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Free Transverse Vibrations of Non-Homogeneous Circular Plates of Linearly Varying Thickness*

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Abstract: This paper analyses free transverse axisymmetric vibrations of non-homogeneous circular plates of linearly varying thickness on the basis of classical plate theory. The non-homogeneity of plate material is assumed to arise due to variation in Young's Modulus and density along radial direction. An approximate solution has been obtained using Ritz method. The basis functions have been chosen to satisfy the essential boundary conditions. Convergence to four digit exactitude is demonstrated for first three natural frequencies of plates. Numerical results are presented for circular plates with clamped, simply supported and free boundary conditions. This study investigates the effect of various parameters namely non-homogeneity parameter, density parameter and taper parameter on the natural frequencies of circular plates for first three modes of vibration. Results in some special cases are compared with existing solutions available from analytical and other numerical methods, which show an excellent agreement.

Keywords: Vibrations, non-homogeneous, circular plate, linearly varying thickness, Ritz method.

2010 MS Classification No.: 74H45

1. Introduction

Plates of tapered thickness are commonly used as structural elements in constructions of ships, aircrafts, automobiles and other vehicles. Therefore it is important to predict the dynamic behavior of such plate type structures to ensure good design. In the literature, a considerable amount of work has

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been reported on vibrations of homogeneous circular plates of variable thickness, and few of them are reported in ref.[1-6]. In various technological situations, particularly in space shuttle, high speed aircrafts, missile technology and microelectronics, certain parts have to operate under elevated temperatures which causes non-uniform heating in case of variable thickness structural components, resulting in non-homogeneity of the material, i.e. the elastic constants of the material become functions of space variable. Very few models representing the behavior of non-homogeneous materials have been reported in the literature. The earliest model was proposed by Bose[7], where Young's modulus and density are supposed to vary with radius vector. Biswas[8] in his model considered exponential variations for torsional rigidity and the material density. Rao et al.[9] dealing with vibration of non-homogeneous isotropic thin plates have assumed linear variations for young's modulus and density. In a series of papers, Tomar et al.[10-13] have analyzed the dynamic behavior of non-homogenous isotropic plates of variable thickness of different geometries. The non-homogeneity of plate material is assumed to arise due to variations of and density exponentially along one direction taking some parameter for variation of Young's modulus and density. Gupta et al.[6] considered a more general model for non-homogeneity of plate material where Young's modulus and density i. e. $E=E_0e^{\mu x}$, $\rho = \rho_0 e^{\eta x}$ are assumed to vary exponentially in radial direction in different manner. In all the studies Poisson's ratio have been assumed to be constant.

This paper investigates the natural frequencies of linearly tapered circular plates taking into account, the non-homogeneity which arises due to variations in and density of the plate material, applying Ritz method. The basis functions based upon static deflection for isotropic plates have been chosen. Convergence and comparison studies have also been presented to verify the accuracy of present method.

2. Analysis

Consider a thin circular plate of radius a, thickness h(r), density ρ, elastically restrained against rotation and translation by springs of stiffness k_φ and k referred to cylindrical polar coordinate (r, θ, z), where the axis of the plate is taken as the line r = 0 and its middle surface as the plane z = 0. The maximum kinetic energy of the plate is given by

$$T\max = \frac{1}{2}\omega^2 \int_0^a \int_0^{2\pi} \rho h W^2 r\,d\theta\,dr \,, \tag{2.1}$$

where W is the transverse deflection, ρ is the mass density and ω is the frequency in rad/s.
The maximum strain energy of the plate is given by

(2.2)
$$U\max=\frac{1}{2}\int_0^a\int_0^{2\pi}D\left[\left(\frac{\partial^2W}{\partial r^2}\right)^2+2\nu\frac{\partial^2W}{\partial r^2}\left(\frac{1}{r}\frac{\partial W}{\partial r}\right)+\left(\frac{1}{r}\frac{\partial W}{\partial r}\right)^2\right]rdrd\theta$$
$$+\frac{1}{2}\mathrm{a}k_\varphi\int_0^{2\pi}\left(\frac{\partial W}{\partial r}\right)^2d\theta+\frac{1}{2}ak\int_0^{2\pi}W^2d\theta.$$

where $1/k_\varphi$ is the rotational flexibility of the spring and

$$D=\frac{Eh^3}{12\left(1-\nu^2\right)},$$

is flexural rigidity of the plate, E and ν are Young's modulus and Poisson's ratio of the plate material.

3. Method of Solution: RITZ Method

Ritz method requires that the functional

(3.1)
$$J(W)=U\max-T\max=\frac{1}{2}\int_0^a\int_0^{2\pi}D\left[\left(\frac{\partial^2W}{\partial r^2}\right)^2+2\nu\frac{\partial^2W}{\partial r^2}\left(\frac{1}{r}\frac{\partial W}{\partial r}\right)+\left(\frac{1}{r}\frac{\partial W}{\partial r}\right)^2\right]rdrd\theta$$
$$+\frac{1}{2}\mathrm{a}k_\varphi\int_0^{2\pi}\left(\frac{\partial W}{\partial r}\right)^2d\theta+\frac{1}{2}ak\int_0^{2\pi}W^2d\theta-\frac{1}{2}\omega^2\int_0^a\int_0^{2\pi}\rho hW^2rd\theta dr,$$

be minimized.
For non-homogeneity of the plate material, let us assume that E and ρ are the functions of space variable r. Now (3.1) becomes

(3.2)
$$\mathrm{J}\left(\mathrm{W}\right)=\frac{1}{24\left(1-\nu^2\right)}\int_0^a\int_0^{2\pi}Eh^3\left[\left(\frac{\partial^2W}{\partial r^2}\right)^2+2\nu\frac{\partial^2W}{\partial r^2}\left(\frac{1}{r}\frac{\partial W}{\partial r}\right)+\left(\frac{1}{r}\frac{\partial W}{\partial r}\right)^2\right]rdrd\theta$$
$$+\frac{1}{2}\mathrm{a}k_\varphi\int_0^{2\pi}\left(\frac{\partial W}{\partial r}\right)^2d\theta+\frac{1}{2}ak\int_0^{2\pi}W^2d\theta-\frac{1}{2}\omega^2\int_0^a\int_0^{2\pi}\rho hW^2rd\theta dr.$$

Introducing non-dimensional variables $\overline{W}=\frac{W}{a},\mathrm{R}=\frac{r}{a}$ and $\mathrm{H}=\frac{h}{a}$ together with linear variation in thickness i.e. H = h_0 (1+αR) and assuming exponential variation (following Tomar et. al.[10-13]) for the non-homogeneity of the material in radial direction as follows:

(3.3) $$E = E_0 e^{\mu R}, \quad \rho = \rho_0 e^{\eta R}.$$

Assuming the deflection function as

(3.4) $$W = \sum_{i=0}^{m} A_i W_i(R) = \sum_{i=0}^{m} A_i (1+\alpha_i R^4 + \beta_i R^2) R^{2i},$$

where A_i are undetermined coefficients $\overline{W} = W/a$ and α_i, β_i are unknown constants to be determined from boundary conditions (Leissa [14]),

(3.5) $$k_\varphi \frac{dW(1)}{dR} = -(1+\alpha)^3 \left[\frac{d^2W}{dR^2} + \nu \left(\frac{1}{R} \frac{dW}{dR} \right) \right]_{R=1},$$

(3.6) $$kW(1) = (1+\alpha)^3 \left[\frac{d}{dR} \left(\frac{d^2W}{dR^2} \right) + \left(\frac{1}{R} \frac{dW}{dR} \right) \right]_{R=1}.$$

Using non-dimensional variables $\overline{W}$ and R along with relations (3.3) and (3.4), the functional J(W) given by (3.2) becomes

(3.7) $$J(\overline{W}) = \frac{D_{r_0}}{2} \int_0^1 \int_0^{2\pi} e^{\mu R} (1+\alpha R)^3 \left\{ \left(\frac{\partial^2 \overline{W}}{\partial R^2} \right)^2 + 2\nu \frac{\partial^2 \overline{W}}{\partial R^2} \left(\frac{1}{R} \frac{\partial \overline{W}}{\partial R} \right) + \left(\frac{1}{R} \frac{\partial \overline{W}}{\partial R} \right)^2 \right\} R\,dR\,d\theta$$
$$+ K \int_0^{2\pi} \overline{W}^2(1)\,d\theta + K\varphi \int_0^{2\pi} \left(\frac{\partial \overline{W}(1)}{\partial R} \right)^2 d\theta - \Omega^2 \int_0^1 \int_0^{2\pi} e^{\eta R} (1+\alpha R) \overline{W}^2 R\,d\theta\,dR.$$

where $$D_{r_0} = \frac{E_0 h_0^{\,3}}{12(1-\nu^2)}, \quad K = \frac{a^3 k}{D_{r_0}}, \quad K_\varphi = \frac{a k_\varphi}{D_{r_0}}, \quad \Omega^2 = \frac{\rho_0 a^4 \omega^2 h_0}{D_{r_0}}.$$

The minimization of the functional J(W) given by (3.7) requires

(3.8) $$\frac{\partial J(\overline{W})}{\partial A_i} = 0, \qquad i = 0, 1, 2, \ldots, m.$$

This leads to a system of homogeneous equations in A_i, i = 0, 1, 2, …, m, whose non-trivial solution leads to the frequency equations

(3.9) $$|A - \Omega^2 B| = 0,$$

where A = [a_{ij}] and B = [b_{ij}] are square matrices of order (m+1) given by

(3.10) $$a_{ij} = \int_0^1 e^{\mu R}(1+\alpha R)^3 \left[W_i'' W_j'' + 2\nu W_i'' \left(\frac{W_j'}{R}\right) + \left(\frac{W_i'}{R}\right)\left(\frac{W_j'}{R}\right)\right] R dR$$
$$+ K_\varphi W_i'(1) W_j'(1) + K W_i(1) W_j(1),$$

and

(3.11) $$b_{ij} = \int_0^1 e^{\eta R}(1+\alpha R) W_i W_j R dR, \text{ for } i = 0, 1, 2, \ldots, m; \ j = 0, 1, 2, \ldots, m.$$

4. Numerical Results and Discussion

The frequency equation (3.9) has been solved to obtain first three natural frequencies for various values of plate parameters such as taper parameter α (= 0.0, ±0.1, ± 0.5), non-homogeneity parameter μ(= -0.5 , 0.0 , 1.0), and density parameter η (= -0.5 , 0.0, 1.0). The Poisson's ratio ν has been fixed as 0.3. Three boundary conditions namely clamped (K_φ = 10^{20}, K = 10^{20}), simply-supported (K_φ = 0, K = 10^{20}), and free(K_φ = 0, K = 0) have been considered.

The convergence study has been carried out for circular plates with ν = 0.3 for different sets of plate parameters. The convergence graphs for clamped, simply-supported and free plates are shown in figures 1(a, b, c) for α = -0.5, μ = -0.5, η = -0.5. It is observed that 17 terms of admissible function give first three frequency parameter at least accurate to four significant digits.

Numerical results are presented in Tables 1-3 and figures 2-4, Tables 1-3 give the values of frequency parameter **Ω** for different values of non-homogeneity parameter μ, density parameter η, and taper parameter α for linearly varying thickness plates for first three modes of vibration for clamped, simply-supported and free plates respectively. It is observed that the frequency parameter **Ω** for free plate is smaller than that for clamped plate and greater than that for simply-supported plate. The frequency parameter **Ω** is found to increase with the increase in non-homogeneity parameter μ as well as taper parameter α, while it decreases with the increase in density parameter η.

Figures 2(a, b, c) show the plots for frequency parameter **Ω** versus taper parameter α for μ = -0.5, η = 1.0 and μ = 1.0, η = -0.5 for clamped, simply-supported and free plate for the first three modes of vibration, respectively. It is observed that frequency parameter **Ω** increases with increasing values of taper parameter α. The rate of increase of Ω with α is more pronounced in case of clamped plate as compared to simply-supported and free plates. The frequency parameter for $\mu = 1.0$, $\eta = -0.5$ are higher than that for $\mu = -0.5$, $\eta = 1.0$. Also the rate of increase of Ω with increasing values of α in all the three plates becomes higher and higher with increase in number of modes.

Figures 3(a, b, c) depict the variation of frequency parameter Ω versus non-homogeneity parameter μ for $\eta = -0.5$, $\alpha = -0.5, 0.5$ for clamped, simply-supported and free plate for the first three modes of vibrations, respectively. It is observed that frequency parameter Ω increases with increasing values of μ. The rate of increase for free plate is higher than simply-supported plate but lesser than clamped plate. The rate of increase of Ω with μ increases by increasing α for all the three plates. This rate of increase gets pronounced with the increase in number of modes.

Figures 4(a, b, c) show the effect of density parameter η on frequency parameter Ω for $\mu = 1.0$, $\alpha = -0.5, 0.5$ for all the three plates vibrating in fundamental, second and third mode respectively, it is found that frequency decreases with increasing value of density parameter. The rate of decrease for clamped plate is higher than that for simply-supported plate and less than that for free plate. The rate of decrease gets pronounced in higher and higher modes.

Table 4 shows a comparison of results for homogeneous ($\mu = 0.0$, $\eta = 0.0$) circular plate of uniform thickness ($\alpha = 0.0$) with exact solutions given by Leissa[14] and approximate solutions obtained by Ansari[15] using Ritz method and Azimi[16] using receptence method. Table 5 gives a comparison of results for homogeneous circular plate of linearly varying thickness with those obtained by Lal[17] using frobenius method, Singh and Saxena[1] and Gutierrez et. al[18] using Rayleigh-Ritz method for clamped and simply-supported plate. An excellent agreement of results shows the versatility of present technique.

5. Conclusion

The present paper analyzes free transverse vibrations of non-homogeneous circular plates of linearly varying thickness on the basis of classical plate theory, using Ritz method. The frequency parameter increases with increasing values of taper parameter α as well as non-homogeneity parameter μ, while it decreases with increasing values of density parameter η. These results presented here have been known for the first time and can be benchmark for design engineer to have desired frequency with four decimal exactitude, by a proper choice of plate parameters i.e. taper parameter, non-homogeneity parameter and density parameter. The accuracy of the approach has been verified by demonstrating a close agreement to our results with those of exact solutions and obtained by various techniques: differential quadrature method receptence method, frobenius method, Rayleigh-Ritz method.

Table 1: Values of frequency parameter Ω for clamped plate

α	μ=-0.5			μ=0.0			μ=1.0		
	η=-0.5	η=0.0	η=1.0	η=-0.5	η=0.0		η=1.0	η=-0.5	η=0.0
	I								
-0.5	5.7823	5.2676	4.3141	6.7376	6.1504	5.0589	9.1707	8.4058	6.9743
-0.1	8.7588	8.0053	6.6021	10.2651	9.4027	7.7906	14.1380	13.0084	10.8804
0.0	9.5005	8.6879	7.1733	11.1464	10.2158	8.4746	15.3791	14.1597	11.8597
0.1	10.2429	9.3712	7.7453	12.0288	11.0301	9.1598	16.6210	15.3122	12.8406
0.5	13.2232	12.1153	10.0431	15.5730	14.3022	11.9150	21.6007	19.9361	16.7811
	II								
-0.5	26.9194	23.7356	18.3158	30.8689	27.3003	21.1908	40.4535	35.9954	28.2756
-0.1	36.6128	32.5249	25.4691	41.9476	37.3763	29.4418	54.8302	49.1502	39.1806
0.0	38.9153	34.6170	27.1791	44.5753	39.7711	31.4115	58.2295	52.2669	41.7754
0.1	41.1865	36.6819	28.8692	47.1658	42.1338	33.3573	61.5775	55.3385	44.3361
0.5	50.0418	44.7439	35.4855	57.2558	51.3481	40.9670	74.5900	67.2920	54.3284
	III								
-0.5	62.6438	54.9912	42.0592	71.6302	63.0612	48.5050	93.0924	82.4331	64.1358
-0.1	83.3343	73.6638	57.1214	94.9453	84.1680	65.6374	122.4932	109.2119	86.1412
0.0	88.1704	78.0372	60.6645	100.3867	89.1041	69.6613	129.3342	115.4552	91.2930
0.1	92.9159	82.3316	64.1483	105.7237	93.9486	73.6158	136.0391	121.5774	96.3510
0.5	111.2162	98.9146	77.6392	126.2838	112.6362	88.9131	161.8258	145.1506	115.8773

Table 2: Values of frequency parameter Ω for simply-supported plate

α	μ=-0.5			μ=0.0			μ=1.0		
	η=-0.5	η=0.0	η=1.0	η=-0.5	η=0.0	η=1.0	η=-0.5	η=0.0	η=1.0
	I								
-0.5	3.4712	3.1222	2.4868	3.9408	3.5498	2.8360	5.0531	4.5644	3.6678
-0.1	4.5717	4.1088	3.2674	5.1823	4.6637	3.7187	6.6773	6.0239	4.8275
0.0	4.8363	4.3455	3.4540	5.4854	4.9351	3.9330	7.0874	6.3917	5.1187
0.1	5.0992	4.5807	3.6392	5.7881	5.2061	4.1467	7.5007	6.6722	5.4118
0.5	6.1453	5.5152	4.3736	7.0035	6.2927	5.0018	9.1878	8.2733	6.6048
	II								
-0.5	21.1645	18.5522	14.1696	24.156	21.2386	16.3177	31.3380	27.7240	21.5609
-0.1	27.8135	24.5825	19.0880	31.6689	28.0774	21.9374	40.8279	36.4285	28.8265
0.0	29.4088	26.0326	20.2762	33.4691	29.7200	23.2937	43.0952	38.5133	30.5755
0.1	30.9879	27.4689	21.4547	35.2503	31.3465	24.6387	45.3368	40.5759	32.3087
0.5	37.1931	33.1210	26.1047	42.2449	37.7424	29.9426	54.1246	48.6740	39.1336
	III								
-0.5	53.3295	46.6826	35.5298	60.8745	53.4405	40.9041	78.8013	69.5828	53.8779
-0.1	69.8658	61.6111	47.5877	79.3885	70.2127	54.5460	101.7924	90.5562	71.1777
0.0	73.7512	65.1256	50.4382	83.7329	74.1561	57.7665	107.1727	95.4733	75.2498
0.1	77.5698	68.5820	53.2491	88.0008	78.0323	60.9364	112.4538	100.3024	79.2540
0.5	92.3436	81.9716	64.1491	104.4969	93.0347	73.2383	132.8272	118.9568	94.7628

Table 3: Values of frequency parameter Ω for free plate

α	μ=-0.5			μ=0.0			μ=1.0		
	η=-0.5	η=0.0	η=1.0	η=-0.5	η=0.0	η=1.0	η=-0.5	η=0.0	η=1.0
	I								
-0.5	7.6512	6.5544	4.8080	8.5256	7.3210	5.3957	10.5669	9.1148	6.7773
-0.1	8.9375	7.7133	5.7385	9.9909	8.4618	6.4550	12.5548	10.9019	8.2046
0.0	9.2861	8.0258	5.9841	10.3962	9.0031	6.7388	13.1162	11.4020	8.5971
0.1	9.6442	8.3443	6.2332	10.8120	9.3728	7.0280	13.6951	11.9166	8.9996
0.5	11.1358	9.6649	7.2577	12.5552	10.9165	8.2260	16.1445	14.0871	10.6864
	II								
-0.5	28.9597	25.1820	18.9596	32.9018	28.6948	21.7315	42.1928	37.0232	28.3819
-0.1	36.7090	32.2078	24.6890	41.5197	36.5407	28.1839	52.7591	46.7212	36.4920
0.0	38.5853	33.9100	26.0798	43.6084	38.4432	29.7511	55.3259	49.0769	38.4637
0.1	40.4450	35.5977	27.4598	45.6790	40.3297	31.3063	57.8713	51.4133	40.4205
0.5	47.7624	42.2439	32.9048	53.8276	47.7595	37.4426	67.8918	60.6162	48.1399
	III								
-0.5	64.7302	56.4713	42.7250	73.7406	64.5149	49.0844	94.9876	83.5836	64.3256
-0.1	83.4123	73.3355	56.3457	94.4897	83.3153	64.3843	120.3233	106.7132	83.4354
0.0	87.8164	77.3162	59.5708	99.3786	87.7502	68.0039	126.2885	112.1634	87.9490
0.1	92.1468	81.2321	62.7467	104.1843	92.1118	71.5671	132.1497	117.5209	92.3897
0.5	108.9063	96.4038	75.0805	122.7728	108.9985	85.3926	154.7983	138.2368	109.5711

Table 4: Comparison of frequency parameter Ω for homogeneous (μ = 0.0,η = 0.0) circular plate of uniform thickness (α = 0.0)

mode	ν = 0.3					
	Clamped plate		Simply-supported plate		Free plate	
I	10.21588 10.2158•	10.2158* 10.216◊	4.9351 4.977•	4.9351* 4.935◊	9.0031	9.0031*
II	39.7711 39.7711•	39.7711* 39.7711◊	29.7200 29.76•	29.7200* 29.720◊	38.4432	38.4432*
III	89.1041 89.104•	89.1041* 89.103◊	74.1561 74.20•	74.1561* 74.156◊	87.7502	87.7502*

* Values taken from Sharma[6], ● Values taken from Leissa[14], ◊ Values taken from Azimi[16]

Table 5:Comparison of frequency parameter Ω for homogeneous (μ=0.0,η=0.0) circular plate of linear thickness

α	Clamped plate					
	I		II		III	
-0.5	6.1504 6.1522°	6.1504* 6.1504•	27.3003 27.3006°	27.3002* 27.300•	63.0612 63.0605°	63.0611* 63.062•
-0.1	9.4027 9.4016° 9.402◊	9.4027* 9.4027•	37.3763 37.3742° 37.376◊	37.3763* 37.376•	84.168 84.1188°	84.1680* 84.168•
0.1	11.0301 11.0297° 11.03◊	11.0301* 11.030•	42.1338 42.1408° 42.133◊	42.1337* 42.134•	93.9486 93.9014°	93.9486* 93.949•
0.5	14.3022 14.3033°	14.3021* 14.302•	51.3481 51.3588°	51.3480* 51.349•	112.636 112.4586°	112.6360* 112.64•
	S-S plate					
	I		II		III	
-0.5	3.5498 3.5507°	3.5498* 3.5498•	21.2386 21.2419°	21.2386* 21.239•	53.4405 53.4095°	53.4404* 53.441•
-0.1	4.6637 4.6627° 4.664◊	4.6637* 4.6637•	28.0774 28.0765° 28.078◊	28.0774* 28.077•	70.2127 70.2104°	70.2127* 70.213•
0.1	5.2061 5.2065° 5.206◊	5.2061* 5.2061•	31.3465 31.3467° 31.346◊	31.3465* 31.346•	78.0323 78.0254°	78.0323* 78.032•
0.5	6.2927 6.2908°	6.2927* 6.2928•	37.7424 37.7414°	37.7423* 37.743•	93.0347 92.7375°	93.0342* 93.042•

* Values taken from Sharma[6], ° Values taken from Lal[17], • Values taken from Singh and Saxena[1], ◊ Values taken from Gutierrez et al .[18]

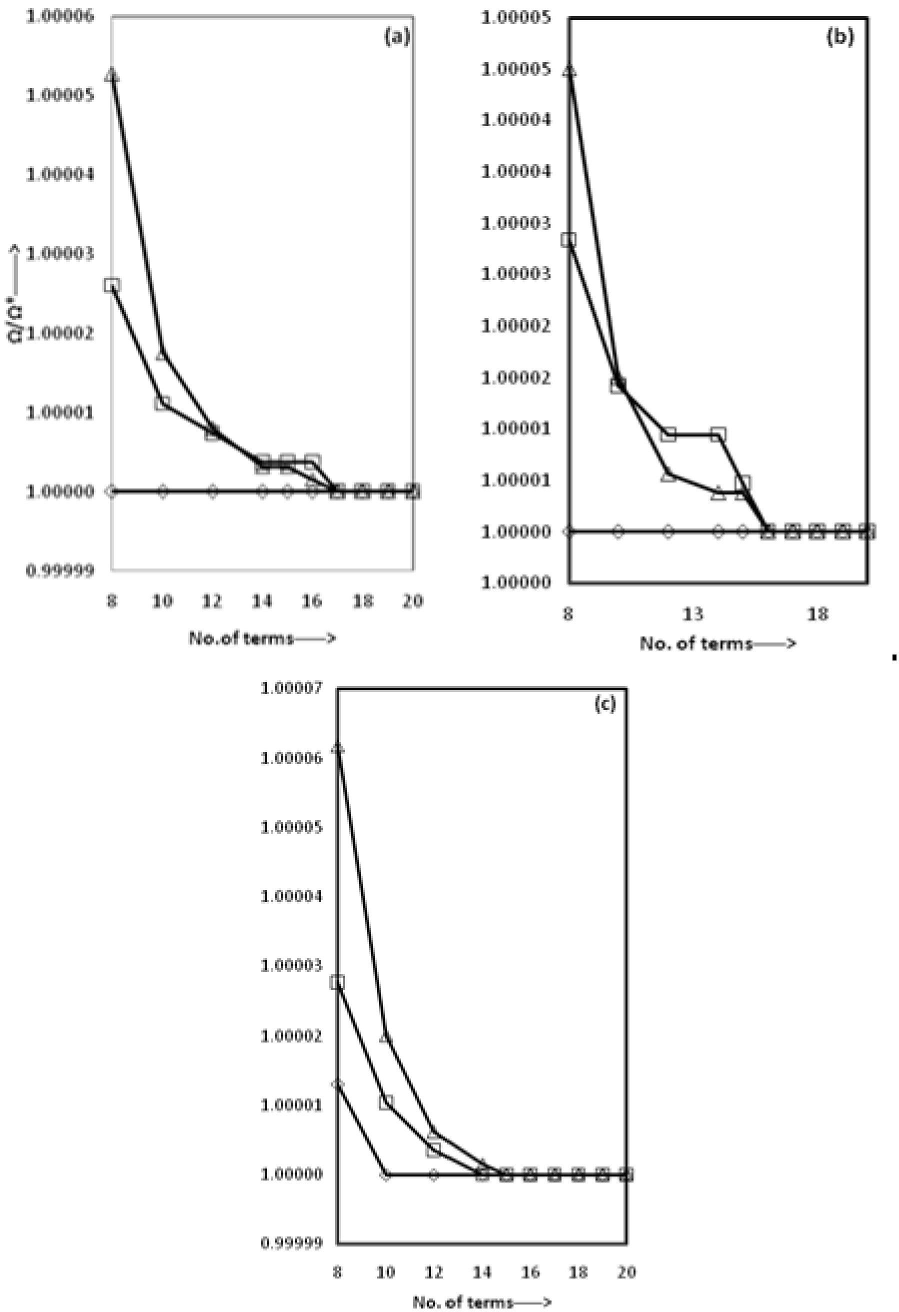

Fig. 1. Convergence of the normalized frequency parameter **Ω/Ω*** for the first three modes of vibration for $\eta = -0.5$, $\mu = -0.5$, $\alpha = -0.5$ for (a) clamped, (b) Simply-supported, (c) free plate. ◊, Fundamental mode; □, Second mode; Δ, Third mode. Ω^* the frequency using 20 terms.

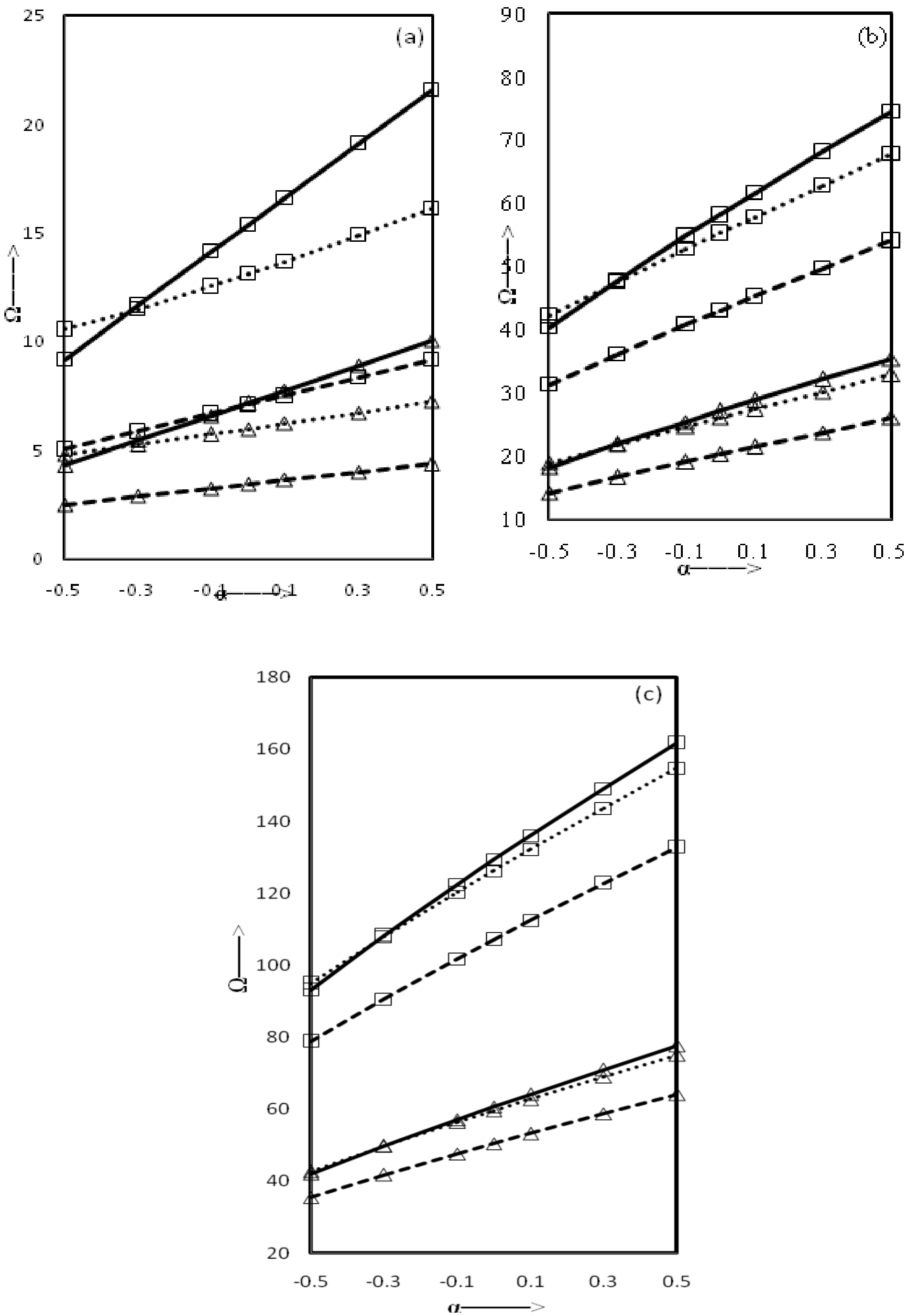

Fig. 2. Frequency parameter Ω for clamped, simply-supported and free plate vibrating in (a) fundamental (b) second and (c) third mode ————, clamped ; -----------, simply-supported; ………, free. Δ, μ = -0.5, η =1.0; □, μ = 1.0, η =-0.5;

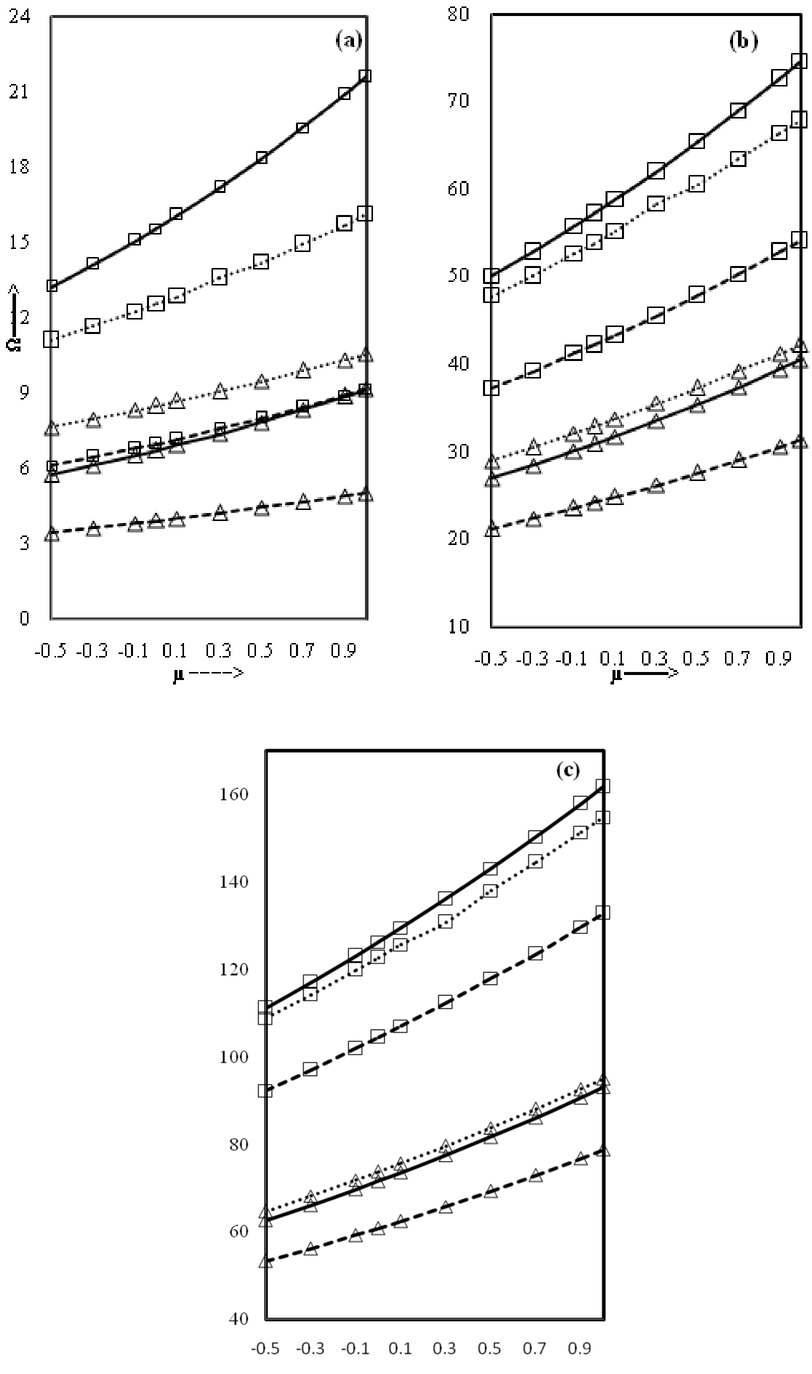

Fig. 3. Frequency parameter Ω for clamped, simply-supported and free plate vibrating in (a) fundamental (b) second and (c) third mode for η=-0.5.

————, clamped ; ---------, simply-supported; …………, free. Δ, α = -0.5; □, α = 0.5;

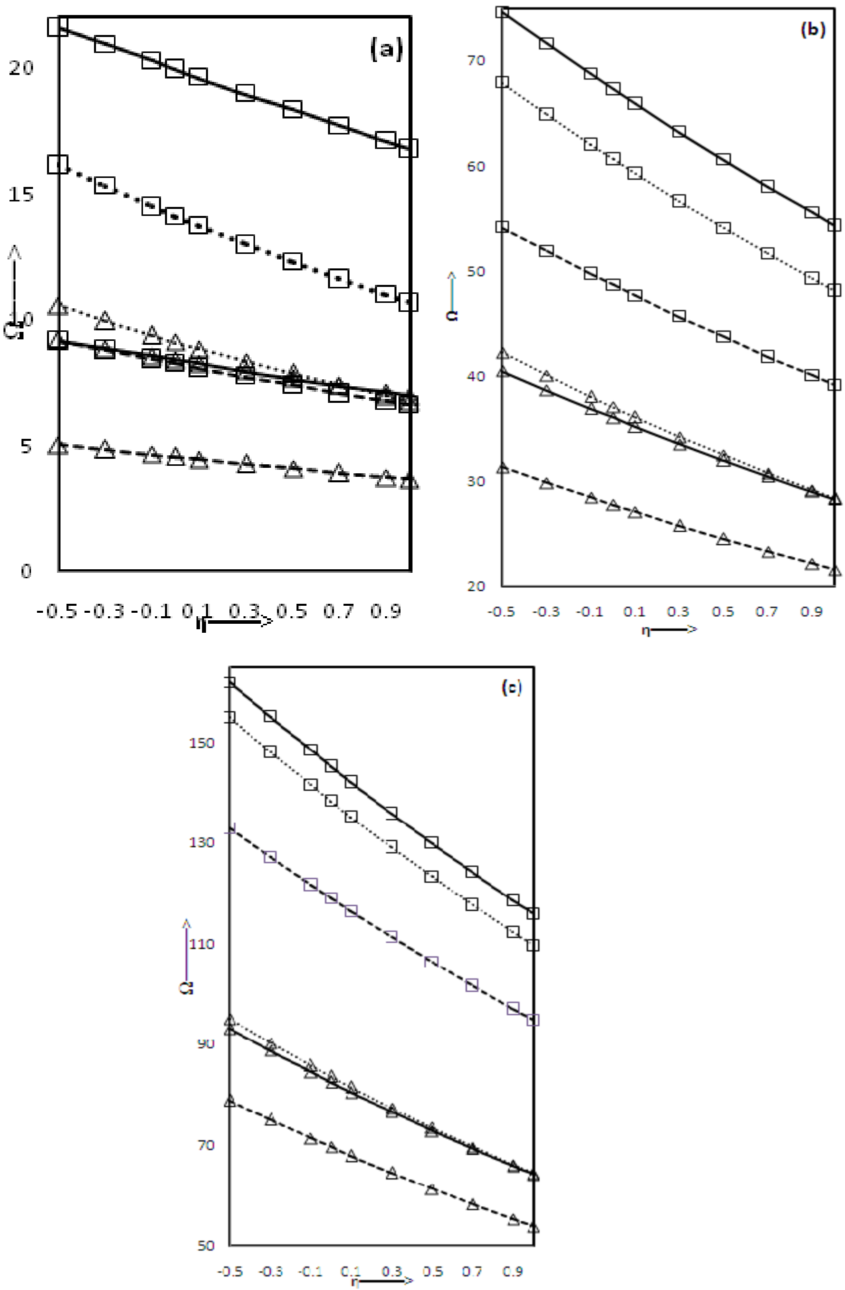

Fig. 4. Frequency parameter Ω for clamped, simply-supported and free plate vibrating in (a) fundamental (b) second and (c) third mode for μ = 1.0

———, clamped ; --------, simply-supported; ………free. Δ, α = -0.5; □, α = 0.5;

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