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On Projective Transformations Between Special Finsler Spaces

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Abstract. Amur and Desai [1]. studied the projective transformation between Riemannian spaces. Adati and Miyazawa[2] discussed such transformations between recurrent, symmetric, projective recurrent and projective symmetric spaces in detail. Few results of these authors were extended to recurrent Finsler spaces by Sinha and Faruqi [3]. The aim of the present paper is to extend the results of Adati and Miyazawa to Finsler spaces and to generalize the results of Sinha and Faruqi. The notation for Berwald's covariant differentiation differs from that of Rund[4] and Matsumoto[5]

1. Preliminaries

Let F_n be an n-dimensional Finsler space equipped with a metric function* F satisfying the required conditions[4], the corresponding symmetric metric tensor g and the Berwald's connection G. Let $\overline{F}_n$ be another Finsler space with metric function $\overline{F}$, metric tensor $\overline{g}$ and the Berwald's connection $\overline{G}$ such that $\overline{F}_n$ is obtained by a projective transformation of F_n, i.e., the F_n and the $\overline{F}_n$ are in geodesic correspondence. The projective transformation is characterized by the relation of Berwald's connection coefficients of F_n and $\overline{F}_n$ such that

$$\overline{G}^i_{jk} = G^i_{jk} - p_j \delta^i_k - p_k \delta^i_j - p_{jk} \dot{x}^i, \tag{1.1}$$

where $p_j = \dot{\partial}_j p$, $p_{jk} = \dot{\partial}_k p_j$ and $\dot{\partial}_j = \partial / \partial \dot{x}^j$. The function $p(x, \dot{x})$ is arbitrary and positively homogeneous of degree one in $\dot{x}^i$. Because of its homogeneity in $\dot{x}^i$, it satisfies

$$\dot{x}^k p_{jk} = \dot{x}^k \dot{\partial}_k p_j = 0, \; p_j \dot{x}^j = p. \tag{1.2}$$

The covariant derivative of an arbitrary tensor T^i_j for the connection G is given by

$$\mathcal{B}_m T^i_j = \partial_m T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_{ms} \dot{x}^s + T^r_j G^i_{rm} - T^i_r G^r_{jm}, \; \partial_m = \partial / \partial x^m, \tag{1.3}$$

Berwald constructed the tensor H^i_{jk} and the curvature tensor H^i_{jkh} from deviation tensor H^i_j

* Unless Otherwise stated, all the geometric objects are supposed to be functions of the line-elements $(x^j, \dot{x}^j)$. The indices $i, j, k \cdots$ take positive integer values from 1 to n.

as follows:

(1.4) a) $H^i_{jk}=\frac{1}{3}\left(\dot\partial_j H^i_k-\dot\partial_k H^i_j\right)$ b) $H^i_{jkh}=\dot\partial_j H^i_{kh}$

The curvature tensor H^i_{jkh}, defined above, is skewsymmetric in last two lower indices k and h and is positively homogeneous of degree zero in $\dot x^i$'s. The tensors H^i_{jk} and H^i_j satisfy

(1.5) a) $H^i_{jkh}\dot x^j=H^i_{kh}$ b) $H^i_{kh}\dot x^k=H^i_h$ c) $H^i_h\dot x^h=0.$

The projective curvature tensor W^i_{jkh} and the tensors W^i_{jk} and W^i_j satisfy the following:

(1.6) a) $W^i_{jk}=\frac{1}{3}\left(\dot\partial_j W^i_k-\dot\partial_k W^i_j\right)$ b) $W^i_{jkh}=\dot\partial_j W^i_{kh}$, c) $W^i_{jkh}\dot x^j=W^i_{kh}$,

d) $W^i_{jk}\dot x^j=W^i_k$, e) $W^i_j\dot x^j=0$, f) $W^i_i=0$, g) $W^i_{ik}=-W^i_{ki}=0$,

h) $W^i_{ikh}=W^i_{jih}=W^i_{jki}=0.$

The projective curvature tensor W^i_{jkh} is skew- symmetric in last two lower indices and is positively homogeneous of degree zero in $\dot x^i$'s. A Finsler space is said to be of scalar curvature if it is isotropic at each point[4]

2. Two Lemmas

Let us assume that there exists a projective transformation from a non-flat Finsler space F_n to a non-flat Finsler space $\overline{F}_n$. This means that the connection coefficients of F_n and $\overline{F}_n$ are related by (1.1).

The covariant derivative of the projective deviation tensor $\overline{W}^i_k$ of $\overline{F}_n$ is given by

$$\overline{\mathcal{B}}_m\overline{W}^i_k=\partial_m\overline{W}^i_k-\left(\dot\partial_r\overline{W}^i_k\right)\overline{G}^r_{sm}\dot x^s+\overline{W}^r_k\overline{G}^i_{rm}-\overline{W}^i_r\overline{G}^r_{km},$$

which, in view of (1.1) and the invariance of the projective deviation tensor W^i_k under a projective transformation, gives

(2.1) $$\overline{\mathcal{B}}_m\overline{W}^i_k=\left(\partial_m W^i_k-\left(\dot\partial_r W^i_k\right)G^r_{sm}\dot x^s+W^r_k G^i_{rm}-W^i_r G^r_{km}\right)$$

$$+p\left(\dot\partial_m W^i_k\right)+2p_m W^i_k-\delta^i_m p_r W^r_k-p_{rm}W^r_k\dot x^i+p_k W^i_m.$$

Using the formula (1.3) in (2.1), we have

$$(2.2)\qquad \bar{\mathcal{B}}_m \bar{W}^i_k = \mathcal{B}_m W^i_k + p\,\dot{\partial}_m W^i_k + 2p_m W^i_k - \delta^i_m p_r W^r_k - p_{rm} W^r_k \dot{x}^i + p_k W^i_m .$$

Now we propose the following:

Lemma 2.1. *If there exists a projective transformation from a non-flat Finsler space F_n to another non-flat Finsler space $\bar{F}_n$, then the invariance of the tensor $\mathcal{B}_m W^i_k$ implies atleast one of the following*

(i) *the transformation is affine,*

(ii) *both the spaces are of scalar curvature.*

Proof. If the tensor $\mathcal{B}_m W^i_k$ is invariant under a projective transformation, i.e. $\bar{\mathcal{B}}_m \bar{W}^i_k = \mathcal{B}_m W^i_k$, the equation (2.2) reduces to

$$(2.3)\qquad p\,\dot{\partial}_m W^i_k + 2p_m W^i_k - \delta^i_m p_r W^r_k - p_{rm} W^r_k \dot{x}^i + p_k W^i_m = 0 .$$

Transvecting (2.3) by $\dot{x}^m$, and using the Euler's theorem for homogeneous functions and the equation (1.2), we have

$$(2.4)\qquad 4p\, W^i_k - \dot{x}^i p_r W^r_k = 0.$$

Transvecting (2.4) by p_i and using $p_i \dot{x}^i = \dot{x}^i \dot{\partial}_i p = p$, we find $p\, p_r W^r_k = 0$; which implies atleast one of the following

$$(2.5)\qquad \text{a) } p = 0, \qquad \text{b) } p_r W^r_k = 0.$$

If (2.5a) holds, the transformation is affine. If (2.5a) does not hold, we must have (2.5b). Using (2.5b) in (2.4), we get $p\, W^i_k = 0$; which implies $W^i_k = 0$. Since the space F_n is non-flat and the condition $W^i_k = 0$ implies that the space is of scalar curvature (Szabo[6]; Matsumoto[7]; Pandey[8]), the space F_n is of scalar curvature. In view of Matsumoto's theorem[7], the space $\bar{F}_n$ is also of scalar curvature. This completes the proof.

Lemma 2.2. *If there exists a projective transformation from a non-flat Finsler space F_n to another non-flat Finsler space $\bar{F}_n$ then the condition*

$$(2.6)\qquad \bar{\mathcal{B}}_m \bar{W}^i_k - \mathcal{B}_m W^i_k = L_m W^i_k,$$

where L_m is a covariant vector field, implies atleast one of the following:

(i) *both the spaces are of scalar curvature,*

(ii) *the condition* $L_m \dot{x}^m = 4p$ *holds.*

Proof. Let us assume that there exists a projective transformation from a non-flat Finsler space F_n to another non-flat Finsler space $\bar{F}_n$ and the condition (2.6) holds good. In view of (2.6), the equation (2.2) may be written as

$$L_m W_k^j = p\,\dot{\partial}_m W_k^j + 2p_m W_k^j - \delta_m^i p_r W_k^r - p_{rm} W_k^r \dot{x}^i + p_k W_m^j \tag{2.8}$$

Contracting the indices i and m in (2.8) and using (1.2), we have

$$L_r W_k^r = p\,\dot{\partial}_r W_k^r - (n-2)\, p_r W_k^r. \tag{2.9}$$

Contracting the indices i and j in (1.6a) and using (1.6f) and (1.6g), we get $\dot{\partial}_r W_k^r = 0$. Hence we may write (2.9) as

$$\{L_r + (n-2)\, p_r\} W_k^r = 0. \tag{2.10}$$

Transvecting (2.8) by $\dot{x}^m$ and using $\dot{x}^m \dot{\partial}_m W_k^j = 2W_k^j$ and (1.2), we get

$$\left(L_m \dot{x}^m - 4\,p\right) W_k^j = -\dot{x}^i p_r W_k^r. \tag{2.11}$$

Transvecting (2.11) by p_i and using $p_i \dot{x}^i = p$, we have

$$\left(L_m \dot{x}^m - 3p\right) p_r W_k^r = 0. \tag{2.12}$$

The condition (2.12) implies atleast one of the following

$$\text{a) } p_r W_k^r = 0, \quad \text{b) } L_r \dot{x}^r - 3p = 0. \tag{2.13}$$

We claim that (2.13a) holds good. If not, suppose $p_r W_k^r \neq 0$. Then we must have (2.13b). From (2.11) and (2.13b) we have

$$p\, W_k^j = \dot{x}^i p_r W_k^r. \tag{2.14}$$

Transvecting (2.14) by L_j, we have

$$p\, L_r W_k^r = L_s \dot{x}^s p_r W_k^r. \tag{2.15}$$

Multiplying (2.10) by p and using (2.15), we find

$$L_r \dot{x}^r + (n-2)\, p = 0, \tag{2.16}$$

since $p_r W^r_k \neq 0$. Using (2.13b) in (2.16) we get $(n+1)p = 0$. Being the dimension of the space, n can not be -1. Therefore we have $p = 0$; which implies $p_r = \dot{\partial}_r p = 0$. This gives $p_r W^r_k = 0$, a contradiction. Therefore, our supposition is wrong. Thus, we must have (2.13a). In view of (2.13a), (2.11) reduces to $(L_m \dot{x}^m - 4p) W^i_k = 0$. This implies atleast one of the following

(2.17) a) $L_m \dot{x}^m = 4p$, b) $W^i_k = 0$.

The condition (2.17b) implies that the space F_n and $\overline{F}_n$ are of scalar curvature.

3. Projective Recurrent Spaces

In this section we discuss the projective transformation (1.1) from a projective recurrent Finsler space F_n to another projective recurrent Finsler space $\overline{F}_n$ characterized by the conditions[9]

(3.1) $$\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh}, \quad W^i_{jkh} \neq 0$$

and

(3.2) $$\overline{\mathcal{B}}_m \overline{W}^i_{jkh} = \overline{\lambda}_m \overline{W}^i_{jkh}, \quad \overline{W}^i_{jkh} \neq 0$$

respectively. The covariant vectors λ_m and $\overline{\lambda}_m$ are said to be recurrence vectors. Projective recurrent Finsler spaces are actually non-flat spaces because the vanishing of curvature tensor implies the vanishing of W^i_{jkh} and for a projective recurrent space $W^i_{jkh} \neq 0$. Transvecting (3.1) and (3.2) by $\dot{x}^j \dot{x}^k$ and using (1.6c) and (1.6d), we have

(3.3) $$\overline{\mathcal{B}}_m \overline{W}^i_h = \overline{\lambda}_m \overline{W}^i_h, \quad \overline{W}^i_h \neq 0$$

and

(3.4) $$\mathcal{B}_m W^i_h = \lambda_m W^i_h, \quad W^i_h \neq 0.$$

Since the projective deviation tensor W^i_k is invariant with respect to a projective transformation, .e., $\overline{W}^i_k = W^i_k$, we have

(3.5) $$\overline{\mathcal{B}}_m \overline{W}^i_k - \mathcal{B}_m W^i_k = \left(\overline{\lambda}_m - \lambda_m\right) W^i_k.$$

If the recurrence vectors are same, we have

$$\bar{\mathcal{B}}_m \bar{W}_k^i - \mathcal{B}_m W_k^i = 0,$$

i.e. the tensor $\mathcal{B}_m W_k^i$ is invariant. Hence, in view of Lemma 2.1, we have atleast one of the following

(i) F_n and $\bar{F}_n$ are of scalar curvature,

(ii) the transformation is affine.

Since $\bar{F}_n$ and F_n are projective recurrent spaces, the projective deviation tensors $\bar{W}_k^i$ and W_k^i can not vanish identically. Therefore F_n and $\bar{F}_n$ can not be of scalar curvature. Hence the (ii) holds. This leads to:

Theorem 3.1. *The projective transformation from a projective recurrent space F_n to another projective recurrent space $\bar{F}_n$ with same recurrence vectors is necessarily affine.*

If the recurrence vectors $\bar{\lambda}_m$ and λ_m are not equal, *i.e.* $\bar{\lambda}_m \neq \lambda_m$; suppose $\bar{\lambda}_m - \lambda_m = L_m$. Then we have $\bar{\mathcal{B}}_m \bar{W}_k^i - \mathcal{B}_m W_k^i = L_m W_k^i$. Since $\bar{W}_k^i = W_k^1 \neq 0$, $\bar{F}_n$ and F_n are not of scalar curvature. Therefore, in view of Lemma 2.2, we have $L_m \dot{x}^m = 4p$ i.e. $\left(\bar{\lambda}_m - \lambda_m\right)\dot{x}^m = 4p$. Thus, we have :

Theorem 3.2. *If a projective recurrent Finsler space F_n with recurrence vector λ_m is transformed to a projective recurrent Finsler space $\bar{F}_n$ with recurrence vector $\bar{\lambda}_m$ by the projective transformation* (1.1), *then we have* $\left(\bar{\lambda}_m - \lambda_m\right)\dot{x}^m = 4p$.

4. Recurrent Spaces

This section is devoted to the study of the projective transformation (1.1) from a recurrent Finsler space F_n to another recurrent Finsler space $\bar{F}_n$ characterized by

$$\mathcal{B}_m H_{jkh}^i = \lambda_m H_{jkh}^i, \quad H_{jkh}^i \neq 0 \tag{4.1}$$

and

$$\bar{\mathcal{B}}_m \bar{H}_{jkh}^i = \bar{\lambda}_m \bar{H}_{jkh}^i, \quad \bar{H}_{jkh}^i \neq 0 \tag{4.2}$$

respectively. The covariant vectors λ_m and $\bar{\lambda}_m$ are recurrence vectors of F_n and $\bar{F}_n$ respectively. It has been observed that the projective deviation tensor W_k^i of a recurrent space is recurrent with same recurrence[9] vector. In view of this result, we find

$$\text{a) } \mathcal{B}_m W_h^i = \lambda_m W_h^i, \qquad \text{b) } \bar{\mathcal{B}}_m \bar{W}_h^i = \bar{\lambda}_m \bar{W}_h^i . \tag{4.3}$$

It has been proved by Pandey[10] that a recurrent Finsler space of scalar curvature does not exist. We also know that a Finsler space is of zero projective curvature if and only if it is of scalar curvature[6-8]. From these two theorems we may conclude that a recurrent Finsler space can not have zero projective curvature, and hence zero projective deviation tensor. Thus, $W^i_k \neq 0$ and $\overline{W}^i_k \neq 0$. Proceeding in the similar way of proof of Theorems 3.1 and 3.2, we may prove:

Theorem 4.1. *The projective transformation from a recurrent Finsler space F_n to another recurrent Finsler space $\overline{F}_n$ with same recurrence vectors is necessarily affine.*

Theorem 4.2. *If a recurrent Finsler space F_n with recurrence vector λ_m is transformed to a recurrent space $\overline{F}_n$ with recurrence vector $\overline{\lambda}_m$ by the projective transformation* (1.1), *we have* $\overline{\lambda}_m - \lambda_m = 4p_m$.

5. Projective Symmetric Spaces

In this section we will study the projective transformation from a non-flat projective symmetric space F_n to another non- flat projective symmetric space $\overline{F}_n$ characterized by

$$(5.1) \qquad \text{a) } \mathcal{B}_m W^i_{jkh} = 0, \quad \text{b) } \overline{\mathcal{B}}_m \overline{W}^i_{jkh} = 0,$$

respectively[11]. Transvection of (5.1a) and (5.1b) by $\dot{x}^j \dot{x}^k$ gives

$$(5.2) \qquad \text{a) } \mathcal{B}_m W^i_h = 0, \quad \text{b) } \overline{B}_m \overline{W}^i_h = 0,$$

which shows that $\overline{\mathcal{B}}_m \overline{W}^i_h = \mathcal{B}_m W^i_h$. Therefore, in view of Lemma 2.1, we may conclude:

Theorem 5.1. *If there exists a projective transformation from a non-flat projective symmetric space F_n to another non-flat projective symmetric space $\overline{F}_n$, then we have atleast one of the following conditions:*

(i) *the transformation is affine,*

(ii) *both spaces F_n and $\overline{F}_n$ are of scalar curvature.*

6. Symmetric Spaces

Let us discuss the projective transformation from a non-flat symmetric space F_n to another non-flat symmetric space $\overline{F}_n$ characterized by

$$(6.1) \qquad \text{a) } \mathcal{B}_m H^i_{jkh} = 0, \quad \text{b) } \overline{\mathcal{B}}_m \overline{H}^i_{jkh} = 0,$$

respectively. The projective deviation tensor W^i_h of a symmetric Finsler space is a covariant constant[12]. Hence we have $\overline{\mathcal{B}}_m \overline{W}^i_h = \mathcal{B}_m W^i_h = 0$; which, in view of Lemma 2.1

implies atleast one of the following:

(i) the transformation is affine,

(ii) both spaces are of scalar curvature.

But according to Pandey[10] a symmetric Finsler space F_n $(n>2)$ of scalar curvature is a Riemannian space of constant Riemannian curvature provided it is non-flat. Hence we conclude:

Theorem 6.1. *Let two non-flat symmetric spaces F_n $(n>2)$ and $\overline{F}_n$ $(n>2)$ be related by a projective transformation. Then we have atleast one of the following:*

(i) *the transformation is affine,*

(ii) *Finsler spaces F_n and $\overline{F}_n$ are Riemannian spaces of constant Riemannian curvature.*

7. Different Type of Spaces

In this section we discuss the projective transformation from a non-flat Finsler space F_n to another non-flat Finsler space $\overline{F}_n$ such that

(i) F_n is recurrent and $\overline{F}_n$ is symmetric.

(ii) F_n is projective recurrent and $\overline{F}_n$ is projective symmetric.

(iii) F_n is projective recurrent and $\overline{F}_n$ is recurrent,

(iv) F_n is recurrent and $\overline{F}_n$ is projective symmetric.

(v) F_n is symmetric and $\overline{F}_n$ is projective symmetric.

Case (i). Suppose the Finsler space F_n is recurrent and $\overline{F}_n$ is symmetric. Since the projective deviation tensor W_k^j is recurrent in a recurrent space and covariant constant in a symmetric space, we have

$$\text{(7.1)} \qquad \text{a) } \mathcal{B}_m W_k^j = \lambda_m W_k^j, \qquad \text{b) } \overline{\mathcal{B}}_m \overline{W}_k^j = 0,$$

λ_m being recurrence vector. Thus

$$\text{(7.2)} \qquad \overline{\mathcal{B}}_m \overline{W}_k^j - \mathcal{B}_m W_k^j = -\lambda_m W_k^j.$$

In view of Lemma 2.2, we have atleast one of the following

(i) F_n and $\overline{F}_n$ both are of scalar curvature,

(ii) $-\lambda_m = 4p_m$.

Pandey[10] proved that a recurrent Finsler space of scalar curvature does not exist. In view of this result (i) is not possible. Hence, we may conclude :

Theorem 7.1. *Let a recurrent Finsler space F_n be related with a symmetric Finsler*

space $\overline{F}_n$ *by a projective transformation. Then the recurrence vector* λ_m *satisfies* $\lambda_m = -4p_m$.

Case (ii). Suppose F_n is projective recurrent and $\overline{F}_n$ projective symmetric. Then we have $\mathcal{B}_m W^j_k = \lambda_m W^j_k$ and $\overline{\mathcal{B}}_m \overline{W}^j_k = 0$. This means $\overline{\mathcal{B}}_m \overline{W}^j_k - \mathcal{B}_m W^j_k = -\lambda_m W^j_k$. Therefore, in view of Lemma 2.2, we have atleast one of the following:

(i) both spaces are of scalar curvature,

(ii) $-\lambda_m = 4p_m$.

But for a projective recurrent space $W^j_k \neq 0$. Thus condition (i) does not hold. Hence, we may conclude:

Theorem 7.2. *Let a projective recurrent space* F_n *be related with a projective symmetric Finsler space* $\overline{F}_n$ *by a projective transformation. Then the recurrence vector* λ_m *satisfies* $\lambda_m = -4p_m$.

Case (iii). Suppose the Finsler space F_n is projective recurrent and $\overline{F}_n$ is recurrent. Since the projective deviation tensor is recurrent in a projective recurrent space as well as in a recurrent space we have

a) $\mathcal{B}_m W^j_k = \lambda_m W^j_k$, $W^j_k \neq 0$

b) $\overline{\mathcal{B}}_m \overline{W}^j_k = \overline{\lambda}_m \overline{W}^j_k$, $\overline{W}^j_k \neq 0$,

where λ_m and $\overline{\lambda}_m$ are recurrence vectors. Thus

$$\overline{\mathcal{B}}_m \overline{W}^j_k - \mathcal{B}_m W^j_k = \overline{\lambda}_m \overline{W}^j_k - \lambda_m W^j_k. \tag{7.4}$$

since the projective deviation tensor is a invariant under a projective transformation i.e., $\overline{W}^j_k = W^j_k$,

$$\overline{\mathcal{B}}_m \overline{W}^j_k - \mathcal{B}_m W^j_k = \left(\overline{\lambda}_m - \lambda_m\right) W^j_k. \tag{7.5}$$

If $\overline{\lambda}_m \neq \lambda_m$, in view of Lemma 2.2, we have atleast one of the following

(i) F_n and $\overline{F}_n$ both are of scalar curvature,

(ii) $\overline{\lambda}_m - \lambda_m = 4p_m$.

But recurrent as well as projective recurrent spaces can not be of scalar curvature. Therefore, we conclude:

Theorem 7.3. *Let a projective recurrent Finsler space* F_n *with recurrence vector* λ_m *be transformed to a recurrent Finsler space* $\overline{F}_n$ *with recurrence vector* $\overline{\lambda}_m$ *by a projective transformation. If* $\overline{\lambda}_m \neq \lambda_m$, *we have* $\overline{\lambda}_m - \lambda_m = 4p_m$.

If $\bar{\lambda}_m = \lambda_m$. Then $\bar{\mathcal{B}}_m \bar{W}_k^j - \mathcal{B}_m W_k^j = 0$. Since a recurrent or a projective recurrent Finsler space can not be of scalar curvature, we conclude that the transformation is necessarily affine. Hence we have :

Theorem 7.4. *The projective transformation from a projective recurrent Finsler space F_n to a recurrent Finsler space $\bar{F}_n$ with same recurrence vectors is necessarily affine.*

Case (iv). Suppose F_n is recurrent and $\bar{F}_n$ is projective symmetric. Since the projective deviation tensor W_k^j is recurrent in a recurrent space and it is a covariant constant in a projective symmetric space, we have $\bar{\mathcal{B}}_m \bar{W}_k^j - \mathcal{B}_m W_k^j = -\lambda_m W_k^j$. Since a recurrent space can not be of scalar curvature, the Lemma 2.2 leads to

Theorem 7.5. *If a recurrent Finsler space F_n with recurrence vector λ_m is transformed to a projective symmetric space $\bar{F}_n$ by a projective transformation, the recurrence vector satisfies $\lambda_m = -4p_m$.*

Case (v). In this case F_n is symmetric and $\bar{F}_n$ is projective symmetric. Since the projective deviation tensor is covariant constant in both the spaces, $\bar{\mathcal{B}}_m \bar{W}_k^j - \mathcal{B}_m W_k^j = 0$. In view of Lemma 2.1, we have atleast one of the following

(i) the transformation is affine,

(ii) both the spaces are of scalar curvature.

Pandey[10] proved that a symmetric Finsler space of scalar curvature is a Riemannian space of constant Riemannian curvature. In view of this result, we may conclude:

Theorem 7.6. *If a symmetric Finsler space F_n is transformed to a projective symmetric space $\bar{F}_n$ by a projective transformation, we have atleast one of the following*

(i) *the transformation is affine,*

(ii) *F_n is a Riemannian space of constant Riemannian curvature and $\bar{F}_n$ is of scalar curvature.*

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