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Framed Metric Submanifold of Kähler and Nearly Kähler Manifolds

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Abstract. The conditions for the framed metric structure induced on the submanifold of a Kähler manifold and for the induced framed metric sructure on the submanifold of a Kähler manifold to be normal are obtained. Few algebraic relations in the framed metric submanifold of a Kähler manifold to be normal with $\underset{x}{U}$ as a killing vector have also been derived. Conditions for a manifold V_n to be a framed metric submanifold of a nearly Kähler manifold V_m are obtained. Some results in a framed metric submanifold of a nearly Kähler manifold are obtained and conditions for this manifold to be totally geodesic and having $\underset{x}{U}$ as a killing vector have been investigated.

1. Introduction

Let us consider an even dimensional differentiable manifold V_m of differentiability class C^∞ with a vector- valued real linear function F of class C^∞, satisfying

$$F^2 + I_m = 0 \tag{1.1}$$

The V_m is said to be an almost complex manifold and $\{F\}$ is said to give an almost complex structure to V_m. If the almost complex manifold is endowed with an almost complex structure F and a Hermite metric G such that

$$G(FX, FY) = G(X, Y)\,. \tag{1.2}$$

then $\{F, G\}$ is said to give to V_m an almost Hermite structure and the manifold V_m is called an almost Hermite manifold[1]

An almost Hermite manifold for which

$$(E_X F)\,Y = 0, \tag{1.3}$$

is satisfied, is called a Kähler manifold, where E is the Riemannian connexion on V_n

An almost Hermite manifold for which

$$(E_X F)\,Y + (E_Y F)\,X = 0, \tag{1.4}$$

is satisfied, is called a nearly Kähler or an almost Tachibana manifold. Let us consider an n-dimensional differentiable manifold V_n of differentiability class C^∞, where $n = r + s$.

and r is even with f-structure of rank r. Let there exist on V_n, s vector fields $\underset{x}{U}$ and s l-forms $\overset{x}{u}$, such that

$$(1.5)\qquad \text{a)}\ f^2 + I_n = \overset{x}{u} \otimes \underset{x}{U},$$

equivalently

$$(1.5)\qquad \text{b)}\ \overline{X} + X = \overset{x}{u}(X)\,\underset{x}{U}, \quad \text{c)}\ fX = \overline{X}$$

$$(1.5)\qquad \text{d)}\ \overline{\underset{x}{U}} = 0, \quad \text{e)}\ \overset{x}{u}(\underset{y}{U}) = \delta^x_y, \quad \text{f)}\ \overset{x}{u} \circ f = 0.$$

where X is an arbitrary vector field X in V_n. Then we say that f-structure has complemented frames and V_n is said to be a globally-framed f-manifold or simply a framed manifold[1]. Let a metric tensor g be defined in a framed manifold V_n, satisfying

$$(1.6)\qquad \text{a)}\ g(\overline{X}, \overline{Y}) = g(X, Y) - \overset{x}{u}(X)\,\overset{x}{u}(Y), \quad \text{b)}\ \overset{x}{u}(X) \overset{def}{=} g(X, \underset{x}{U})$$

Then the system $\{f, \underset{x}{U}, \overset{x}{u}, g\}$ is said to give to V_n a framed metric structure and the manifold V_n is called a framed metric manifold[1].

Let us put

$$(1.7)\qquad 'f(X, Y) = g(\overline{X}, Y)$$

Then $'f(X, Y) = -'f(Y, X)$, i.e. $'f$ is skew-symmetric in X and Y, and $'f(\overline{X}, \overline{Y}) = 'f(X, Y)$, i.e. $'f$ is hybrid in X and Y.

A framed manifold V_n is said to be normal[1] if $[f, f] + d\overset{x}{u} \otimes \underset{x}{U} = 0$, where $[f, f]$ is the Nijenhuis tensor of the structure f. This equation is equivalent to

$$(D_{fX}f)\,Y - (D_{fY}f)\,X + f(D_Y f)\,X - f(D_X f)\,Y + [(D_X \overset{x}{u})(Y) - (D_Y \overset{x}{u})(X)]\,\underset{x}{U} = 0.$$

Let us now consider the two differentiable manifolds V_m and V_n of class C^∞ with structures $\{F, G\}$ and $\{f, g\}$ and dimensions m and n respectively. Let b be the inclusion map defined by $b : V_n \to V_m$, such that $p \in V_n \Rightarrow bp \in V_m$. Inclusion map b induces a

Jacobian map B, defined by $B : T_n^1 \rightarrow T_m^1$ where T_n^1 is tangent space to V_n at a point p and T_m^1 is tangent space to V_m at the point bp such that X in V_n at $p \Rightarrow BX$ in V_m at bp.

Let g be the induced metric tensor in V_n, then $(G(BX, BY))\, o\, b = g(X, Y)$. Let $N_\alpha : \alpha = n+1, \cdots, m$ be a system of C^∞ mutually orthogonal unit normal vector fields in V_n at p. Then

$$\text{(1.8)} \qquad \text{a)} \ (G(N_\alpha, BX))\, o\, b = 0, \quad \text{b)} \ (G(N_\alpha, N_\beta))\, o\, b = \delta^\alpha_\beta = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}.$$

Let E be the Riemannian connexion in V_m and let D be the induced Riemannian connexion in V_n. The Gauss equations and the Weingarten equations are given by[1]

$$\text{(1.9)} \qquad E_{BX} BY = BD_X Y + {}'\underset{\alpha}{H}(X, Y)\, \underset{\alpha}{N},$$

$$\text{(1.10)} \qquad E_{BX} \underset{\alpha}{N} = - B\underset{\alpha}{H}X + \underset{\alpha\beta}{L}(X)\, \underset{\beta}{N},$$

where ${}'\underset{\alpha}{H}$ are the symmetric bilinear functions in V_n and

$$\text{(1.11)} \qquad g(\underset{\alpha}{H}X, Y) \overset{def}{=} {}'\underset{\alpha}{H}(X, Y) = g(X, \underset{\alpha}{H}\, Y).$$

$\underset{\alpha}{H}$ are known as Weingarten maps and $\underset{\alpha\beta}{L}$ are third fundamental forms in V_n. If the submanifold be totally geodesic[1], then

$$\text{(1.12)} \qquad {}'\underset{\alpha}{H}(X, Y) = 0.$$

The transformations of BX and $\underset{\alpha}{N}$ by F are represented in the forms of tangential and normal parts as follows:

$$\text{(1.13)} \qquad \text{a)} \ FBX = \overline{BX} + \overset{x}{u}(X)\, \underset{\alpha}{N}. \quad \text{b)} \ F\underset{\alpha}{N} = - B\underset{x}{U},$$

The conditions that V_n be a framed metric submanifold of a Kähler manifold V_m are[3]

$$\text{(1.14)} \quad \text{a)} \ (D_X f)\, Y = \overset{x}{u}(Y)\, \underset{\alpha}{H}X - {}'\underset{\alpha}{H}(X, Y)\, \underset{x}{U}, \quad \text{b)} \ (D_X \overset{x}{u})\, Y = - {}'\underset{\alpha}{H}(X, \bar{Y}) - \underset{\beta\alpha}{L}(X)\, \overset{x}{u}(Y).$$

The condition that on the framed metric submanifold of a Kähler manifold, $\underset{x}{U}$ is a Killing vector is[3]

$$\overline{\underset{\alpha}{H}X} - \underset{\beta\alpha}{L}(X)\,\underset{x}{U} = 0. \tag{1.15}$$

If the framed metric structure on the submanifold of a Kähler manifold is totally geodesic[3], then

(1.16) a) $(D_X f)\,Y = \overset{x}{u}(Y)\,\underset{\alpha}{H}X$, b) $(D_X {'f})(Y, Z) = 0$,

c) $(D_X \overset{x}{u})(Y) = -\underset{\beta\alpha}{L}(X)\,\overset{x}{u}(Y)$, d) $(D_X \overset{x}{u})(\overline{Y}) = 0$.

2. Submanifold of Kähler Manifold

Theorem 2.1. *The condition for the framed metric structure induced on the submanifold of a Kähler manifold to be normal is*

$$\underset{\alpha}{H}\overline{X} = \overline{\underset{\alpha}{H}X} + \underset{\beta\alpha}{L}(X)\,\underset{x}{U}. \tag{2.1}$$

PROOF. Let

$$M(X, Y) = [f, f] + d\overset{x}{u} \otimes \underset{x}{U}, \tag{2.2}$$

which is equivalent to

$$M(X, Y) = (D_{fX}f)\,Y - (D_{fY}f)\,X + f(D_Y f)\,X - f(D_X f)\,Y + [(D_X \overset{x}{u})(Y) - (D_Y \overset{x}{u})\,X]\,\underset{x}{U}.$$

Substituting the equations (1.14) a) and (1.14) b) in this equation we get

$$M(X, Y) = \overset{x}{u}(Y)\,[\underset{\alpha}{H}\overline{X} - \overline{\underset{\alpha}{H}X} - \underset{\beta\alpha}{L}(X)\,\underset{x}{U}] - \overset{x}{u}(X)\,[\underset{\alpha}{H}\overline{Y} - \overline{\underset{\alpha}{H}Y} - \underset{\beta\alpha}{L}(Y)\,\underset{x}{U}].$$

If the equation (2.1) holds, we have $M(X, Y) = 0$, i.e. $[f, f] - d\,\overset{x}{u} \otimes \underset{x}{U} = 0$. Hence the statement.

Theorem 2.2. *If the induced framed metric structure on the submanifold of a Kähler manifold is normal, then*

(2.3) a) $(D_{\bar X} f)\bar Y = -{}'\underset{\alpha}{H}(X, Y)\underset{x}{U} + \overset{x}{u}(Y)\overset{x}{u}(\underset{\alpha}{H}X)\underset{x}{U},$

b) $(D_{\bar X}\overset{x}{u})(Y) = -{}'\underset{\alpha}{H}(X, Y) + \overset{x}{u}(Y)\,{}'\underset{\alpha}{H}(X, \underset{x}{U}) - \underset{\beta\alpha}{L}(\bar X)\overset{x}{u}(Y),$

c) $(D_{\bar X} f)\bar Y = (D_{\bar X}\overset{x}{u})(Y)\underset{x}{U} + \underset{\beta\alpha}{L}(\bar X)\overset{x}{u}(Y)\underset{x}{U}.$

PROOF. Barring X and Y in equation (1.14a) and using equations (1.5d), (1.6), (1.11) and (2.1), we get

(2.4)
$$(D_{\bar X} f)\bar Y = {}'\underset{\alpha}{H}(X, \bar{\bar Y})\underset{x}{U}.$$

Now, using equations (1.5a), (1.6b) and (1.11) in this equation, we get the equation (2.3a). Barring X in equation (1.14b), we get

(2.5)
$$(D_{\bar X}\overset{x}{u})(Y) = -{}'\underset{\alpha}{H}(\bar X, \bar Y) - \underset{\beta\alpha}{L}(\bar X)\overset{x}{u}(Y).$$

Using the equations (1.5a), (1.6a), (1.11) and (2.1) in equation (2.5), we have the equation (2.3b). Comparing the equations (2.3a) and (2.3b), we get the equation (2.3c).

Theorem 2.3. *If the framed metric submanifold of a Kähler manifold is normal with* $\underset{x}{U}$ *as a killing vector, then*

(2.6) a) $\underset{\alpha}{H}\bar X = 2\overline{\underset{\alpha}{H}X},$ b) $\underset{\alpha}{H}\bar X = 2\underset{\beta\alpha}{L}(X)\underset{x}{U}.$

PROOF. From equations (1.15) and (2.1), we get the equations (2.6).

Theorem 2.4. *If the totally geodesic framed metric submanifold of a Kähler manifold is normal, then*

(2.7) a) ${}'\underset{\alpha}{H}(X, Y)\underset{x}{U} = \overset{x}{u}(Y)\overset{x}{u}(\underset{\alpha}{H}X)\underset{x}{U},$ b) $(D_{\bar X}\overset{x}{u})(Y)\underset{x}{U} = -\underset{\beta\alpha}{L}(\bar X)\overset{x}{u}(Y)\underset{x}{U}.$

PROOF. Barring X and Y in equation (1.16a) and using equation (1.5d), we get $(D_{\bar X} f)\bar Y = 0$. Using this equation in equations (2.3a) and (2.3c), we get the equations (2.7).

3. Submanifold of Nearly Kähler Manifold

Theorem 3.1. *The conditions that* V_n *be a framed metric submanifold of a nearly Kähler manifold* V_m *are*

(3.1) a) $(D_X f)\,Y + (D_Y f)\,X = \overset{x}{u}(Y)\,\underset{\alpha}{H}X + \overset{x}{u}(X)\,\underset{\alpha}{H}Y - 2\,'\underset{\alpha}{H}(X, Y)\,\underset{x}{U},$

b) $(D_X \overset{x}{u})(Y) + (D_Y \overset{x}{u})(X) = -\,'\underset{\alpha}{H}(fX, Y) - '\underset{\alpha}{H}(X, fY)\,\underset{\beta\alpha}{L}(Y)\,\overset{x}{u}(X) - \underset{\beta\alpha}{L}(X)\,\overset{x}{u}(Y).$

PROOF. Let V_m be a nearly Kähler manifold, then in consequence of the equation (1.4), we get

(3.2) $$(E_{BX}F)\,BY + (E_{BY}F)\,BX = 0.$$

Substituting from the equations (1.19), (1.10), and (1.13) in equation (3.2) and separating tangential and normal parts, we have the equation (3.1).

Theorem 3.2. *Let V_n be a framed metric submanifold of a nearly Kähler manifold V_m, then we have*

(3.3)

a) $(D_X\,'f)(Y, Z) + (D_Y\,'f)(X, Z) = \overset{x}{u}(Y)\,'\underset{\alpha}{H}(X, Z) + \overset{x}{u}(X)\,'\underset{\alpha}{H}(Y, Z) - 2\,'\underset{\alpha}{H}(X, Z)\,\overset{x}{u}(Z),$

b) $(D_{\bar X} \overset{x}{u})(\bar Y) + (D_{\bar Y} \overset{x}{u})(\bar X) = '\underset{\alpha}{H}(\bar X, Y) + '\underset{\alpha}{H}(X, \bar Y) - \overset{x}{u}(Y)\,\overset{x}{u}(\underset{\alpha}{H}\bar X) - \overset{x}{u}(X)\,\overset{x}{u}(\underset{\alpha}{H}\bar Y),$

c) $(D_{\bar X} f)\,\bar Y + (D_{\bar Y} f)\,\bar X = 2\,'f(\underset{\alpha}{H}\bar X, Y)\,\underset{x}{U},$

d) $(D_{\bar X} \overset{x}{u}(\bar Y) + (D_{\bar Y} \overset{x}{u})(\bar X) = 'f(\underset{\alpha}{H}\bar X, \bar Y) + 'f(\underset{\alpha}{H}\bar Y, \bar X),$

e) $'f(\underset{\alpha}{H}\bar X, \bar Y) + 'f(\underset{\alpha}{H}\bar Y, \bar X) + \overset{x}{u}(Y)\,\overset{x}{u}(\underset{\alpha}{H}\bar X) + \overset{x}{u}(X)\,\overset{x}{u}(\underset{\alpha}{H}\bar Y) = '\underset{\alpha}{H}(\bar X, Y) + '\underset{\alpha}{H}(X, \bar Y).$

PROOF. From equation (3.1a), we have

$$g((D_X f)\,Y, Z) + g((D_Y f)\,X, Z) = \overset{x}{u}(Y)\,g(\underset{\alpha}{H}X, Z) + \overset{x}{u}(X)\,g(\underset{\alpha}{H}Y, Z) - 2\,'\underset{\alpha}{H}(X, Y)\,g(\underset{x}{U}, Z).$$

(3.4)

Using equations (1.7), (1.6b) and (1.11) in equation (2.11), we get the equation (3.3a). Barring X and Y in equation (3.1b) and using equations (1.5a), (1.5d), (1.6b) and (1.11) in the resulting equation, we have the equation (3.3b). the remaining part of the theorem follows similarly.

Theorem 3.3. *If the induced framed metric structure on the submanifold of a nearly Kähler manifold is totally geodesic, then*

(3.5) a) $$(D_X f)\,Y + (D_Y f)\,X = \overset{x}{u}(Y)\,\underset{\alpha}{H}\,X + \overset{x}{u}(X)\,\underset{\alpha}{H}\,Y,$$

b) $$(D_X{}'f)\,(Y, Z) + (D_Y{}'f)\,(X, Z) = 0,$$

c) $$(D_X\,\overset{x}{u}\,(Y) + (D_Y\,\overset{x}{u})\,(X) = -\underset{\beta\alpha}{L}(Y)\,\overset{x}{u}(X) - \underset{\beta\alpha}{L}(X)\,\overset{x}{u}(Y),$$

d) $$(D_{\bar{X}}\,\overset{x}{u})\,(\bar{Y}) + (D_{\bar{Y}}\,\overset{x}{u})\,(\bar{X}) = 0.$$

PROOF. Using equation (1.12) in equation (3.1a), we get the equation (3.5a). From equation (3.5a), we have

(3.6) $$g\,((D_X f)\,Y, Z) + g\,((D_Y f)\,X, Z) = \overset{x}{u}(Y)\,g\,(\underset{\alpha}{H}\,X, Z) + \overset{x}{u}(X)\,g\,(\underset{\alpha}{H}\,Y, Z).$$

Using equations (1.7), (1.11) and (1.12) in equation (3.6), we get the equation (3.5b). From equations (1.12) and (3.1b), we have the equation (3.5c). Barring X and Y in equation (3.5c), we get the equation (3.5d).

Theorem 3.4. *The condition that on the framed metric submanifold of a nearly Kähler manifold,* $\underset{x}{U}$ *is a killing vector is*

(3.7) a) $$\overline{\underset{\alpha}{H}\,X} - \underset{\beta\alpha}{L}(X)\,\underset{x}{U} = 0, \quad \text{or b)} \quad \underset{\alpha}{H}\,\bar{X} + \underset{\beta\alpha}{L}(X)\,\underset{x}{U} = 0.$$

PROOF. Using equations (1.6) and (1.11) in equation (3.1b), we get

(3.8) $$(D_X\,\overset{x}{u})\,(Y) + (D_Y\,\overset{x}{u})\,(X) = g\,(\overline{\underset{\alpha}{H}\,Y} - \underset{\beta\alpha}{L}(Y)\,\underset{x}{U}, X) + g\,(\overline{\underset{\alpha}{H}\,X} - \underset{\beta\alpha}{L}(X)\,\underset{x}{U}, Y).$$

Now, if $\overline{\underset{\alpha}{H}\,X} - \underset{\beta\alpha}{L}(X)\,\underset{x}{U} = 0$, then from equation (3.8), we have

$$(D_X\,\overset{x}{u})\,(Y) + (D_Y\,\overset{x}{u})\,(X) = 0, \quad \text{or} \quad (D_X\,\overset{x}{u})\,(X) = 0.$$

Hence we have equation (3.7a). Similarly, from equations (1.6), (1.11) and (3.1b), we have the equation (3.7b).

Theorem 3.5. *If $\underset{x}{U}$ is a killing vector on a framed metric submanifold of a nearly Kähler manifold, then*

$$(3.9)\qquad \text{a)}\quad (D_X f)\,\bar{Y} + (D_{\bar{Y}} f)\, X = \overset{x}{u}(X)\,\underset{\alpha}{H}\,\bar{Y} + 2\,\underset{\beta\alpha}{L}(X)\,\overset{x}{u}(Y)\,\underset{x}{U},$$

$$\text{b)}\quad (D_{\bar{X}} f)\,\bar{Y} + (D_{\bar{Y}} f)\,\bar{X} = 0,$$

$$\text{c)}\quad (D_X \overset{x}{u})(Y) + (D_Y \overset{x}{u})(X) = \underset{\beta\alpha}{L}(X)\,\overset{x}{u}(Y) - \underset{\beta\alpha}{L}(Y)\,\overset{x}{u}(X),$$

$$\text{d)}\quad (D_{\bar{X}} \overset{x}{u})(\bar{Y}) + (D_{\bar{Y}} \overset{x}{u})(\bar{X}) = 0.$$

PROOF. Barring Y in equation (3.1a) and using equation (1.5d), we get

$$(3.10)\qquad (D_X f)\,\bar{Y} + (D_{\bar{Y}} f)\, X = \overset{x}{u}(X)\,\underset{\alpha}{H}\,\bar{Y} - 2\,{}'\underset{\alpha}{H}(X, \bar{Y})\,\underset{x}{U}.$$

Using equations (1.6), (1.11) and (3.7a) in equation (3.10), we get the equation (3.9a). Barring X and Y in equation (3.9a) and using equation (1.5d) in the resulting equation, we have the equation (3.9b). Using equations (1.6), (1.11) and (3.7) in equation (3.1b), we get the equation (3.9c). Barring X and Y in equation (3.9c) and using equation (1.5d), we get the equation (3.9d).

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