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Sharp Inequalities for the Generalized Steiner-Gutman Index

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Abstract: For any connected graph $G = (V, E)$ with real numbers a, b and a positive integer k, the Steiner distance is denoted by $d_G(S)$ for a set of vertices $S \subseteq V(G)$ is defined as the minimum size of connected subgraphs that include a given set of vertices S with $S = k$. In this article, we introduce a new version of the Steiner-Gutman index for a graph G, defined it as

$$SG^k_{(a,b)}(G) = \sum_{S\subseteq V(G),|S|=k}\left(\prod_{v_i\in S} d_G(v_i)\right)^a d_G(S)^b$$

where a and b are any real numbers. In this paper, we obtained some best possible inequalities and their characterizations in terms of the order, size, minimum / maximum degree, and diameter of G. Also, the comparisons of $SG^k_{(a,b)}(G)$ with other graphical indices are obtained.

Keywords: Wiener index, Steiner index, Steiner gutman index.
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1. Introduction

The graphs considered in this paper are undirected, simple, finite, and connected. The graph $G = (V, E)$ has p-vertices and q-edges, where $V = V(G)$ and $E = E(G)$ represent the vertex and edge collections, respectively. The degree of a vertex v_i is defined as the number of vertices adjacent to it and is denoted by $d_G(v_i)$. If a vertex is adjacent to only one edge, it is called a pendant vertex. The distance between two vertices in a graph is given by $d_G(v_i, v_j)$, the shortest path length between v_i and v_j. The greatest distance between any two vertices in a graph G is called the diameter of the graph and is denoted by $diam(G)$. For undefined notations in this paper, we refer to the paper[1,2].

The Wiener index, the first distance-based graph invariant, was introduced by Harold Wiener[3] in 1947. Wiener's research revealed connections between the boiling points of paraffins and their molecular structure. The Wiener index, represented as $W(G)$, is calculated by adding the distances between all pairs of vertices within a connected graph G. In other words

$$W(G) = \sum_{v_i, v_j \subseteq V(G)} d_G(v_i, v_j).$$

In 1989, Chartrand[4] introduced the Steiner distance of a graph. This distance is represented by $d_G(S)$ and measures the connectivity of a subset of vertices S in a graph G. The minimum number of edges is needed to connect all the vertices in S, where $S \subseteq V(G)$.

In 1994, Gutman[5] proposed the Gutman index of a connected graph G and is defined as

$$Gut(G) = \sum_{\{v_i, v_j\} \in V(G)} d(v_i) d(v_j) d(v_i, v_j).$$

This index is a numeric measure of the molecular branching of a chemical compound represented by a molecular graph.

In 2018, Mao[6] introduced the Steiner Gutman index of a connected graph G. This index measures the molecular branching and connectivity of a molecular graph simultaneously. It is defined as

$$SG^k(G) = \sum_{S \subseteq V(G), |S| = k} \left(\prod_{v_i \in S} d_G(v_i) \right) d_G(S),$$

where k is the size of the subset S.

Analogously, we now defined the Generalized Steiner Gutman index for a connected graph G as

$$SG^k_{(a,b)}(G) = \sum_{S \subseteq V(G), |S| = k} \left(\prod_{v_i \in S} d_G(v_i) \right)^a d_G(S)^b,$$

where a and b are any real numbers.

For historical developments, applications, and mathematical properties of graphical indices and its related concepts, see the papers[7-24] and the references cited therein.

2. Bounds and Charactirization

To prove the next couple of results, we use the following definition of Narumi and Katayama[25].

The Narumi–Katayama index of a graph G is defined as

$$NK(G) = \prod_{v_i \in V(G)} d_G(v_i).$$

Theorem 2.1. *Let G be a connected graph and $|\ S\ | = p$ with real numbers a, b. Then*

$$SG^p_{(a,b)}(G) = \left(NK(G)\right)^a (p-1)^b.$$

Proof. Let G be a connected graph and $|S| = p$ with real numbers a, b. Then $d_G(S) = p - 1$ and $\prod_{v_i \in S} d_G(v_i) = NK(G)$. Therefore

$$SG^k_{(a,b)}(G) = \sum_{S=V(G)} \left(\prod_{v_i \in S} d_G(v_i) \right)^a d_G(S)^b$$
$$= \left(NK(G)\right)^a (p-1)^b$$

Theorem 2.2. *Let G be a connected graph and $S \subset V(G)$ with $|S| = p - 1$, $a \geq 0$ and $b \geq 0$. Then*

$$p\left(\frac{NK(G)}{\Delta(G)}\right)^a (p-2)^b \leq SG^{p-1}_{(a,b)}(G) \leq p\left(\frac{NK(G)}{\delta(G)}\right)^a (p-1)^b.$$

Proof. Let G be a connected graph with $|S| = p - 1$ for $S \subset V(G)$ and each subset S of $V(G)$ satisfies

$$\frac{NK(G)}{\Delta(G)} \leq \prod_{v_i \in S} d_G(v_i) \leq \frac{NK(G)}{\delta(G)}.$$

This inequality remains the same if we raise a positive power a to each side:

$$\left(\frac{NK(G)}{\Delta(G)}\right)^a \leq \left[\prod_{v_i \in S} d_G(v_i)\right]^a \leq \left(\frac{NK(G)}{\delta(G)}\right)^a. \tag{2.1}$$

The value of $d_G(S)$ is either $(p-2)$ or $(p-1)$, and for $b \geq 0$, the following inequality holds:

$$(p-2)^b \leq d_G(S)^b \leq (p-1)^b. \tag{2.2}$$

Taking the product of equations (2.1) and (2.2) for each subset S of $V(G)$, we have

$$\sum_{\substack{S \subset V(G) \\ |S|=p-1}} \left(\frac{NK(G)}{\Delta(G)}\right)^a (p-2)^b \leq \sum_{\substack{S \subset V(G) \\ |S|=p-1}} \left[\prod_{v_i \in S} d_G(v_i)\right]^a d_G(S)^b$$
$$\leq \sum_{\substack{S \subset V(G) \\ |S|=p-1}} \left(\frac{NK(G)}{\delta(G)}\right)^a (p-1)^b,$$

On simplification, we have the desired result.

By Theorem 2.2, the following Theorem's can be obtained and we omit their proofs.

Theorem 2.3. *Let G be a connected graph and $S \subset V(G)$.*

(i) If $a \leq 0$ and $b \geq 0$, then

$$p\left(\frac{NK(G)}{\delta(G)}\right)^a (p-2)^b \leq SG^{p-1}_{(a,b)}(G)$$
$$\leq p\left(\frac{NK(G)}{\Delta(G)}\right)^a (p-1)^b.$$

(ii) If $a \leq 0$ and $b \leq 0$, then

$$p\left(\frac{NK(G)}{\delta(G)}\right)^{a}(p-1)^{b} \leq SG_{(a,b)}^{p-1}(G) \leq p\left(\frac{NK(G)}{\Delta(G)}\right)^{a}(p-2)^{b}.$$

(iii) If $a \geq 0$ and $b \leq 0$, then

$$p\left(\frac{NK(G)}{\Delta(G)}\right)^{a}(p-1)^{b} \leq SG_{(a,b)}^{p-1}(G) \leq p\left(\frac{NK(G)}{\delta(G)}\right)^{a}(p-2)^{b}.$$

Theorem 2.4. *Let G be a connected graph and $S \subset V(G)$ with $2 \leq n \leq p-1$.*

(i) If $a \geq 0$ and $b \geq 0$, then

$$p\left(\frac{NK(G)}{n\Delta(G)}\right)^{a}(p-n-1)^{b} \leq SG_{(a,b)}^{p-n}(G) \leq p\left(\frac{NK(G)}{n\delta(G)}\right)^{a}(p-n)^{b}.$$

(ii) If $a \leq 0$ and $b \geq 0$, then

$$p\left(\frac{NK(G)}{n\delta(G)}\right)(p-n-1)^{b} \leq SG_{(a,b)}^{p-n}(G) \leq p\left(\frac{NK(G)}{n\Delta(G)}\right)^{a}(p-n)^{b}.$$

(iii) If $a \leq 0$ and $b \leq 0$, then

$$p\left(\frac{NK(G)}{n\delta(G)}\right)^{a}(p-n)^{b} \leq SG_{(a,b)}^{p-n}(G) \leq p\left(\frac{NK(G)}{n\Delta(G)}\right)^{a}(p-n-1)^{b}.$$

(iv) If $a \geq 0$ and $b \leq 0$, then

$$p\left(\frac{NK(G)}{n\Delta(G)}\right)^{a}(p-n)^{b} \leq SG_{(a,b)}^{p-n}(G) \leq p\left(\frac{NK(G)}{n\delta(G)}\right)^{a}(p-n-1)^{b}.$$

Theorem 2.5. *Let G and G' be the connected graphs, and G' be obtained by deleting an edge in G. If $a \leq 0$ and $b \geq 0$, then*

$$SG_{(a,b)}^{k}(G) \leq SG_{(a,b)}^{k}(G').$$

Proof. Let G and G' be the connected and $S \subseteq V(G)$ with $|S| = k$. Then the removal of the edge will not decrease the value $d_G(S)$ since G' is obtained by the removal of an edge in a graph G, which implies that $d_G(S) \leq d_{G'}(S)$ for some $S \subseteq V(G)$. For every subset $S \subseteq V(G)$ with $b \geq 0$, we have

$$d_G(S)^b \leq d_{G'}(S)^b \tag{2.3}$$

If v_i and v_j are adjacent in G, then the same vertices need not be adjacent in G'. For each $S \subseteq V(G)$, we have

$$\prod_{v_i \in S} d_G(v_i) \geq \prod_{v_i \in S} d_{G'}(v_i).$$

Now, raising the power with $a \leq 0$ on both sides,

$$\left[\prod_{v_i \in S} d_G(v_i)\right]^a \leq \left[\prod_{v_i \in S} d_{G'}(v_i)\right]^a. \tag{2.4}$$

Taking the product of equations (2.3) and (2.4) for each subset S of $V(G)$, we have The product of the above equations results

$$\sum_{S \subseteq V(G), |S|=k} \left[\prod_{v_i \in S} d_G(v_i)\right]^a d_G(S)^b \leq \sum_{S \subseteq V(G'), |S|=k} \left[\prod_{v_i \in S} d_{G'}(v_i)\right]^a d_{G'}(S)^b,$$

$$SG^k_{(a,b)}(G) \leq SG^k_{(a,b)}(G').$$

Corollary 2.1. *Let G and G' be the connected graphs and G' is obtained by deleting an edge in G. If $a \geq 0$ and $b \leq 0$, then*

$$SG^k_{(a,b)}(G) \geq SG^k_{(a,b)}(G').$$

Theorem 2.6. *Let G be a connected graph and $S \subseteq V(G)$ with $a \geq 0$ and $b \geq 0$. Then*

$$\binom{p}{k} \delta(G)^{ka}(k-1)^b \leq SG^k_{(a,b)}(G) \leq \binom{p}{k} \Delta(G)^{ka}(p-1)^b$$

Proof. Let G be a connected graph and $S \subseteq V(G)$ with $|S| = k$. For each $S \subseteq V(G)$ and $a,\ b \geq 0$, we have

$$\delta(G)^{ka} \leq \left[\prod_{v_i \in S} d_G(S)\right]^a \leq \Delta(G)^{ka} \tag{2.5}$$

and

$$(k-1)^b \leq d_G(S)^b \leq (p-1)^b. \tag{2.6}$$

The product of equation (2.5) and equation (2.6), for each $S \subseteq V(G)$, we have

$$\binom{p}{k} \delta(G)^{ka}(k-1)^b \leq SG^k_{(a,b)}(G) \leq \binom{p}{k} \Delta(G)^{ka}(p-1)^b$$

By Theorem 2.6, the following Theorem's can be obtained and we omit their proofs.

Theorem 2.7. Let G be a connected graph and $S \subseteq V(G)$.

(i) If $a \geq 0$ and $b \leq 0$, then

$$\binom{p}{k}\delta(G)^{ka}(p-1)^b \le SG^k_{(a,b)}(G) \le \binom{p}{k}\Delta(G)^{ka}(k-1)^b.$$

(ii) If $a \le 0$ and $b \ge 0$, then

$$\binom{p}{k}\Delta(G)^{ka}(k-1)^b \le SG^k_{(a,b)}(G) \le \binom{p}{k}\delta(G)^{ka}(p-1)^b.$$

(iii) $a, b \le 0$, then

$$\binom{p}{k}\Delta(G)^{ka}(p-1)^b \le SG^k_{(a,b)}(G) \le \binom{p}{k}\delta(G)^{ka}(k-1)^b.$$

Theorem 2.8. *Let G be a connected graph with $diam(G) = 2$, $k \ge 3$ and $a \le 0,\ b \ge 0$. Then*

$$\begin{aligned}\binom{p-1}{k-1}(p-1)^a(k-1)^b \quad &+ \binom{p-1}{k}k^b \\ &\le SG^k_{(a,b)}(G) \\ &\le \binom{p-2}{k}(p-1)^{ka}(k-1)^b \\ &+2\binom{p-2}{k-1}(p-1)^{(k-1)a}(p-2)^a(k-1)^b \\ &+\binom{p-2}{k-2}(p-1)^{(k-2)a}(p-2)^{2a}(k-1)^b.\end{aligned}$$

Proof. Let G be a connected graph, and let $S \subseteq V(G)$ have a $diam(G) = 2$. If $a \le 0$, the removing an edge from G will decreases the values of $d_G(v_i)$ and $\left(d_G(v_i)\right)^a$. Similarly, if $b \le 0$, not increase the value of $d_G(S)$ and not decrease the value $\left(d_G(S)\right)^b$.

The star attains the lower bound with $a \ge 0$ and $b \le 0$.

$$\binom{p-1}{k-1}(p-1)^a(k-1)^b + \binom{p-1}{k}k^b \le SG^k_{(a,b)}(G). \tag{2.7}$$

We need to reverse the above mentioned process to find the upper bound. For this purpose, we add an edge, which increases the values of $d_G(v_i)^a$ and does not decrease the value of $d_G(S)$. If we delete one edge in a complete graph, we get the graph with the maximum possible number of edges with $diam(G) = 2$, giving us the upper bound.

$$\begin{aligned}SG^k_{(a,b)}(G) \quad \le &\sum_{\substack{S\subseteq V(G),\\ \forall v_i : d_G(v_i)=p-1}} \left[\prod_{v_i\in S}(\mathrm{p}-1)\right]^a d_G(S)^b \\ &+ \sum_{\substack{S\subseteq V(G),\\ \text{exists } v_i : d_G(v_i)\ne p-1}} \left[\prod_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b\end{aligned}$$

$$\begin{aligned}SG^k_{(a,b)}(G) \quad &\le \left[\binom{p-2}{k}(p-1)^{ka} + 2\binom{p-2}{k-1}(p-1)^{(k-1)a}(p-2)^a\right](k-1)^b \\ &+\binom{p-2}{k-2}(p-1)^{(k-2)a}(p-2)^{2a}(k-1)^b.\end{aligned} \tag{2.8}$$

By equation (2.7) and equation (2.8), the desired result follows.

Lemma 2.1. *Let G^* be a connected graph with maximum possible edges and $diam(G^*) = 3$. Then*

$$\begin{aligned} SG_{(a,b)}^{k}(G^*) &= \binom{p-2}{k}(p-3)^{ka}(k-1)^{b} \\ &+ \binom{d_{G^*}(v_1)}{k-1} d_G(v_1)^a (p-3)^{a(k-1)}(k-1)^b \\ &+ \binom{d_{G^*}(v_2)}{k-1} d_G(v_1)^a (p-3)^{a(k-1)}k^b \\ &+ \binom{d_{G^*}(v_2)}{k-1} d_G(v_2)^a (p-3)^{a(k-1)}(k-1)^b \\ &+ \binom{d_{G^*}(v_1)}{k-1} d_G(v_2)^a (p-3)^{a(k-1)}k^b \\ &+ \binom{d_{G^*}(v_1)}{k-2} (p-3)^{a(k-1)}\left(k^b-(k-1)^b\right) \\ &+ \binom{d_{G^*}(v_2)}{k-2} (p-3)^{a(k-1)}\left(k^b-(k-1)^b\right) \\ &+ \binom{p-2}{k-2}(p-3)^{a(k-1)}(k-1)^b, \end{aligned}$$

where a, b are real numbers and $d_G(v_1) + d_G(v_2) = p - 2$.

Proof. Consider a graph G^* with $diam(G^*) = 3$ and the maximum possible edges. Since $diam(G^*) = 3$, a pair of vertices (v_1, v_2) must exist such that the distance between them is 3. Hence, the remaining $(p-2)$-vertices in the graph must be adjacent to each other, and each of these $(p-2)$-vertices must be connected to the vertices of either v_1 or v_2. Therefore, the partition of $(p-2)$-vertices into sets V_1 and V_2 is such that vertices in V_1 are adjacent to vertex v_1 with $|V_1| = d_{G^*}(v_1)$ and vertices in V_2 are adjacent to vertex v_2 with $|V_2| = d_{G^*}(v_2)$. This implies that, the degree of vertices v_1 and v_2 is at least 1, and $d_{G^*}(v_1) + d_G(v_4) = p - 2$, while all other vertices have degree $(p-3)$.

Thus, the value of $d_{G^*}(S)$ is $k-1$ for $S \subseteq V(G^*)$ with $|S| = k$, if any of the following conditions hold,

(i) the vertices in S are chosen from $(p-2)$-vertices, excluding v_1 and v_2.

(ii) the vertices in S include v_1 and v_2, along with at least one vertex from the partitions V_1 and V_2.

(iii) the vertices in S include v_1 but not v_2, and at least one vertex from V_1, and vice versa.

In all other cases, $d_{G^*}(S) = k$.

For $d_{G^*}(v_1) \geq d_{G^*}(v_2) \geq k = |S|$, then

$$\begin{aligned} SG_{(a,b)}^{k}(G^*) &= \sum_{S\subseteq V_1\cup V_2}\left[\prod_{v_i\in S} d_{G^*}(v_i)\right]^a d_{G^*}(S)^b \\ &+ \sum_{\substack{S\subseteq V_1\cup\{v_1\},\\ v_1\in S}}\left[\prod_{v_i\in S} d_{G^*}(v_i)\right]^a d_{G^*}(S)^b \\ &+ \sum_{\substack{S\subseteq V_2\cup\{v_1\},\\ v_1\in S}}\left[\prod_{v_i\in S} d_{G^*}(v_i)\right]^a d_{G^*}(S)^b \\ &+ \sum_{\substack{S\subseteq V_1\cup\{v_2\},\\ v_1\in S}}\left[\prod_{v_i\in S} d_{G^*}(v_i)\right]^a d_{G^*}(S)^b \\ &+ \sum_{\substack{S\subseteq V_2\cup\{v_2\},\\ v_1\in S}}\left[\prod_{v_i\in S} d_{G^*}(v_i)\right]^a d_{G^*}(S)^b \\ &+ \sum_{\substack{S\subseteq\{v_1,v_2\}\cup V_1\cup V_2,\\ v_1,v_2\in S}}\left[\prod_{v_i\in S} d_{G^*}(v_i)\right]^a d_{G^*}(S)^b. \end{aligned}$$

Based on the above facts, the desired result follows. □

Lemma 2.2. *Let $S_{m,n}$ be a double star with $m \geq n \geq k$ and $d_G(v_1) + d_G(v_2) = p$. Then*

$$\begin{aligned} SG_{(a,b)}^{k}(S_{m,n}) &= \left(\binom{m-1}{k} + \binom{n-1}{k}\right)k^b + \binom{p-2}{k-2}(mn)^a(k-1)^b \\ &+ \left[\binom{m+n-2}{k} - \binom{m-1}{k} - \binom{n-1}{k}\right](k+1)^b \\ &+ \sum_{h=1}^{k-1}\binom{m}{h}\binom{n}{k-h-1}(m^a+n^a)k^b \\ &+ \binom{m-1}{k-1}\left(m^a(k-1)^b + n^a k^b\right) \\ &+ \binom{n-1}{k-1} m^a k^b\left(m^a k^b + n^a(k-1)^b\right), \end{aligned}$$

where the double star graph $S_{m,n}$ is a tree obtained by joining the center of two stars S_m and S_n with an edge.

Proof. Let $S_{m,n}$ be a double star with $m \geq n \geq k$ and $d_G(v_1) + d_G(v_2) = p$. If the central vertices of stars S_m and S_n are v_1 and v_2, respectively, with V_1 is the set of pendant vertices adjacent to v_1, and V_2 is the set of vertices adjacent to v_2. Therefore, the degree of vertices v_1 and v_2 are m and n, respectively.

For any $S \subseteq V(G)$, the value of $d_G(S)$ is lies between $k-1$ and $k+1$, we have

$$
\begin{aligned}
SG_{(a,b)}^{k}(S_{m,n}) &= \sum_{S\subseteq V_1}\left[\prod_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b + \sum_{S\subseteq V_2}\left[\prod_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b \\
&+ \sum_{\substack{S\subseteq V_1\cup V_2,\\ S\nsubseteq V_1, S\nsubseteq V_2}}\left[\prod_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b + \sum_{\substack{S\subseteq V_1\cup\{v_1\},\\ v_1\in S}}\left[\prod_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b \\
&+ \sum_{\substack{S\subseteq V_2\cup\{v_1\},\\ v_1\in S}}\left[\prod_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b + \sum_{\substack{S\subseteq V(G)-\{v_2\},\\ v_1\in S,\\ S\nsubseteq V_1\cup\{v_1\},\\ S\nsubseteq V_2\cup\{v_1\}}}\left[\prod_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b \\
&+ \sum_{\substack{S\subseteq V_1\cup\{v_1\},\\ v_2\in S}}\left[\prod_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b + \sum_{\substack{S\subseteq V_2\cup\{v_2\},\\ v_2\in S}}\left[\prod_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b \\
&+ \sum_{\substack{S\subseteq V(G)-\{v_1\},\\ v_2\in S,\\ S\nsubseteq V_1\cup\{v_2\},\\ S\nsubseteq V_2\cup\{v_2\}}}\left[\prod_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b + \sum_{\substack{S\subseteq V(G)\\ \{v_1,v_2\in\}}}\left[\prod_{v_i\in S} d_G(v_i)\right]^a d_G(S)^b
\end{aligned}
$$

$$
\begin{aligned}
SG_{(a,b)}^{k}(S_{m,n}) &= \binom{m-1}{k}k^b + \binom{n-1}{k}k^b \\
&+ \left[\binom{m+n-2}{k} - \binom{m-1}{k} - \binom{n-1}{k}\right](k+1)^b \\
&+ \binom{m-1}{k-1}m^a(k-1)^b + \binom{n-1}{k-1}m^a k^b \\
&+ \sum_{h=1}^{k-1}\binom{m}{h}\binom{n}{k-h-1}m^a k^b \\
&+ \binom{m-1}{k-1}n^a k^b + \binom{n-1}{k-1}n^a(k-1)^b \\
&+ \sum_{h=1}^{k-1}\binom{m}{h}\binom{n}{k-h-1}n^a k^b + \binom{p-2}{k-2}(mn)^a(k-1)^b
\end{aligned}
$$

Based on the above facts, the desired result follows.

Observation 2.1. Let G be a connected graph and $S \subseteq V(G)$ with $|S| = k$, $a \leq 0$, $b \geq 0$ and $diam(G) = 3$. Then

(i) If $a \geq 0$, $b \leq 0$, then

$$SG_{(a,b)}^{k}(K_{m,n}) \leq SG_{(a,b)}^{k}(G) \leq SG_{(a,b)}^{k}(G^*)$$

(ii) If $a \leq 0$, $b \geq 0$, then

$$SG_{(a,b)}^{k}(G^*) \leq SG_{(a,b)}^{k}(G) \leq SG_{(a,b)}^{k}(K_{m,n})$$

3. Comparision among degree-distance based indices

If $f(x,y) = \left(\prod_{i=1}^{k} a_i\right)^x b^y$ is a function in two variables, where $a_i \in \{1,2,\dots,p-1\}$, $b \in \{k-1,k,\dots,p-1\}$ and x and y are real numbers, then $f(x,y)$ is a strictly increasing function. We have the following inequalities among the existing indices for a fixed value of a, b, and k. For fixing $a = 0$ and $k = 2$,

$$SG^2_{(0,-2)}(G) < SG^2_{(0,-1)}(G) < SG^2_{(0,1)}(G)$$

$$\sum_{\{v_i,v_j\}\subseteq V(G)} \frac{1}{d\left(v_i,v_j\right)^2} < \sum_{\{v_i,v_j\}\subseteq V(G)} \frac{1}{d\left(v_i,v_j\right)} < \sum_{\{v_i,v_j\}\subseteq V(G)} d\left(v_i,v_j\right)$$

$$H^2(G) < H(G) < W(G).$$

Theorem 3.1. *Let G be a connected graph and $S \subseteq V(G)$ with $2 \le k \le p-2$. Then*

$$Gut(G) \le SG^k_{(a,b)}(G).$$

Proof. Let G be a connected graph and $S \subseteq V(G)$ with $2 \le k \le p-2$. Now for any any $\{v_i,v_j\} \subseteq V(G)$, there exist $S \subseteq V(G)$ such that $\{v_i,v_j\} \subseteq S$ and $d_G(v_i)d_G\left(v_j\right) \le \prod_{v_i\in S} d_G\,(v_i)$. Which is also satisfies $d_G\left(v_i,v_j\right) \le d_G(S)$, we have

$$\sum_{v_i,v_j\subseteq V(G)} \left(d_G(v_i)d_G\left(v_j\right)\right)d_G\left(v_i,v_j\right) \le \sum_{S\subseteq V(G)} \left(\prod_{v_i\in S} d_G(v_i)\right) d_G(\mathrm{S})$$

$$Gut(G) \le SG^k_{(a,b)}(G).$$

4. Conclusion and Open problems

For any two real numbers a and b, the Generalized Steiner gutman index of a graph lies on the claim that their special cases, for pertinently chosen values of the parameters a, b, and k, with the vast majority of previously considered vertex degree-distance based topological indices. This research raises several questions and observations from a comparative advantage, applications, and mathematical standpoint.

(i) When the values of $a \ge b \ge 0$ are fixed, the complete graph reaches the maximum value of $SG^k_{(a,b)}(G)$. The path or the star can reach the minimum value of $SG^k_{(a,b)}(G)$ depending on the value of a. However, the bounds will be reversed if we choose $a \le b \le 0$.

(ii) If G is a tree with diameter d, then the number of pendent vertices varies between $\left\lceil \frac{p}{d} \right\rceil$ to $(p-d+1)$. However, the bounds for the tree with diameter d are still an open problem.

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References

1. F. Buckley and F. Harary, *Distance in graphs,* Addison-Wesley, Redwood, 1990.
2. F. Harary; *Graph Theory*, Addison Wesley, Reading Ma USA, 1969.
3. H. Wiener; Structural determination of paraffin boiling points, *Journal of the American chemical society*, 69(1) (1947), 17-20.
4. G. Chartrand; Steiner distance in graphs, casopis pro pestovani matematiky, 144(4) (1989), 399-410.
5. I. Gutman; Selected properties of the Schultz molecular topological index, *J. Chem. Inf. Comput. Sci.*, 34 (1994), 1087-1089.
6. Y. Mao, K. C. Das; Steiner Gutman index, MATCH Commun, *Math. Comput. Chem*., 79(3) (2018), 779-94.
7. S. A. Basha, T. N. Asha and B. Chaluvaraju; Generalized Schultz and Gutman Indices, *Iranian Journal of Mathematical Chemistry*, 13(4) (2022), 301-316.
8. B. Chaluvaraju, H. S. Boregowda and I. N. Cangul; Some Inequalities for the First General Zagreb Index of Graphs and Line Graphs, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci*., 91 (2021), 79-88.
9. B. Chaluvaraju and S. A. Basha; Different versions of atom-bond connectivity indices of some molecular structures: Applied for the treatment and prevention of COVID-19, *Polycyclic Aromatic Compounds*, 42(6) (2022), 3748–3761.
10. P. Dankelmann; The Steiner k-Wiener index of graphs with given minimum degree, *Discrete Applied Mathematics*, 268 (2019), 35–43
11. P. Dankelmann, R. Ortrud, Oellermann and C. S. Henda; The average Steiner distance of a graph, *Journal of Graph Theory*, 22(1) (1996), 15–22
12. P. Dankelmann, I. Gutman, S. Mukwembi and C. S. Henda; On the degree distance of a graph, *Discrete Applied Mathematics*, 157(13) (2009), 2773–2777.
13. P. Dankalmann and R. Entringer; Average distance, minimum degree, and spanning trees, *Journal of Graph Theory*, 33(1) (2000), 1-13.
14. I. Gutman; On Steiner degree distance of trees, *Appl. Math. Comput*. 283 (2016), 163-167.
15. I. Gutman; On Steiner degree distance of trees, *Applied Mathematics and Computation*, 283 (2016), 163-167.

16. I. Gutman, B. Furtula and K. C. Das; On some degree-and-distance-based graph invariants of trees, *Applied Mathematics and Computation*, 289 (2016), 1-6.
17. I. Gutman and Y. N. Yen; The sum of all distances in bipartite graphs, *Mathematica Slovaca*, 45, 45(4) (1995), 327-334.
18. H. Hua and S. Zhang; On the reciprocal degree distance of graphs, *Discrete Applied Mathematics*, 160(7-8) (2012), 1152-1163.
19. Y. Mao, W. Zhao and I. Gutman; Steiner Wiener index of graph products, *Transactions on Combinatorics*, 5(3) (2016), 39-50.
20. Y. Mao, P. Dankelmann and Z. Wang; Steiner diameter, maximum degree and size of a graph, *Discrete Mathematics*, 344(8) (2021), 112468.
21. Y. Mao, C. Melekian and E. Cheng; A note on the Steiner-diameter of a graph, *International Journal of Computer Mathematics: Computer Systems Theory*, 3(1) (2018), 41-46.
22. I. Redzepovic, Y. Mao, Z. Wang and B. Furtula; Steiner degree distance indices: Chemical applicability and bounds, *International Journal of Quantum Chemistry*, 120(12) (2020), e26209.
23. B. Sarveshkumar, V. R. Kulli and B. Chaluvaraju; Some inequalities on generalized degree based indices: An (a, b)-KA Indices and Coindices, *Proceedings of the Jangjeon Mathematical Society*, 26(1) (2023): 43-53.
24. X. Zhang; Reciprocal sterner degree distance, *Utilitas Mathematica*, 113 (2019).
25. H. Narumi, M. Katayama; Simple topological index. a newly devised index characterizing the topological nature of structural isomers of saturated hydrocarbons, *Mem. Fac. Engin. Hokkaido Univ*., 16 (1984), 209-214.