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Instability of a Rotating Sphere in Axial Flow*

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Abstract In many astrophysical and geophysical situations such as in the theories of sunspot magnetic fields, heating of solar corona and the instability of stellar astrospheres in magnetic fields, the instability of fluid of variable density is of considerable importance. Therefore in this paper, the hydromagnetic instability of a rotating sphere in axial flow has been studied. Assuming the fluid to be permeated by a uniform vertical magnetic field, the solution has been obtained through the use of a variational principle. The dispersion relation has been obtained for a fluid in which the density is stratified exponentially along the direction of the magnetic field. The dispersion relation has been solved graphically and it is found that the growth rate of the unstable perturbations decreases with the effect of viscosity showing thereby stabilizing influence of viscosity. The growth rate is, however, found to increase with the effects of rotation, compressibility and magnetic resistivity. Therefore, rotation, compressibility and magnetic resistivity are destabilizing the system.

Keywords: Instability, compressibility, magnetic resistivity and viscosity.

Mathematics Subject Classification: 76E07, 76E19, 76E25

1. Introduction

In many astrophysical situations such as in the theories of sunspot magnetic fields, heating of solar corona and the instability of stellar astrospheres in magnetic fields, the instability of fluid of variable density is of considerable importance. A comprehensive account of these investigations was given by Chandrasekhar[1].

Ariel[2] has investigated the instability of an inviscid compressible layer of a fluid of variable density in the presence of a uniform vertical magnetic field. Bhatia[3] studied the combined influence of viscosity and compressibility on the Rayleigh-Taylor instability of a stratified fluid.

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Sharma[4] has studied the Rayleigh-Taylor instability of compressible rotating finitely conducting inviscid plasma of variable density.

The flow field for the steady laminar incompressible boundary layer on a sphere rotating in axial flow has been studied theoretically by Schlichting[5], Hoskin[6], Lee et al.[7], Kumari and Nath[8] and El-Shaarawi et al.[9] and by experimentally by Luthander and Rydberg[10], and El-Shaarawi[11]. Their results showed marked influence of rotation on laminar separation, drag, and the critical Reynolds number, for which the drag coefficient decreases abruptly. Axial flow with the uniform velocity is directed from left to right along the axis of rotation as shown in Fig.1

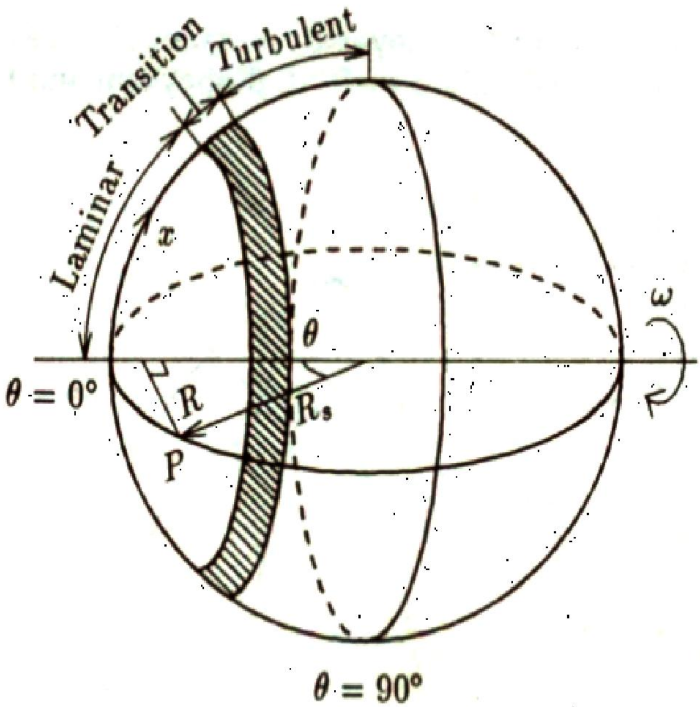

Fig. 1

Therefore, it is interesting to examine the combined effect of magnetic resistivity, viscosity and rotation of sphere on the instability of compressible fluid of variable density.

2. Perturbation Equation

The relevant linearized perturbation equations governing the motion of conducting viscous compressible rotating fluid are

$$(2.1)\qquad \rho\frac{\partial}{\partial t}\vec{u}=-\nabla\delta p+\left(\nabla\times\vec{h}\right)\times\vec{H}+\vec{g}\,\delta\rho+2\rho\left(\vec{u}\times\vec{\Omega}\right)+\mu\nabla^2\vec{u}+\frac{1}{3}\mu\nabla\left(\nabla.\vec{u}\right),$$

$$(2.2)\qquad \frac{\partial}{\partial t}\delta\rho=-\rho\nabla.\vec{u}-\left(\vec{u}.\nabla\right)\rho,$$

$$(2.3)\qquad \frac{\partial}{\partial t}\delta p=\left(\vec{u}.\nabla\right)p-C^2\left\{\frac{\partial}{\partial t}\delta\rho+\left(\vec{u}.\nabla\right)\rho\right\},$$

$$(2.4)\qquad \frac{\partial\vec{h}}{\partial t}=\nabla\times\left(\vec{u}\times\vec{H}\right)+\eta\nabla^2\vec{h},$$

$$(2.5)\qquad \nabla.\vec{h}=0,$$

where $\delta\rho,\delta p,\vec{h}\left(h_x,h_y,h_z\right)$ and $\vec{u}\left(u,v,w\right)$ are the perturbations, respectively, in density ρ, pressure p, magnetic field $\vec{H}$ and velocity $\vec{u}$. Here $\vec{g}\left(0,0,-g\right)$ is the gravity, μ is the coefficient of viscosity, η is the magnetic resistivity, C is the velocity of sound and $\vec{\Omega}$ is the angular velocity of sphere.

Assuming that, the ambient magnetic field is uniform and is acting along the vertical direction i.e. $\vec{H}=\left(0,0,H\right)$ and that the sphere is rotating about z-axis i.e. $\vec{\Omega}=\left(0,0,\Omega\right)$, we seek solutions of the above equations by analyzing the disturbance in terms of normal modes, whose dependence on space coordinates x, y and z and time t is of the form

$$(2.6)\qquad F\left(z\right)\exp\left(ik_x x+ik_y y+n\,t\right),$$

where $F\left(z\right)$ is some function of z, k_x and k_y are the horizontal wave numbers $\left(k_x^2+k_y^2=k^2\right)$ and n (may be complex) is the frequency of the harmonic disturbance.
Using (2.6) in equation (2.1)-(2.5), we get

$$(2.7)\qquad n\rho u=-ik_x\delta p+H\left(Dh_x-ik_y h_z\right)+2\rho v\Omega+\mu\left(D^2-k^2\right)u+\frac{1}{3}\mu ik_x\left(\nabla.\vec{u}\right),$$

$$(2.8)\quad n\rho v = -ik_y\delta p + H\left(Dh_y - ik_y h_z\right) - 2\rho u\Omega + \mu\left(D^2 - k^2\right)v + \frac{1}{3}\mu ik_y\left(\nabla.\vec{u}\right),$$

$$(2.9)\quad n\rho w = -D\delta p - g\delta\rho + \mu\left(D^2 - k^2\right)w + \frac{1}{3}\mu D\left(\nabla.\vec{u}\right),$$

$$(2.10)\quad n\delta\rho = -\rho\left(\nabla.\vec{u}\right) - w\left(D\rho\right),$$

$$(2.11)\quad n\delta p = -\rho C^2\nabla.\vec{u} + \rho g w,$$

$$(2.12)\quad \left[n - \eta\left(D^2 - k^2\right)\right]h_x = HDu,$$

$$(2.13)\quad \left[n - \eta\left(D^2 - k^2\right)\right]h_y = HDv,$$

$$(2.14)\quad \left[n - \eta\left(D^2 - k^2\right)\right]h_z = H\left(Dw - \nabla.\vec{u}\right),$$

$$(2.15)\quad ik_x h_x + ik_y h_y + Dh_z = 0.$$

where D stands for the operator $\frac{d}{dz}$.

Eliminating some of the variables from the above equations, we obtain

$$(2.16)\quad n^2\rho w - D\left(\rho C^2 Dw\right) + \frac{1}{H}\left[g - \frac{1}{3}\mu n D + \rho C^2 D + D\left(\rho C^2\right)\right]$$
$$\left[n - \eta\left(D^2 - k^2\right)\right]h_z + n\mu\left(k^2 - \frac{4}{3}D^2\right)w = 0.$$

and

$$(2.17)\quad \left[n-\eta\left(D^2-k^2\right)\right]\left[n^2+C^2k^2-\frac{n\mu}{\rho}\left(D^2-\frac{4}{3}k^2\right)\right]h_z-nV^2\left(D^2-k^2\right)h_z$$
$$+2nH\Omega\xi-Hk^2\left(C^2Dw-gw+\frac{n\mu Dw}{3\rho}\right)=0.$$

From equation (2.7) and (2.8), we get

$$(2.18)\qquad n\rho\zeta=HD\xi+\frac{2\rho\Omega}{H}\left[n-\eta\left(D^2-k^2\right)\right]h_z+\mu\left(D^2-k^2\right)\zeta,$$

and

$$(2.19)\quad \left[n-\eta\left(D^2-k^2\right)\right]\xi=HD\zeta,$$

where $\zeta=ik_xv-ik_yu$ and $\xi=ik_xh_y-ik_yh_x$ are respectively vertical components of the vector curl $\vec{u}$ and $\vec{h}$.

3. Boundary Conditions

We assume that the fluid under consideration is confined between two planes at $z=0$ and $z=d$. Since at the boundaries the fluid cannot have a normal component of velocity, we have $w=0$ at $z=0$ and $z=d$.
For the electromagnetic conditions at the boundaries, we have either $h_z=0$ or $Dh_z=0$ at $z=0$ and $z=d$, according to weather the boundaries are of perfectly conducting or insulating material.

If we preclude the possibility of a surface charge and surface current on the boundary, we have
$D\zeta=0,\ \xi=0$ at $z=0$ and $z=d$.

4. Variational Principle

Let us suppose that the solutions belonging to the characteristic value n_i are w_i,h_i,ζ_i and ξ_i and solutions corresponding to the characteristic value n_j are w_j,h_j,ζ_j and ξ_j, where the suffix z on h has been dropped for convenience.

Multiplying equation (2.16) for i by w_j and integrating with respect to z from $z=0$ to $z=d$, we get

$$(4.1)\quad n_i^2\int_0^d \rho w_i w_j dz-\int_0^d D\left(\rho C^2 Dw_i\right)w_j dz+n_i\int_0^d \mu\left(k^2-\frac{4}{3}D^2\right)w_i w_j dz$$
$$+\frac{1}{H}\int_0^d\left[g-\frac{1}{3}\mu nD+\rho C^2 D+D\left(\rho C^2\right)\right]\left[n-\eta\left(D^2-k^2\right)\right]h_i w_j dz=0.$$

Integrating by parts and using boundary conditions, we get

$$(4.2)\quad n_i^2\int_0^d \rho w_i w_j dz+\int_0^d \rho C^2 Dw_i Dw_j dz+n_i\int_0^d \mu\left(\frac{4}{3}Dw_i Dw_j+k^2 w_i w_j\right)dz-\frac{n_i n_j^3}{H^2k^2}\int_0^d \rho h_i h_j dz$$
$$-\frac{n_i n_j}{H^2k^2}\int_0^d C^2k^2\rho h_i h_j dz-\frac{n_i n_j^2}{H^2k^2}\int_0^d \mu\left(Dh_i Dh_j+\frac{4}{3}k^2 h_i h_j\right)dz$$
$$-\frac{\left(n_i+n_j\right)n_j^2}{H^2k^2}\int_0^d \rho\eta\left(Dh_i Dh_j+k^2 h_i h_j\right)dz-\frac{\left(n_i+n_j\right)}{H^2k^2}\int_0^d C^2k^2\rho\eta\left(Dh_i Dh_j+k^2 h_i h_j\right)dz$$
$$-\frac{n_i n_j}{k^2}\int_0^d\left(Dh_i Dh_j+k^2 h_i h_j\right)dz$$
$$-\frac{\left(n_i+n_j\right)n_j}{H^2k^2}\int_0^d \mu\eta\left[\frac{4}{3}k^2\left(Dh_i Dh_j+k^2 h_i h_j\right)+\left(D^2h_i D^2h_j+k^2 Dh_i Dh_j\right)\right]dz$$
$$-\frac{n_j^2}{H^2k^2}\int_0^d \eta^2\rho\left[\left(D^2-k^2\right)h_i\left(D^2-k^2\right)h_j\right]dz$$

$$-\frac{1}{H^2k^2}\int_0^d C^2k^2\eta^2\rho\left[\left(D^2-k^2\right)h_i\left(D^2-k^2\right)h_j\right]dz-\frac{n_j}{k^2}\int_0^d \eta\left[\left(D^2-k^2\right)h_i\left(D^2-k^2\right)h_j\right]dz$$
$$-\frac{n_j}{H^2k^2}\int_0^d \mu\eta^2\left[\left(D^2-k^2\right)Dh_i\left(D^2-k^2\right)Dh_j+\frac{4}{3}k^2\left(D^2-k^2\right)h_i\left(D^2-k^2\right)h_j\right]dz$$
$$-\frac{n_i n_j}{k^2}\int_0^d \rho\zeta_i\zeta_j dz-\frac{n_i n_j}{k^2}\int_0^d \xi_i\xi_j dz-\frac{n_j}{k^2}\int_0^d \eta\left(D\xi_i D\xi_j+k^2\xi_i\xi_j\right)dz$$
$$-\frac{n_j}{k^2}\int_0^d \mu\left(D\zeta_i D\zeta_j+k^2\zeta_i\zeta_j\right)dz=0.$$

Setting $i=j$ in equation (4.2) and considering the arbitrary variations δw, $\partial h,\delta\zeta$ and $\delta\xi$ in the corresponding physical quantities w,h,ζ and ξ compatible with boundary conditions and proceeding along usual lines we can show that $\delta n = 0$. Thus equation (4.2) (with $i = j$) provides the variational formulation of the present problem.

Making use of the existence of variational principle, we now treat the problem of instability of fluid in which undisturbed density distribution ρ is given by

$$(4.3) \quad \rho = \rho_1 \exp(\beta z),$$

where ρ_1 denotes the density at the lower boundary and β is the stratification constant.

Accordingly the velocity of sound C is given by

$$(4.4) \quad C = C_0 \exp\left(-\frac{1}{2}\beta z\right).$$

In order to ensure that the density variation within the fluid is small compared to the average density, we make an assumption that

$$(4.5) \quad |\beta d| \ll 1.$$

Now let us take the trail solution for w as

$$(4.6) \quad w(z) = A \exp\left(-\frac{1}{2}\beta z\right) \sin \alpha z,$$

where A is constant and $\alpha = \dfrac{m\pi}{d}, m$ being an integer. The corresponding solution for $h(z), \zeta(z)$ and $\xi(z)$ can be obtained from equations (2.16) – (2.18) by making use of (4.5). These are

$$(4.7) \quad h(z) = AHk^2 e^{\frac{\beta z}{2}} \frac{E_1}{E_2}\left[\left(C_0^2 + \frac{n\upsilon_0}{3}\right)\alpha \cos \alpha z - \left\{g + \frac{\beta}{2}\left(C_0^2 + \frac{n\upsilon_0}{3}\right)\right\}\sin \alpha z\right],$$

$$(4.8) \quad \zeta(z) = \frac{2A\Omega k^2 He^{\frac{\beta z}{2}}\left\{n + \eta\left(\alpha^2 + k^2\right)\right\}^2}{E_2}\left[\left(C_0^2 + \frac{n\upsilon_0}{3}\right)\alpha \cos \alpha z - \left\{g + \frac{\beta}{2}\left(C_0^2 + \frac{n\upsilon_0}{3}\right)\right\}\sin \alpha z\right],$$

$$(4.9)\quad \xi(z)=\frac{2A\Omega k^2He^{-\frac{\beta z}{2}}\left\{n+\eta\left(\alpha^2+k^2\right)\right\}^2}{E_2}\left[\left(C_0^2+\frac{n\upsilon_0}{3}\right)\alpha\cos\alpha z\right.$$
$$\left.-\left\{g+\frac{\beta}{2}\left(C_0^2+\frac{n\upsilon_0}{3}\right)\right\}\sin\alpha z\right],$$

where

$$(4.10)\quad E_1=\left\{n+\eta\left(\alpha^2+k^2\right)\right\}\left\{n+\upsilon_0\left(\alpha^2+k^2\right)\right\}+V^2\alpha^2$$

$$(4.11)\quad E_2=\left[\left\{n+\eta\left(\alpha^2+k^2\right)\right\}n^2+C_0^2k^2+n\upsilon_0\left(\alpha^2+\frac{4}{3}k^2\right)+nV^2\left(\alpha^2+k^2\right)\right]E_1$$
$$+4\Omega^2n\left\{n+\eta\left(\alpha^2+k^2\right)\right\}^2$$

Evaluating the integrals in equation (4.2) by substituting these solutions and writing

$$(4.12)\quad \sigma=\frac{n}{V\alpha},\ S=\frac{\upsilon_0\alpha}{V},\ C_0=\frac{C_0}{V},\ F=\frac{\Omega}{V\alpha},\ G=\frac{g}{V^2\alpha},\ X=1+x^2,$$
$$Y=1+\frac{3}{4}x^2, R=\frac{\eta\alpha}{V},\ x=\frac{k}{\alpha},\ a=\frac{\beta}{\alpha}.$$

we get the dispersion relation as

$$(4.13)\qquad \sum_{i=0}^{12}B_i\sigma^i=0.$$

where the coefficients B_i's are given in the appendix. The parameter S, C, F and R measure respectively the effects of viscosity, compressibility, rotation and magnetic resistivity in terms of Alfven velocity V.

5. Discussion

Stable stratification ($a=\frac{\beta}{\alpha}<0$): Appling the Hurwitz' criterion to the dispersion relation (2.13), we find that as all the terms of this equation become positive when $(a<0)$, the values of σ are either all the real and negative or there are two (or four or six or eight or ten) real and negative values and remaining are complex with negative real parts, thereby implying stability in each case. We thus find that a stable stratification remains stable whether the effects of magnetic resistivity, rotation, compressibility and viscosity are included simultaneously.

Unstable stratification ($a = \frac{\beta}{\alpha} > 0$): Appling the Hurwitz' criterion to the dispersion relation (2.13), we find that when $(a > 0)$, at least one root of σ is always real and positive for all wave numbers x. Since the dispersion relation is quit complex, we have performed graphical calculations to locate the roots of σ for the unstable mode of wave propagation for several values of physical parameters involved. These calculations are presented in Fig. (2 - 5), where we have shown growth rate (positive real value of σ) against wave number x for different values of parameters S, C, F and R taking fixed values of G = 5.0, $a = 0.1$.

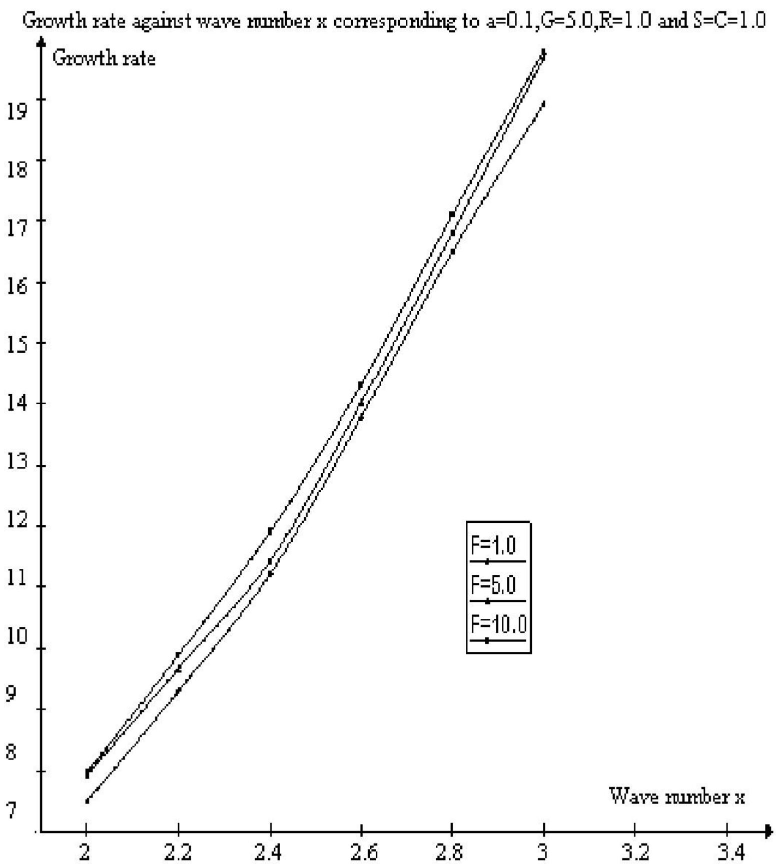

Figure 2

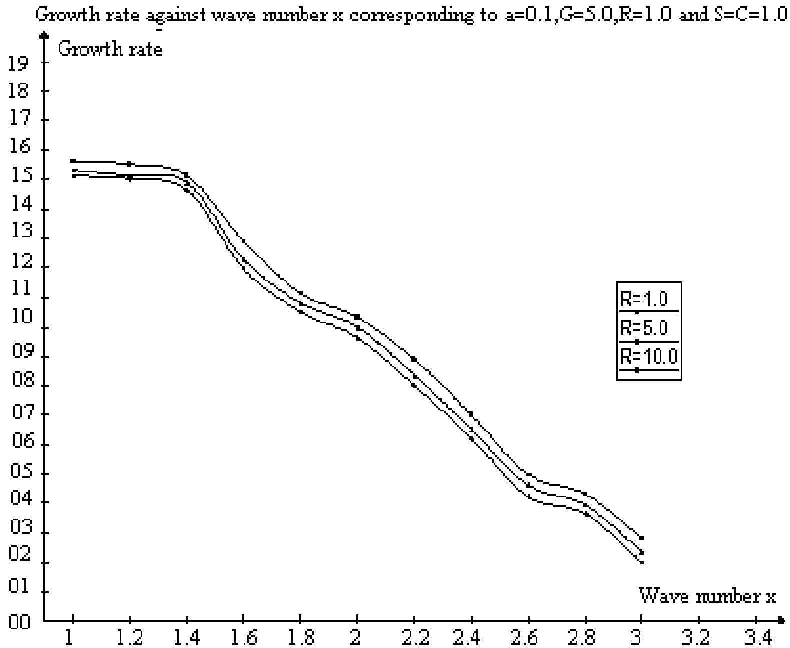

Figure 3

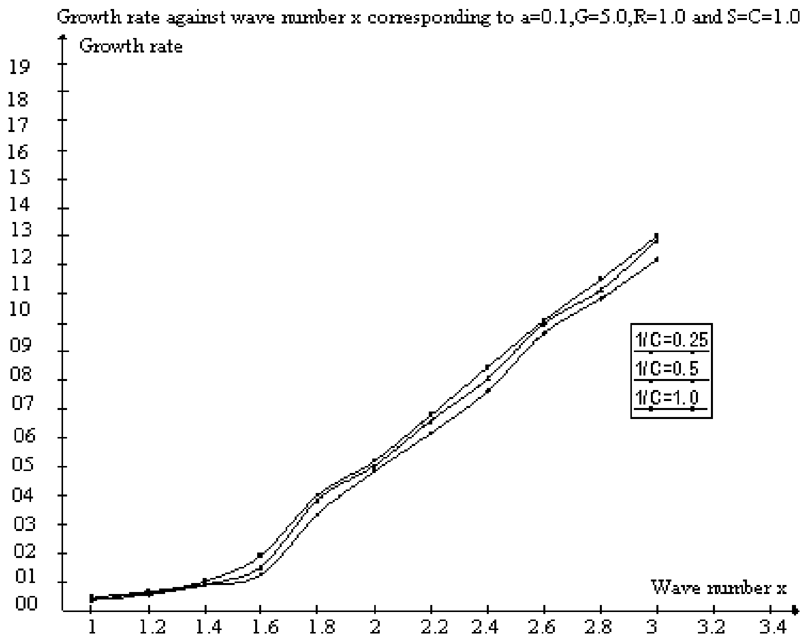

Figure 4

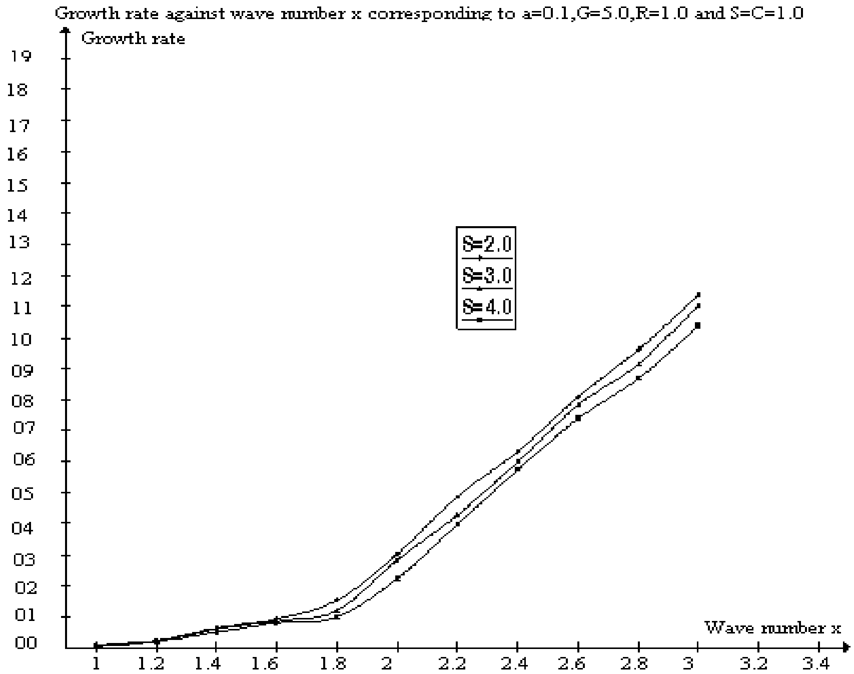

Figure 5

Fig. (2, 3 & 4) shows that for the fixed values of other parameters, the growth rate (positive real root of $\sigma \times 10^2$) increases as the parameter F(characterizing rotation), R(characterizing magnetic resistivity) and $\frac{1}{C}$ (the parameters characterizing compressibility), increase. Thus the rotation, magnetic resistivity and compressibility are destabilizing the system. Fig. (5) show that the growth rate decreases with increasing value of parameter S(characterizing viscosity) implying thereby that the viscosity has a stabilizing influence on the unstable mode of disturbance.

We may thus conclude that the effect of rotation, magnetic resistivity, and compressibility are destabilizing on the stability of a stratified layer of a hydromagnetic fluid configuration. The effect of viscosity is, however, stabilizing on the same configuration.

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