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Spinning Particles in Reissner-nordstrom Space-time

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Abstract. We study the pseudo-classical spinning particles in a spherically symmetric, static, exterior field of a charged distribution of mass described by the Reissner-Nordstrom (RN) space-time. We investigate the generalized equations for spinning space and describe the constants of motion from their solutions. Considering the motion in a plane we obtain exact solutions.

1. Introduction

The spinning point particles are studied in pseudo -classical mechanics which has been developed by supersymmetric extension of the spinless relativistic point particles[1-5]. The spin degrees of freedom are characterized by anti-commuting Grassmann variables. These variables do not admit any direct classical interpretation. The Lagrangian of these models can be made to have a natural interpretation for the path-integral description of the quantum dynamics. The pseudo-classical equations of motion attain physical meaning when averaged over, inside the functional integral[1-10]. The quantum mechanical expectation values of the Grassmann variables for the spinning particles can be obtained by replacing some appropriate combination of them by real numbers[1, 11-15].

The study of the geometry of graded pseudo-manifolds with real number and Grassmann co-ordinates draws special attention. This generalized Riemannian geometry based on spin co-ordinates has a wide mathematical concern. Rietdijk and Van Holten[7-9,13] investigated the general relations between symmetries of graded pseudo-manifolds and motion for spinning point particles elaborately. The anti-commuting Grassmann variables modifies the Killing equations and thereby generalize the Killing vectors. This generalization due to spin is explained by introducing Killing scalars. The constants of motion for the spinning particle are constructed in this respect.

In this paper we investigate the motion of pseudo-classical point particle in the space-time geometry outside of a charged distribution of mass described by the RN metric. The RN metric draws our attention because it is the only asymptotically flat, static, spherically symmetric solution of the coupled Einstein Maxwell equations.

We arrange this paper in the following way. In sec. 2 we summarize the equations relevant to the motion of spinning point particles in curved space-time. In sec. 3 we find the vectors and Killing scalars corresponding to the RN metric. Applying the equations of the previous section we derive the constants of motion. In sec. 4 we consider the motion in a plane and analyze the bound state solutions. The precession of the perihelion is also obtained. In sec. 5 summarizing the results we draw our conclusions. We use the natural units: $\hbar = c = 1$ throughout the paper.

2. Motion in Spinning Space

The geodesic for spinning space can be obtained from the action:

$$S = m \int_1^2 d\tau \left(\frac{1}{2} g_{\mu\nu}(x)\, \dot{x}^{\mu} \dot{x}^{\nu} + \frac{1}{2} i\, g_{\mu\nu}(x)\, \Psi^{\mu} \frac{D\Psi^{\nu}}{D\tau} \right). \tag{1}$$

The constant m has the dimension of mass. The overdot, here and in the following, denotes a derivative with respect to proper time $\frac{d}{d\tau}$. The covariant derivative of Grassmann co-ordinates Ψ^{μ} is

$$\frac{D\Psi^{\mu}}{D\tau} = \dot{\Psi}^{\mu} + \dot{x}^{\lambda} \Gamma^{\mu}_{\lambda\nu} \Psi^{\nu} \tag{2}$$

With this action the space-time interval along the curve is

$$ds^2 = g_{\mu\nu}(x)\, dx^{\mu} dx^{\nu} = -d\tau^2 \tag{3}$$

where $d\tau$ is the corresponding proper time interval. The last equality holds only in the absence of external forces[15].

The trajectories obtained by making the action stationary under variations vanishing at the end points are

$$\frac{D^2 x^{\mu}}{D\tau^2} = \ddot{x}^{\mu} + \Gamma^{\mu}_{\lambda\nu} \dot{x}^{\lambda} \dot{x}^{\nu} = \frac{1}{2i} \Psi^{x} \Psi^{\lambda} R^{\mu}_{x\lambda\nu} \dot{x}^{\nu}, \tag{4}$$

$$\frac{D\Psi^{\mu}}{D\tau} = 0. \tag{5}$$

The anti-symmetric tensor $S^{\mu\nu} = -i\Psi^{\mu}\Psi^{\nu}$ describes the relativistic-spin of the particle[1, 5, 12, 15]. Eqn. (4) implies the existence of a spin dependent gravitational force[11-15]

$$\frac{D^2 x^{\mu}}{D\tau^2} = \frac{1}{2} S^{k\lambda} R^{\mu}_{k\lambda\nu} \dot{x}^{\nu} \tag{6}$$

whereas the eqn. (5) asserts that the spin is covariantly constant: $\frac{DS^{\mu\nu}}{D\tau} = 0$. The space-like components S^{ij} and the time-like components S^{io} of the spin tensor respectively represent the particle's magnetic dipole moment and the electric dipole moment. In the rest frame, the spin components S^{io} for physical fermions (leptons and quarks) should vanish. We thus have the covariant constraint

$$g_{\nu\lambda}(x)\, S^{\mu\nu}\, \dot{x}^{\lambda} = 0 \tag{7}$$

which is equivalent to

$$g_{\mu\upsilon}(x)\, \dot{x}^{\mu}\, \Psi^{\upsilon} = 0 \tag{8}$$

in Grassmann co-ordinates.

There are, in general, two classes of conserved quantities:

(i) The generic constants of motion which exist in any theory;

(ii) The non-generic kind which results from the specific form of the metric $g_{\mu\upsilon}(x)$.

For spinning-particle model defined by the action (1) we get four generic constants of motion[6,7]:

1. The world-line Hamiltonian: $H = \frac{1}{2m} g^{\mu\upsilon}(x)\, p_{\mu}\, p_{\upsilon}$
2. The supercharge: $Q = p_{\mu}\, \Psi^{\mu}$
3. The dual supercharge: $Q^{*} = \frac{1}{2!}\sqrt{-g}\; \varepsilon_{\mu\upsilon\lambda\sigma}\, p^{\mu}\, \Psi^{\upsilon}\, \Psi^{\lambda}\, \Psi^{\sigma}$
4. The chiral charge: $\Gamma_{*} = \frac{1}{4!}\sqrt{-g}\; \varepsilon_{\mu\upsilon\lambda\sigma}\, \Psi^{\mu}\, \Psi^{\upsilon}\, \Psi^{\lambda}\, \Psi^{\sigma}$

The condition (10) implies that $Q = 0$.

3. Reissner-Nordstrom Spinning Space

Applying the results stated in the previous section we now study the motion of a spinning particle in the geometry of a charged gravitating body. A simple exact solution of such a geometry is the RN metric which has the form:

$$d s^{2} = -\left(1 - \frac{\alpha}{r} + \frac{e^{2}}{r^{2}}\right) d t^{2} + \left(1 - \frac{\alpha}{r} + \frac{e^{2}}{r^{2}}\right)^{-1} d r^{2} + r^{2}\left(d\theta^{2} + \sin^{2}\theta\, d\phi^{2}\right) \tag{9}$$

where $\alpha = 2\,GM$ and $e^{2} = Gq^{2}$ with M the total mass and q the total charge of the body, and G is Newton's gravitational constant. The co-ordinate singularities of the metric due to the distinct zeros $r_{+}, r_{-}, (r_{+} > r_{-})$ of $\left(1 - \frac{\alpha}{r} + \frac{e^{2}}{r^{2}}\right)$ correspond to the event horizon and Cauchy horizon, respectively. The RN metric has the four Killing vectors

$$D^{(\delta)} \equiv R^{(\delta)\mu}\, \partial_{\mu},\ \delta = 1, \cdots, 4; \tag{10}$$

where $D^{(1)} = \frac{\partial}{\partial t}, D^{(2)} = \frac{\partial}{\partial \phi} D^{(3)}, D^{(3)} = \cos\phi \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi}, D^{(4)} = \frac{\partial}{\partial \phi}$

These invariances correspond to conservation of energy and angular momentum for bosons. For spinning particle, the sum of the orbital and the spin angular momentum is conserved. The additional spin contribution is contained in the Killing scalars $B^{(\delta)}(x, \psi)$ *defined by*

(11)
$$B^{(\delta)}_{\mu} + \frac{\partial B^{(\delta)}}{\partial \Psi^{\sigma}} \Gamma^{\sigma}_{\mu\lambda} \Psi^{\lambda} = \frac{i}{2} \Psi^{\lambda} \Psi^{\sigma} R_{\lambda\sigma\upsilon\mu} R^{(\delta)\nu} .$$

Inserting the $R^{(\delta)}(x)$ as given by (10), we obtain the corresponding Killing scalars:

(12)
$$B^{(1)} = \left(\frac{\alpha}{2r^2} - \frac{e^2}{2r^3}\right) S^{tr}, B^{(2)} = \frac{\partial}{\partial \phi} B^{(3)} ,$$

$$B^{(3)} = r \cos\phi\, S^{r\theta} - r \sin\theta \cos\theta \sin\phi\, S^{r\phi} + r^2 \sin^2\theta \sin\phi\, S^{\theta\phi} ,$$

$$B^{(4)} = r \sin^2\theta\, S^{r\phi} + r^2 \sin\theta \cos\theta\, S^{\theta\phi} ,$$

The four conserved quantities $J^{(\delta)}$ can now be found easily:

$$J^{(1)} \equiv E = m\left(1 - \frac{\alpha}{r} + \frac{e^2}{r^2}\right)\dot{t} - B^{(1)}$$

$$J^{(2)} = B^{(2)} - m r^2 (\sin\phi\, \dot{\theta} + \sin\theta \cos\theta \cos\phi\, \dot{\phi})$$

$$J^{(3)} = B^{(3)} + m r^2 (\cos\phi\, \dot{\theta} - \sin\theta \cos\theta \sin\phi\, \dot{\phi})$$

(13)
$$J^{(4)} = B^{(4)} + m r^2 \sin^2\theta\, \dot{\phi}$$

Together with these there are four generic constants of motion as described in the previous section. Further, the covariantly Ψ^{μ} as formulated in (5) yields

$$\dot{\Psi}^t = -\frac{\alpha r - 2e^2}{2r^3\left(1 - \frac{\alpha}{r} + \frac{e^2}{r^2}\right)} (\dot{r}\, \Psi^t + \dot{t}\, \Psi^r)$$

$$\dot{\Psi}^r = r\left(1 - \frac{3\alpha}{2r} + \frac{2e^2}{r^2}\right)(\dot{\theta}\, \Psi^{\theta} + \sin^2\theta\, \dot{\phi}\, \Psi^{\phi})$$

$$\dot{\Psi}^{\theta} = -\frac{1}{r}(\dot{r}\,\Psi^{\theta} + \dot{\theta}\,\Psi^{r}) + \sin\theta\cos\theta\,\dot{\phi}\,\Psi^{\phi}$$

(14)
$$\dot{\Psi}^{\phi} = -\left(\frac{1}{r}\,\dot{r} + \cot\theta\,\dot{\theta}\right) - \frac{1}{r}\,\dot{\phi}\,\Psi^{r} - \cot\theta\,\dot{\phi}\,\Psi^{\theta}$$

We thus get twelve equations altogether. Since the motion is geodesic we can consider

(15)
$$H = -\frac{m}{2}$$

or, equivalently

(16)
$$g^{\mu\upsilon}\,p_{\mu}\,p_{\upsilon} + m^{2} = 0\,.$$

Eqn. (15) allows us to write $\dot{r}$ in terms of the other velocities:

$$\dot{t} = \frac{1}{\left(1 - \frac{\alpha}{r} + \frac{e^{2}}{r^{2}}\right)}\left[\frac{E}{m} + \left(\frac{\alpha}{2\,m\,r^{2}} - \frac{e^{2}}{2\,m\,r^{3}}\right)S^{rt}\right]$$

$$\dot{r} = \left\{\left(1 - \frac{\alpha}{r} + \frac{e^{2}}{r^{2}}\right)^{2}\dot{t}^{2} - \left(1 - \frac{\alpha}{r} + \frac{e^{2}}{r^{2}}\right) - r^{2}\left(1 - \frac{\alpha}{r} + \frac{e^{2}}{r^{2}}\right)[\dot{\theta}^{2} + \sin^{2}\theta\,\dot{\phi}^{2}]\right\}^{\frac{1}{2}}$$

(17)
$$\dot{\theta} = \frac{1}{m\,r^{2}}(-J^{(2)}\sin\phi + J^{(3)}\cos\phi - r\,S^{r\theta})$$

$$\dot{\phi} = \frac{1}{mr^{2}\sin^{2}\theta}\,J^{(4)} - \frac{1}{mr}\,S^{r\phi} - \frac{1}{m}\cot\theta\,S^{\theta\phi}$$

The combination

(18)
$$r^{2}\sin\theta\,S^{\theta\phi} = J^{(2)}\sin\theta\cos\phi + J^{(3)}\sin\theta\sin\phi + J^{(4)}\cos\theta$$

implies that there is only the spin angular momentum in the radial direction. From the supersymmetric constraint $Q = 0$ we obtain,

(19)
$$\left(1 - \frac{\alpha}{r} + \frac{e^{2}}{r^{2}}\right)\dot{t}\,\Psi^{t} = \frac{1}{\left(1 - \frac{\alpha}{r} + \frac{e^{2}}{r^{2}}\right)}\dot{r}\,\Psi^{r} + r^{2}(\dot{\theta}\,\Psi^{\theta} + \sin^{2}\theta\,\dot{\phi}\,\Psi^{\phi})\,.$$

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This makes the chiral charge Γ_* and the dual supercharge Q^* to vanish:

$$\Gamma_* = Q^* = 0. \tag{20}$$

Expression (19) satisfies the first part of (14). The remaining three of (14) can be written as

$$\dot{S}^{r\theta} = -\frac{1}{r}\dot{r}\,S^{r\theta} + \sin\theta\cos\theta\,\dot{\phi}\,S^{r\phi} - r\sin^2\theta\left(1-\frac{3\alpha}{2r}+\frac{2e^2}{r^2}\right)\dot{\phi}\,S^{\theta\phi}$$

$$\dot{S}^{r\phi} = \cot\theta\,\dot{\phi}\,S^{r\theta} - \left(\frac{1}{r}\dot{r}+\cot\theta\,\dot{\theta}\right)S^{r\phi} + r\left(1-\frac{3\alpha}{2r}+\frac{2e^2}{r^2}\right)\dot{\theta}\,S^{\theta\phi} \tag{21}$$

$$\dot{S}^{\theta\phi} = \frac{1}{r}\dot{\phi}\,S^{r\theta} - \frac{1}{r}\dot{\theta}\,S^{r\phi} - \left(\frac{2}{r}\dot{r}+\cot\theta\,\dot{\theta}\right)S^{\theta\phi}$$

The equation for $S^{\theta\phi}$ is solved by (18). Using (19) the first part of (17) can be written in the form

$$\dot{t} = \frac{1}{\left(1-\frac{\alpha}{r}+\frac{e^2}{r^2}\right)}\left[\frac{E}{m}+\frac{1}{2E}\left(\alpha-\frac{e^2}{r}\right)(\dot{\theta}\,S^{r\theta}+\sin^2\theta\,\dot{\phi}\,S^{r\phi})\right]. \tag{22}$$

The solutions of the equations of motion for the co-ordinates and spins are obtained by integrating equations (17), (22) and the first two equations of (21).

4. Special Solutions

We here solve the equations given in the previous section for the motion in a plane for which $\theta = \frac{\pi}{2}$. With $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$, the equations (17) and (22) become

$$\dot{t} = \frac{1}{\left(1-\frac{\alpha}{r}+\frac{e^2}{r^2}\right)}\left[\frac{E}{m}+\frac{1}{2E}\left(\alpha-\frac{e^2}{r}\right)\dot{\phi}\,S^{r\phi}\right]$$

$$\dot{r} = \left[\left(1-\frac{\alpha}{r}+\frac{e^2}{r^2}\right)^2\dot{t}^2 - \left(1-\frac{\alpha}{r}+\frac{e^2}{r^2}\right) - r^2\left(1-\frac{\alpha}{r}+\frac{e^2}{r^2}\right)\dot{\phi}^2\right]^{\frac{1}{2}}$$

(23)
$$\dot{\phi} = \frac{1}{mr^2} J^{(4)} - \frac{1}{mr} S^{r\phi}$$

$$(rs^{r\theta})^{\bullet} = -\left(1 - \frac{3\,\alpha}{2\,r} + \frac{2e^2}{r^2}\right) r^2 S^{\theta\phi}\,\dot{\phi}$$

$$(rs^{r\phi})^{\bullet} = 0.$$

The last and the third equations show that the orbital angular momentum and the spin's component perpendicular to the plane of motion are separately conserved:

(24)
$$rS^{r\phi} \equiv \sum, m\, r^2\,\dot{\phi} = J^{(4)} - \sum \equiv L$$

where $\sum$ and L are two constants. The first of (23), then, becomes

(25)
$$dt = \frac{d\tau}{\left(1 - \frac{\alpha}{r} + \frac{e^2}{r^2}\right)}\left[\frac{E}{m} + \frac{\left(\alpha - \frac{e^2}{r}\right)}{2m\,E\,r^3} L \sum\right]$$

Eqn. (25) shows that the time-dilation has an additional contribution from spin-orbit coupling and this dynamical component in the charged gravitational field is less compared to that in charge-free field.

From equations (17), (18) and (23) we derive $S^{\theta\phi}\,\dot{\phi} = 0$.

The motion for $\dot{\phi} \neq 0$ gives

(26)
$$S^{\theta\phi} = 0,\ J^{(2)} = J^{(3)} = 0.$$

Then, also $S^{r\theta} = 0$; that is, the spin is parallel to the orbital angular momentum. We derive from (23) the equation for the orbit of the particle

(27)
$$\frac{1}{r^2}\left(\frac{dr}{d\phi}\right)^2 = \frac{(E^2 - m^2)\,r^2}{L^2} - 1 + \frac{m}{L}\left(\alpha - \frac{e^2}{r}\right)\left(\frac{mr}{L} + \frac{J^{(4)}}{mr}\right).$$

Using the dimensionless variables

(28)
$$\epsilon = \frac{E}{m},\ x = \frac{r}{\alpha},\ 1 = \frac{L}{m\,\alpha},\ \Delta = \frac{\sum}{L}$$

we obtain

$$\left(\frac{du}{d\phi}\right)^2 = -(1+\Delta)\frac{e^2}{\alpha^2}u^4 + (1+\Delta)u^3 - \left(1+\frac{e^2}{l^2\alpha^2}\right)u^2 + \frac{1}{l^2}u - \frac{1-\epsilon^2}{l^2}$$

(29) $$\equiv f(u)$$

where $u = \frac{1}{x}$. In order to analyze the possible motions we need to assign some expectation value to Δ and consider $\Delta << 1$.

We here study the bound state orbits of the spinning point particle for which it is necessary that $\epsilon^2 < 1$. The bound states correspond to quasi-elliptic and circular orbits. We first discuss the circular orbits. For circular orbits:

(30) $$f(u) = f'(u) = 0;$$

and the energy ϵ and the orbital angular momentum 1 for the radius $x_c = \frac{1}{u_c}$ have, respectively, the expressions

(31) $$\epsilon^2_{\alpha\iota\tau} = \left(1 - \frac{R}{P}u_c\right), l^2 = \frac{1}{Pu_c}\left(1 - 2\frac{e^2}{\alpha^2}u_c\right)$$

provided, $P > 0$ where

$$P = 2 - 3(1+\Delta)u_c + 4(1+\Delta)\frac{e^2}{\alpha^2}u_c^2,$$

$$R = 1 - 2(1+\Delta)\left\{1 - \left(3 - 2\frac{e^2}{\alpha^2}u_c\right)\frac{e^2}{\alpha^2}u_c\right\}u_c - 2\frac{e^2}{\alpha^2}u_c.$$

Equation (29) can now be put in the form

$$\left(\frac{du}{d\phi}\right)^2 = (u-u_c)(1+\Delta)\left[-\frac{e^2}{\alpha^2}u^2 + \left(1 - 2\frac{e^2}{\alpha^2}u_c\right)u + \left(\frac{1}{2} - \frac{e^2}{\alpha^2}u_c - \frac{1}{2(1+\Delta)l^2u_c^2}\right)u_c\right\}$$

(32)

The equation (32) provides a circular orbit of radius x_c together with a second kind orbit that starts at a certain aphelion distance and then plunges into the Cauchy horizon, determined by

(33) $$\phi = \mp \frac{1}{\sqrt{(1+\Delta)}} \int \frac{d\xi}{\sqrt{-\frac{e^2}{\alpha^2} + \beta\xi + \gamma\xi^2}}$$

where

$$\xi = (u - u_c)^{-1},\ \beta = 1 - 2\frac{e^2}{\alpha^2} u_c,\ \gamma = \left(\frac{1}{2} - \frac{e^2}{\alpha^2} u_c - \frac{1}{2(1+\Delta) l^2 u_c^2}\right) u_c\,.$$

Equation (33) has the solution

(34) $$\mp\phi = \frac{1}{\sqrt{(1+\Delta)}} \begin{cases} \frac{1}{\sqrt{\gamma}} \log\left[2\left\{\gamma\left(-\frac{e^2}{\alpha^2} + \beta\xi + \gamma\xi^2\right)\right\}^{\frac{1}{2}} + 2\gamma\xi + \beta\right],\ \gamma > 0 \\ -\frac{1}{\sqrt{-\gamma}} \sin^{-1}\left[\frac{2\gamma\xi + \beta}{\sqrt{4\frac{e^2}{\alpha^2}\gamma + \beta^2}}\right],\ \gamma < 0\,. \end{cases}$$

The radius of the stable circular orbit is minimum at the point of inflexion of the function $f(u)$. Supplementing equations (30) with the further equation $f''(u) = 0$ and using (31) we obtain

(35) $$x_c^3 - 3(1+\Delta)x_c^2 + 9(1+\Delta)\frac{e^2}{\alpha^2}x_c - 8(1+\Delta)\frac{e^4}{\alpha^4} = 0$$

for the radius of the minimum stable circular orbit. The second kind orbit associated with the minimum stable circular orbit is determined from

(36) $$\left(\frac{du}{d\phi}\right)^2 = (1+\Delta)(u - u_c)^3\left(1 - 3\frac{e^2}{\alpha^2}u_c - \frac{e^2}{\alpha^2}u\right)$$

which is

(37) $$x = x_c - \frac{4\left(x_c - 4\frac{e^2}{\alpha^2}\right)x_c^3}{(1+\Delta)\left(x_c - 4\frac{e^2}{\alpha^2}\right)^2(\phi - \phi_*)^2 + 4\left(x_c - 3\frac{e^2}{\alpha^2}\right)x_c^2}\,.$$

We thus observe that in the spinning space the orbits of the energy of the particle are modified.

We now investigate the quasi-elliptic orbit. The orbit is described by

$$x = \frac{K}{1 + \varepsilon \cos [\phi - w(\phi)]} \tag{38}$$

where $K = \frac{k}{\alpha}$. k is the semilatus rectum and ε the eccentricity with $0 < \varepsilon < 1$. The function $w(\phi)$ accounts the precession of the perihelion due to relativistic effects. The perihelion and aphelion are described by

$$\phi_{ph}^{(k)} - w\left(\phi_{ph}^{(k)}\right) = 2k\pi, \ \phi_{ah}^{(k)} - w\left(\phi_{ah}^{(k)}\right) = (2k+1)\pi \tag{39}$$

The precession of the perihelion after one revolution is

$$\Delta w \equiv w\left(\phi_{ph}^{(1)}\right) - w\left(\phi_{ph}^{(0)}\right) = \phi_{ph}^{(1)} - \phi_{ph}^{(0)} - 2\pi \equiv \Delta\phi - 2\pi . \tag{40}$$

The energy ϵ at the perihelion / aphelion is given by

$$\epsilon^2 = 1 - \left(\frac{1 \pm \varepsilon}{k}\right) + \left(l^2 + \frac{e^2}{\alpha^2}\right)\left(\frac{1 \pm \varepsilon}{K}\right)^2 - l^2 (1 + \Delta)\left(\frac{1 \pm \varepsilon}{K}\right)^3 + l^2 (1 + \Delta)\frac{e^2}{\alpha^2}\left(\frac{1 \pm \varepsilon}{K}\right)^4 \tag{41}$$

The constancy of ϵ gives

$$l^2 = \frac{K^2\left(K - 2\frac{e^2}{\alpha^2}\right)}{2K^2 - (1 - \Delta)\left[(3 + \varepsilon^2) K - 4\frac{e^2}{\alpha^2}(1 + \varepsilon^2)\right]}. \tag{42}$$

Using the above results and introducing $x = \phi - w(\phi)$, the equation of motion (27) can be written in the form

$$d\phi = \frac{dx}{\sqrt{A\,(a - b\cos x + c\cos^2 x)}} \tag{43}$$

where

$$A = \frac{K}{K - 2\frac{e^2}{\alpha^2}}, \ F = \frac{1 + \Delta}{K}$$

$$a = 1 - \exists F, \ \exists = \frac{1}{K^2}\left\{3K\left(K - 3\frac{e^2}{\alpha^2}\right) + 8\frac{e^4}{\alpha^4} - 2\varepsilon^2\frac{e^4}{\alpha^4}\right\}$$

$$b = D\,F,\; D = \varepsilon\,\frac{1}{K^2}\left\{K\left(K - 6\,\frac{e^2}{\alpha^2}\right) + 8\,\frac{e^4}{\alpha^4}\right\}$$

$$c = N\,F,\; N = \varepsilon^2\,\frac{1}{K^2}\left(K - 2\,\frac{e^2}{\alpha^2}\right)\frac{e^2}{\alpha^2}.$$

Then, $\Delta\,\phi$ as defined in (40) becomes

(44)
$$\Delta\,\phi = \frac{1}{\sqrt{A\,a}}\int_0^{2x}\frac{d\,x}{\sqrt{1 - \left(\frac{b}{a}\cos x - \frac{c}{a}\cos^2 x\right)}}.$$

With application of the expansion

(45)
$$(1-t)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} T(n)\,t^n,\; T(n) = \frac{1}{2^{2n}}\binom{2n}{n}$$

and the integrals

(46)
$$\int_0^{2x} d\,x\cos^{2n}x = 2\,\pi\,T(n),\; \int_0^{2x} d\,x\cos^{2n-1}x = 0.$$

we derive from (44) an expression for $\Delta\,\phi$ as a series in F from which we obtain

(47)
$$\begin{aligned}\Delta\,\phi = 2\,\pi\,(1-\tau)^{\frac{1}{2}}\Bigg\{&1 + 3\,\eta\left[1 - \frac{18+\varepsilon^2}{12}\tau + \frac{8-\varepsilon^2}{12}\tau^2\right](1+\Delta)\\ &+ \frac{3}{4}\eta^2\Bigg[18\left(1 - \frac{3}{2}\tau + \frac{4-\varepsilon^2}{6}\tau^2\right)^2 + \varepsilon^2(1 - 3\,\tau + 2\,\tau^2)^2\\ &- \frac{5}{3}\varepsilon^2\,\tau\,(1-\tau)\left(1 - \frac{72+9\,\varepsilon^2}{80}\tau + \frac{32+\varepsilon^2}{80}\tau^2\right)\\ &+ \frac{3}{16}\varepsilon^4\,\tau^2\,(1-\tau)^2\Bigg](1+\Delta)^2 + \cdots\Bigg\}\end{aligned}$$

where $\tau = \dfrac{Q^2}{M\,k}$ and $\eta = \dfrac{M\,G}{k}$.

For $\Delta = 0$, the lowest-order contribution to the precession of the perihelion is given by the second term in the expansion. The spin of a particle contributes to this lowest- order precession which we find by keeping terms of first order in Δ.

5. Concluding Remarks

The usual Killing equations and Noether's theorem for particles are generalized with the spin degrees of freedom to study the motion in spinning space. The antisymmetric Grassmann variables characterize the spin degrees of freedom and by this model the pseudo-classical limit of the Dirac equation is described. Our results in this paper can be applied to the formal aspects of the motion of electrons or possibly massive neutrinos (or photinos, gravitinos, etc.) in the external gravitational field of a heavy spherically symmetric charged object. Such formal aspects include the proof of spin-orbit coupling and the corresponding fine splitting, resulting from dependence of the energy on the values and relative orientation of the orbital and spin angular momentum, and this predicts that in a charged gravitational field the time-dilation, the perihelion precession for boundstate orbits, the circular orbits and the orbits of plunging the particle into the Cauchy horizon are spin-dependent. The deflection of a particle in a charged gravitational field is also spin-dependent. Thus the Stern-Gerlach-type forces have the gravitational analogue.

Although the charge term $\frac{e^2}{r^2}$ in RN metric can be ignored at large r, this metric deserves attention as a simple example of an exact solution of the Einstein Maxwell's equations. Setting $e = 0$ in the results obtained here we get corresponding results of Schwarzchild space-time. The motion of spinning particles in Schwarzchild metric was described in [9] and our results with $e = 0$ agrees with them.

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