\omega &= \int_A^{\tau} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \, |g(\omega, t)| \, d\omega \\
&\quad + \int_{\tau}^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \, |g(\omega, \pi) - h(\omega, t)| \, d\omega \\
\leq \int_A^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \, |g(\omega, \pi)| \, d\omega &+ \int_A^{\tau} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \, |g(\omega, t)| \, d\omega \\
&\quad + \int_{\tau}^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \, |h(\omega, t)| \, d\omega .
\end{aligned}$$

We have already proved (4.1). Hence it is enough to prove that, uniformly in $0 < t < \pi$,

$$I_1 = \int_A^{\tau} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \, |g(\omega, t)| \, d\omega = O(1) \tag{4.3}$$

and

$$I_2 = \int_{\tau}^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \, |h(\omega, t)| \, d\omega = O(1). \tag{4.4}$$

By Lemma 2,

$$I_1 = O\left( t^{\delta} \int_A^{\tau} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2}\, \lambda(\omega)\, \frac{(\log\omega)^{\Delta}}{\omega}\, d\omega \right)$$

$$= O\left( t^{\delta} \int_A^{\tau} \frac{d\omega}{\omega} \right) \left( \lambda'(\omega) = \frac{\lambda(\omega)}{(\log\omega)^{\Delta}} \left( 1 - \frac{\Delta}{\log\omega} \right) \right)$$

$$= O\left( t^{\delta} \log\tau \right)$$

$$= O\left( t^{\delta} \frac{1}{t^{\delta}} \right) = O(1).$$

Also, by Lemma 3,

$$I_2 = O\left( t^{-1} \int_{\tau}^{\infty} \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2}\, \frac{\lambda(\omega)}{\omega}\, d\omega \right)$$

$$= O\left( t^{-1} \int_{\tau}^{\infty} \frac{d\omega}{\omega(\log\omega)^{\Delta}} \right)$$

$$= O\left( t^{-1} \frac{1}{(\log\tau)^{1/\delta}} \right)$$

$$= O\left( t^{-1} \frac{1}{t^{-1}} \right) = O(1).$$

Thus (4.3) and (4.4) are proved. This completes the proof of the theorem.

## References

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