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Free Vibration Analysis of Circular Plate of Variable Thickness Resting on Pasternak Foundation*

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Abstract: In this paper, differential quadrature method (DQM) has been employed for free vibration analysis of circular plate of parabolically varying thickness resting on Pasternak foundation. The governing differential equation of motion of such plates has been derived using Hamilton's energy principle. Mode shapes and natural frequencies for different values of parameters have been presented for first three modes of vibration for two boundary conditions namely clamped and simply supported. The convergence studies have been carried out to fix the grid points required for achieving four decimal accuracy. Also, for some special cases the solutions are verified by comparing them with the published results and are found to be in excellent agreement.

Keywords: Circular Plate, Variable Thickness, Pasternak Foundation, Differential Quadrature Method.

2010 MS Classification No.: 74H45

1. Introduction

The study of vibration of plates are often encountered in engineering applications and their use in machine design, nuclear reactor technology, naval and aerospace structures are quite common. The problem of plates resting on an elastic foundation finds application in foundation engineering such as floor slabs of multi-storey buildings, foundation of deep wells and storage tanks, pavement slabs of roads and air fields. In many structural components, it is desirable or even necessary to vary the thickness of plate.

A lot of literature is available on the plates of uniform/non-uniform thickness resting on Pasternak foundation[1-7], to mention a few. Most of the studies have been devoted to rectangular plates and very little has been reported for circular plates.

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In this paper, a differential quadrature technique has been used for analysis of vibration of a circular plate with variable thickness resting on Pasternak foundation. The analysis is based on classical plate theory. An attractive advantage of the DQM, introduced by Bellman et al.[8-9], is that it can produce the acceptable accuracy of numerical results and therefore can be very useful for rapid evaluation in engineering design.

2. Mathematical Formulation

Consider an isotropic homogeneous circular plate of radius a, thickness h(r) and density ρ resting on Pasternak foundation with spring and shear stiffness K_f and G_f respectively, referred to cylindrical polar coordinate system (r, θ, z). Small deflection axisymmetric motion of such a plate is governed by the equation[7]

$$\text{(2.1)}\quad Dw_{,rrrr}+2\frac{\left(D+rD_{,r}\right)}{r}w_{,rrr}+\frac{\left(-D+\left(2+\nu\right)rD_{,r}+r^2D_{,rr}-r^2G_f\right)}{r^2}w_{,rr}+\frac{\left(D-rD_{,r}+r^2\nu D_{,rr}-r^2G_f\right)}{r^3}w_{,r}+K_fw+\rho hw_{,tt}=0,$$

where a comma followed by a suffix represents the partial differentiation with respect to that variable and $D=\dfrac{Eh^3}{12\left(1-\upsilon^2\right)}$ is the flexural rigidity of the plate, w the transverse deflection, t the time, E, ν are the Young's modulus and Poisson's ratio of the material of the plate.

Introducing the non-dimensional variables x = r/a, $\overline{w}$ = w/a , $\overline{h}$ = h/a together with the parabolic thickness variation along radial direction, i.e.

$$\text{(2.2)}\quad \overline{h}=h_0(1+\alpha x^2),$$

(2.1) reduces to

$$\text{(2.3)}\quad P_0\frac{d^4W}{dx^4}+P_1\frac{d^3W}{dx^3}+P_2\frac{d^2W}{dx^2}+P_3\frac{dW}{dx}+P_4W=0\ ,$$

where $\overline{w}(x,t)$ = $W(x)e^{i\omega t}$ (for harmonic vibrations), ω is the radian frequency, h_o is thickness of plate at the center, α is the taper parameter,

$$P_0=1,\qquad P_1=\frac{2}{x}\left[1+3A\right],$$

$$P_2 = \frac{1}{x^2}\left[-1+3(2+\upsilon)A+3A+6A^2-\frac{Gx^2}{\left(1+\alpha x^2\right)^3}\right],$$

$$P_3 = \frac{1}{x^3}\left[1-3A+\upsilon\left(3A+6A^2\right)-\frac{Gx^2}{\left(1+\alpha x^2\right)^3}\right],$$

$$P_4 = \frac{K}{\left(1+\alpha x^2\right)^3}-\frac{\Omega^2}{\left(1+\alpha x^2\right)^2} \quad , A=\frac{2\alpha x^2}{\left(1+\alpha x^2\right)} \ , K=\frac{aK_f}{D_0} \ ,$$

$$G=\frac{G_f}{aD_0} \ , D_0=\frac{E\,h_0^3}{12\left(1-\upsilon^2\right)} \ , \ \Omega^2=\frac{12\rho a^2\omega^2\left(1-\upsilon^2\right)}{E\,h_0^2} \quad .$$

Equation (2.3), which is fourth order linear differential equation with variable coefficients, involving several parameters becomes quite complex and therefore its exact solution is not possible. Equation (2.3) together with boundary conditions at the edge x=1 and regularity condition at the center x=0 constitutes boundary value problem, which has been solved numerically employing differential quadrature method (DQM).

3. Method of Solution

Let x_1, x_2 , ….., x_m be the m grid points in the applicability range [0,1] of the plate. The DQ method approximates the n^{th} order derivative of W(x) w.r.t. x at discrete point x_i as

(3.1) $$W_x^{(n)}(x_i)=\sum_{j=1}^{m} c_{ij}^{(n)} W(x_j) \ , \quad \text{i, j } = 1,2,\ldots..,m,$$

where the weighting coefficients $c_{ij}^{(n)}$ are given by

(3.2) $$c_{ij}^{(1)}=\frac{M^{(1)}(x_i)}{\left(x_i-x_j\right)M^{(1)}(x_j)} \ , \quad \text{i,j} = 1,2,\ldots..,m, \text{ but } j \neq I,$$

where

(3.3) $$M^{(1)}(x_i) = \prod_{\substack{j=1 \\ j\neq i}}^{m} \left(x_i - x_j\right),$$

(3.4) $$c_{ij}^{(n)} = n\left(c_{ii}^{(n-1)} c_{ij}^{(1)} - \frac{c_{ij}^{(n-1)}}{x_i - x_j}\right),$$ i, j = 1,2,…..,m, but j≠i and n=2,3,…..

and

(3.5) $$c_{ii}^{(n)} = -\sum_{\substack{j=1 \\ j\neq i}}^{m} c_{ij}^{(n)},$$ i=1,2,……,m .

Now, discretizing eq.(2.3) at the grid point x = x_i and substituting the values of first four derivatives of W from eq.(2.4), we get

(3.6) $$\sum_{j=1}^{m} (P_0 c_{ij}^{(4)} + P_{1,i} c_{ij}^{(3)} + P_{2,i} c_{ij}^{(2)} + P_{3,i} c_{ij}^{(1)}) W(x_j) + P_{4,i} W(x_i) = 0$$ for i=2,3,,(m-2).

The satisfaction of eq. (3.6) at (m-3) grid points x_i , i=2, 3, ….., (m-2) together with the regularity condition at the center provides a set of (m-2) equations in terms of unknowns $W_j(\equiv W(x_j))$, j=1, 2, ……, m. The resulting system of equations can be written in the matrix form as

(3.7) $$[B][W^*] = [0],$$

where B and W^* are matrices of order (m-2) x m and m x 1, respectively. The (m-2) internal grid points chosen for collocation are the zeros of shifted Chebyshev polynomial of order (m-2) with orthogonality range (0,1) given by

(3.8) $$x_k = \frac{1}{2}\left[1 + \cos\left(\frac{2k-1}{m-2}\frac{\pi}{2}\right)\right] \quad \mathrm{k} = 1, 2, \ldots (\mathrm{m}-2) \quad .$$

4. Boundary Conditions and Frequency Equations

By satisfying the relations

(i) $W = \frac{dW}{dx} = 0$: for clamped edge, and

(ii) $W = \frac{d^2W}{dx^2} + \frac{\upsilon}{x}\frac{dW}{dx} = 0$: for simply–supported edge,

a set of two homogeneous equations in terms of W_j is obtained. These equations together with field eq. (3.7) give a complete set of m equations in m unknowns. For a clamped plate, the above set of homogeneous equations can be written as

(4.1) $$\left[\frac{B}{B^c}\right]\left[W^*\right] = [0],$$

where B^c is a matrix of order 2 x m.

For a non-trivial solution of eq. (4.1), the frequency determinant must vanish and hence

(4.2) $$\left|\frac{B}{B^c}\right| = 0.$$

Similarly for simply supported plate, the frequency determinant can be written as

(4.3) $$\left|\frac{B}{B^s}\right| = 0.$$

5. Results and Discussion

The frequency equations (4.2) and (4.3) are transcendental in nature, from which infinitely many roots of Ω can be obtained. Frequencies for first three modes of vibration have been computed in both the cases of boundary conditions in view of their importance. Results have been computed accurate to fourth decimal place for the following values of plate parameters: α=-0.5(0.1)0.5; K=0(100)500; G=0(5)25; υ=0.3.

In general, the accuracy of the results is increased by increasing number of grid points. To guarantee the accuracy of frequency obtained by DQM, it is necessary to conduct some convergence studies to determine the numbers of grid points required to attain four decimal exactitude for first three modes of vibration. Table 1 is such a study for clamped and simply supported plates of parabolicallyvarying thickness. The table lists first three natural frequencies for different sets of parameters viz. α=-0.1, G=25, K=500; α=0.5, G=25, K=500; α=-0.2, G=5, K=300; α=0.3, G=5, K=100. In most of the cases, a monotonic convergence is observed. One notices that the first three natural frequencies require at least 15 grid points for four digit exactitude.

Table 1 Convergence of Normalized Frequency Parameter Ω for First Three Modes of Vibrations

m	Clamped Plate			Simple supported plate		
	I	II	III	I	II	III
	α= - 0.1, G = 25, K = 500					
12	27.7060	52.9324	99.3728	26.1208	46.1095	87.0511
13	27.7059	52.9324	99.3740	26.1208	46.1095	87.0563
14	27.7059	52.9324	99.3742	26.1208	46.1095	87.0556
15	27.7059	52.9324	99.3741	26.1208	46.1095	87.0554
16	27.7059	52.9324	99.3741	26.1208	46.1095	87.0554
17	27.7059	52.9324	99.3741	26.1208	46.1095	87.0554
18	27.7059	52.9324	99.3741	26.1208	46.1095	87.0554
19	27.7059	52.9324	99.3741	26.1208	46.1095	87.0554
20	27.7059	52.9324	99.3741	26.1208	46.1095	87.0554
α= 0.5, G = 25, K = 500						
12	28.7458	59.7614	115.6990	24.7893	48.2303	97.7206
13	28.7458	59.7615	115.6950	24.7893	48.2294	97.627
14	28.7458	59.7616	115.6960	24.7893	48.2297	97.6162
15	28.7458	59.7616	115.6970	24.7893	48.2297	97.6257
16	28.7458	59.7616	115.6970	24.7893	48.2297	97.6257
17	28.7458	59.7616	115.6970	24.7893	48.2297	97.6257
18	28.7458	59.7616	115.6970	24.7893	48.2297	97.6257
19	28.7458	59.7616	115.6970	24.7893	48.2297	97.6257
20	28.7458	59.7616	115.6970	24.7893	48.2297	97.6257
α= - 0.2, G = 5, K = 300						
12	20.5432	42.3635	86.3296	19.1245	35.2765	73.9534
13	20.5432	42.3635	86.3305	19.1245	35.2764	73.9569
14	20.5432	42.3635	86.3305	19.1245	35.2764	73.9570
15	20.5432	42.3635	86.3305	19.1245	35.2764	73.9569
16	20.5432	42.3635	86.3305	19.1245	35.2764	73.9569
17	20.5432	42.3635	86.3305	19.1245	35.2764	73.9569
18	20.5432	42.3635	86.3305	19.1245	35.2764	73.9569
19	20.5432	42.3635	86.3305	19.1245	35.2764	73.9569
20	20.5432	42.3635	86.3305	19.1245	35.2764	73.9569
α=0.3, G=5, K=100						
12	16.8174	47.6855	101.3820	12.2854	36.1687	84.2028
13	16.8174	47.6857	101.3800	12.2854	36.1682	84.1619
14	16.8174	47.6857	101.3820	12.2854	36.1684	84.1532
15	16.8174	47.6857	101.3820	12.2854	36.1684	84.1562
16	16.8174	47.6857	101.3820	12.2854	36.1684	84.1564
17	16.8174	47.6857	101.3820	12.2854	36.1684	84.1563
18	16.8174	47.6857	101.3820	12.2854	36.1684	84.1563
19	16.8174	47.6857	101.3820	12.2854	36.1684	84.1563
20	16.8174	47.6857	101.3820	12.2854	36.1684	84.1563

Table 2: Value of frequency parameter Ω for clamped plate

Mode	α	K						
		0	200			500		
		G	G			G		
		0	0	10	25	0	10	25
I	-0.5	6.6320	16.2651	18.2678	20.8848	24.3693	25.7691	27.7019
	-0.3	8.0760	16.6173	18.5634	21.1097	24.3349	25.7143	27.6189
	-0.1	9.5055	17.1357	19.0257	21.5027	24.4647	25.8272	27.7059
	0	10.2158	17.4460	19.3082	21.7514	24.5838	25.9385	27.8051
	0.1	10.9235	17.7857	19.6201	22.0302	24.7352	26.0823	27.9373
	0.3	12.3317	18.5416	20.3211	22.6666	25.1253	26.4576	28.2901
	0.5	13.7310	19.3832	21.1088	23.3920	25.6178	26.9351	28.7458
II	-0.5	30.0152	33.8117	39.0136	45.6885	38.8265	43.4135	49.4836
	-0.3	34.1610	37.2529	41.7257	47.6460	41.4628	45.5197	50.9982
	-0.1	37.9627	40.5844	44.5637	49.9304	44.2267	47.9037	52.9324
	0	39.7711	42.2107	45.9974	51.1398	45.6261	49.1504	53.9933
	0.1	41.5301	43.8126	47.4323	52.3771	47.0292	50.4189	55.0965
	0.3	44.9242	46.9491	50.2929	54.9046	49.8329	52.9960	57.3913
	0.5	48.1833	50.0051	53.1293	57.4702	52.6208	55.5993	59.7616
III	-0.5	69.8624	71.6173	78.0160	86.6786	74.1747	80.3658	88.7945
	-0.3	78.1241	79.5409	84.8042	92.1282	81.6203	86.7565	93.9271
	-0.1	85.5877	86.7862	91.3539	97.8030	88.5536	93.0343	99.3741
	0	89.1041	90.2194	94.5297	100.6458	91.8670	96.1034	102.1253
	0.1	92.5050	93.5491	97.6424	103.4740	95.0938	99.1235	104.8730
	0.3	99.0173	99.9456	103.6920	109.0630	101.3220	105.0200	110.3260
	0.5	105.2133	106.0509	109.5309	114.5425	107.2953	110.7367	115.6968

Table 2 and 3 present the values of frequency parameter Ω for clamped and simply supported plates respectively for α=-0.5, -0.3, -0.1, 0.0, 0.1, 0.3, 0.5; K=0, 200, 500 and G=0, 10, 25, for first three modes of vibration.

Comparison of frequency parameter Ω with the results obtained by Ansari[10], Azimi[11] and exact results given by Leissa[12] for clamped and simply supported plates of uniform thickness (α=0.0) without foundation (K=0, G=0) , is given in Table 4 for first three modes of vibration.A comparison of our results for clamped and simply supported plates of parabolically varying thickness without foundation (K=0, G=0) is shown in Table 5 with the results obtained by Ansari[10], Lal[13], Gutierrez et al.[14].

Figure 1 depicts the variation of frequency parameter Ω with respect to taper parameter α for clamped and simply supported plates vibrating in fundamental mode. It is observed that in the absence of foundation, frequency parameter Ω increases by increasing the value of taper parameter α for both the plates. In presence of Winkler foundation, frequency parameter Ω increases by increasing α for clamped plate while for simply supported plate it decreases by increasing α. In presence of Pasternak foundation, for clamped plate frequency parameter Ω first decreases and then increases with a local minima in the vicinity of α=-0.3 while frequency parameter Ω decreases continuously by increasing α for simply supported plate.

Table 3: Value of Frequency Parameter Ω for Simply Supported Plate

Mode	α	K						
		0	200			500		
		G	G			G		
		0	0	10	25	0	10	25
I	-0.5	4.0392	15.5894	17.5564	20.1405	24.0689	25.3796	27.2268
	-0.3	4.4034	15.3087	17.2208	19.7376	23.5778	24.8556	26.6570
	-0.1	4.7576	15.0768	16.9303	19.3775	23.1140	24.3620	26.1208
	0	4.9351	14.9785	16.8025	19.2145	22.8988	24.1314	25.8689
	0.1	5.1142	14.8914	16.6861	19.0629	22.6959	23.9127	25.6286
	0.3	5.4787	14.7490	16.4865	18.7937	22.3273	23.5122	25.1846
	0.5	5.8537	14.6467	16.3288	18.5681	22.0067	23.1599	24.7893
II	-0.5	23.8870	28.6439	34.7876	42.1888	34.6074	39.7969	46.3680
	-0.3	26.3765	30.3273	35.5619	42.1746	35.4460	40.0104	45.9807
	-0.1	28.6437	32.0494	36.6444	42.6145	36.5686	40.6561	46.1095
	0	29.7200	32.9132	37.2563	42.9558	37.1925	41.0857	46.3163
	0.1	30.7664	33.7759	37.9007	43.3582	37.8445	41.5669	46.5963
	0.3	32.7863	35.4934	39.2588	44.3045	39.2081	42.6445	47.3281
	0.5	34.7290	37.1968	40.6782	45.3880	40.6244	43.8310	48.2297
III	-0.5	59.9533	62.0274	69.5996	79.4460	65.0210	72.2791	81.7972
	-0.3	66.0394	67.7227	73.7764	81.9962	70.1731	76.0325	84.0317
	-0.1	71.5534	72.9855	78.1248	85.2533	75.0824	80.0874	87.0554
	0	74.1561	75.4925	80.2995	87.0135	77.4540	82.1463	88.7207
	0.1	76.6756	77.9303	82.4604	88.8225	79.7755	84.2063	90.4456
	0.3	81.5074	82.6292	86.7235	92.5210	84.2841	88.3018	94.0019
	0.5	86.1120	87.1299	90.8964	96.2609	88.6351	92.3404	97.6257

Table 4 Comparison of Frequency Parameter for Circular Plate of Uniform Thickness

Mode	**Clamped plate**		**S-S plate**	
I	10.2158	10.2158*	4.9351	4.977*
	10.2158°	10.216$^{\lozenge}$	4.9352°	4.935$^{\lozenge}$
II	39.7711	39.771*	29.72	29.76*
	39.7711°	39.771$^{\lozenge}$	29.7200°	29.720$^{\lozenge}$
III	89.1041	89.104*	74.1561	74.20*
	89.1041°	89.103$^{\lozenge}$	74.1961°	74.156$^{\lozenge}$

Table 5: Comparison of frequency parameter for circular plate of parabolic thickness variation

α	Clamped						S-S					
	I		II		III		I		II		III	
-0.5	6.6320	6.6303*	30.0152	30.0130*	69.8624	69.8709*	4.0392	4.0391*	23.887	23.8884*	59.9533	59.9567*
-0.3	8.0759	8.0748*	34.161	34.1768*	78.1241	78.1086*	4.4034	4.4029*	26.3765	26.3757*	66.0394	66.0258*
	8.0759°	8.076◊	34.1610°	34.161◊			4.4034°	4.403◊	26.3765°	26.376◊		
-0.1	9.5055	9.5055*	37.9627	37.9631*	85.5877	85.5598*	4.7576	4.7562*	28.6437	28.6447*	71.5534	71.5579*
	9.5055°	9.505◊	37.9627°	37.963◊			4.7576°	4.758◊	28.6437°	28.644◊		
0.1	10.9235	10.9223*	41.5301	41.5380*	92.505	92.5123*	5.1142	5.1130*	30.7664	30.7682*	76.6758	76.6675*
	10.9235°	10.924◊	41.5301°	41.529◊			5.1142°	5.114◊	30.7664°	30.768◊		
0.3	12.3317	12.3287*	44.9242	44.9329*	99.0172	99.2534*	5.4787	5.4802*	32.7863	32.7877*	81.5074	81.5172*
	12.3317°	12.332◊	44.9242°	44.921◊			5.4787°	5.479◊	32.7863°	32.786◊		
0.5	13.731	13.7317*	48.1833	48.1822*	105.2133	—	5.8537	5.8509*	34.729	34.7138*	86.112	—

○ values obtained by Ritz method10, * values obtained by Frobenius method13, ◊ values obtained by Rayleigh-Ritz method14

Figure 2 shows the effect of taper parameter α on frequency parameter Ω for clamped and simply supported plate vibrating in second mode. It is seen that in the absence of foundation and in presence of Winkler foundation, frequency parameter Ω increases by increasing α for both the plates. For the plates resting on Pasternak foundation, the effect of increasing α is that frequency parameter Ω increases in case of clamped boundary while frequency parameter Ω first decreases and then increases with a local minima in the vicinity of α=-0.3.

Figure 3 shows the behavior of frequency parameter Ω versus taper parameter α for both the plates vibrating in third mode. It is found that frequency parameter Ω increases by increasing α in all the cases for both the plates.

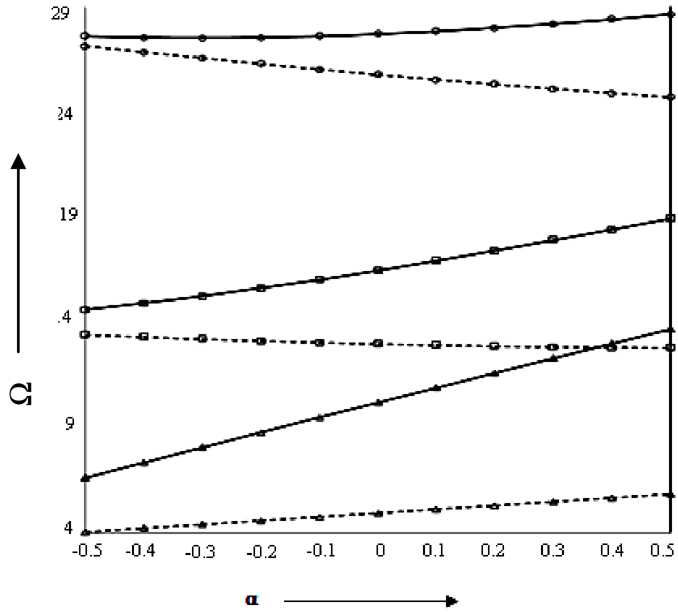

Figure 1: Frequency parameter Ω for plate vibrating in fundamental mode.
Δ : K=0, G=0; □ : K=500, G=0 ; ○ : K=500, G=25.
———— : Clamped Plate ; ------------ : Simply Supported Plate.

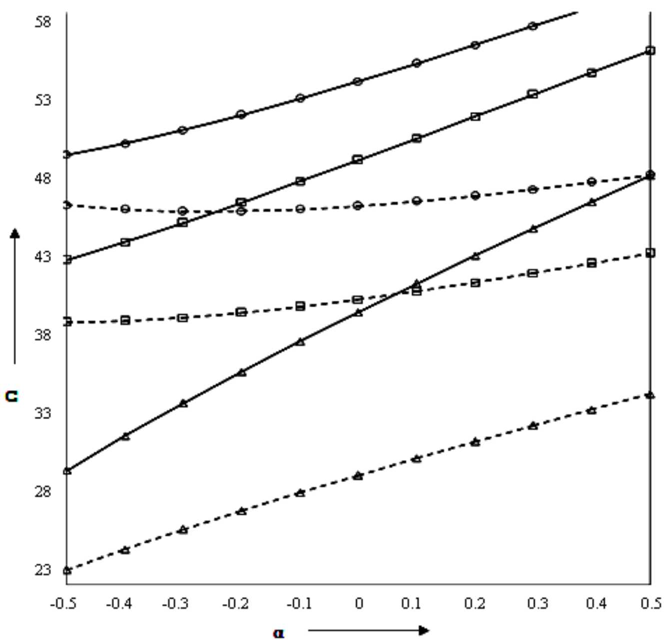

Figure 2: Frequency parameter Ω for plate vibrating in second mode.
Δ : K=0, G=0; □ : K=500, G=0 ; ○ : K=500, G=25.
———— : Clamped Plate ; ------------ : Simply Supported Plate.

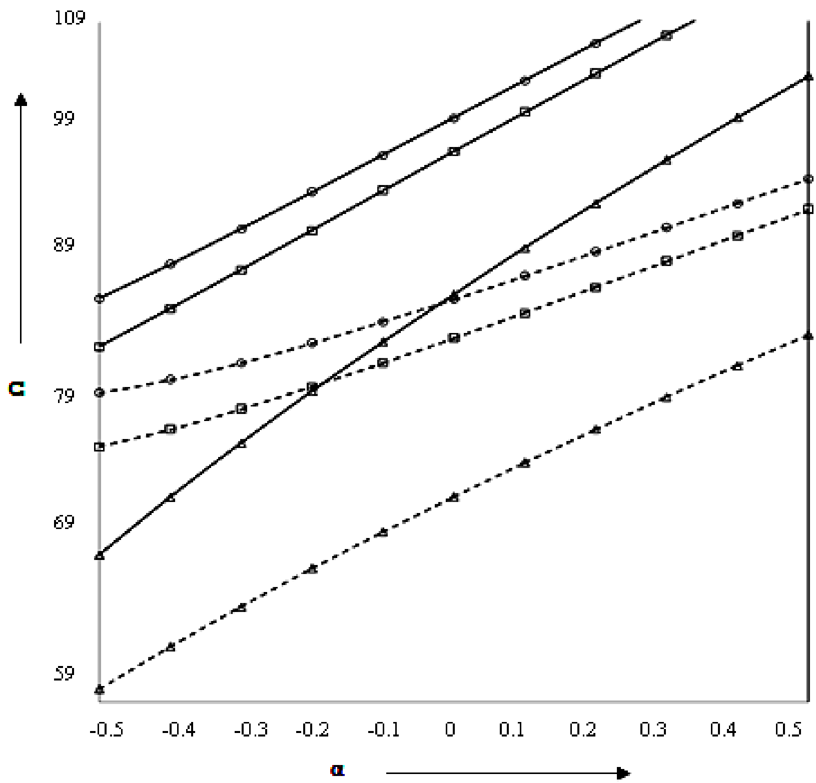

Figure 3: Frequency parameter Ω for plate vibrating in third mode
Δ : K=0, G=0; □ : K=500, G=0 ; ○ : K=500, G=25.
———— : Clamped Plate ; ------------ : Simply Supported Plate.

Also the frequency parameter Ω increases by increasing spring stiffness parameter K as well as shear stiffness parameter G for both the plates. The rate of increase with K as well as G gets reduced by increasing α for both the plates for all the three modes of vibration.

Figures (4 A and B) show the plots of normalized transverse displacements for α=-0.5, 0.5; K=0, G=0; K=500, G=0; K=500; G= 25, for the first three modes of vibration for clamped and simply supported plates respectively . The nodal radii of the plates decrease by increasing K as well as G for α=-0.5 for both the plates except for clamped plate where the radii of nodal circle decreases in the order of no foundation, Pasternak foundation, Winkler foundation for first two modes. Further nodal radii increases by increasing the value of K as well as G for α=0.5 for both the plates.

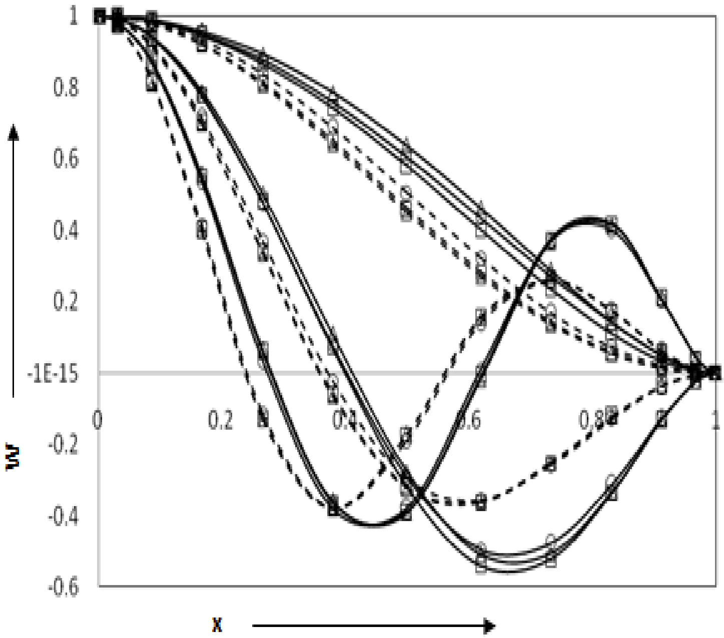

Fig. 4: A. Normalized Transverse Displacement for Clamped Plate.
Δ : K=0, G=0; □ : K=500, G=0; ○ : K=500, G=25.
———— : α= -0.5; ------------------ α=0.5

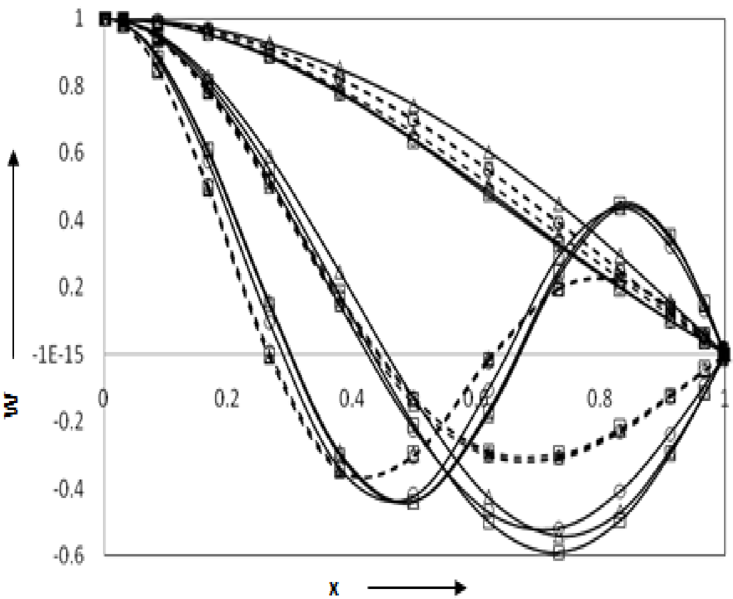

Fig. 4 B. Normalized Transverse Displacement for Simply Supported Plate.
Δ : K=0, G=0; □ : K=500, G=0; ○ : K=500, G=25.
———— : α= -0.5; ---------------- : α=0.5.

5. Conclusion

The vibration of circular plates of parabolically varying thickness resting on Pasternak foundation has been studied in the present paper employing DQM. The method is straight forward and is capable of determining frequencies and mode shapes as close to the exact ones as desired. Results have also been presented for Winkler foundation as a special case of Pasternak foundation and for the plate without foundation. The following conclusions are drawn from numerical results reported in the previous section.

1. The frequency parameter increases by increasing the value of taper parameter α.
2. Frequency parameter Ω increases by increasing the foundation parameters K and G.

The results have been known for the first time and will be useful for further research workers & design engineer.

References

1. C. M. Wang, C. Wang and K. K. Ang, Vibration of initially stressed Reddy plates on a Winkler-Pasternak foundation,J. Sound Vib., 204(2)(1997)203-212.
2. M. H. Omurtag and F. Kadioglu, Free vibration analysis of orthotropic plates resting on Pasternak foundation by mixed finite element formulation. J. Comp. Struc, 67(4)(1998)253-265.
3. H. S. Shen, J. Yang and L. Zhang, Free and forced vibration ofReissner-Mindlin plates with free edges resting on elastic foundations. J. Sound Vib., 244(2)(2001)299-320.
4. C. M. Wang, Y. Xiang and Q. Wang, Axisymmetric buckling of Reddy circular plates on Pasternak foundation. J. Engg. Mech.-ASCE, 127(3) (2001)254-259.
5. P. Malekzadeh and G. Karami, Vibration of non-uniform thick plates on elastic foundation by differential quadrature method. J. Engg. Struc., 26(2004)1473-1482.
6. A. V. T. Leung and B. Zhu, Transverse vibration of Mindlin plates on two parameter foundation by Trapezoidal p- elements, J. Eng. Mech., 131(11)(2005)1140-1145.
7. S. Sharma, Free vibration studies on non-homogeneous circular and annular plates, PhD thesis, Department of Mathematics, I. I. T. Roorkee, Roorkee, India.
8. R. E. Bellman and J. Casti, Differential Quadrature and long term integration. J. Math. Analysis App., 34 (1971) 235-238.
9. R. E. Bellman, B.G. Kashef and J. Casti, Differential quadrature technique for the rapid solution of nonlinear partial differential equation. J. Comp. Phy. 10 (1972) 40-52.

10. A. H. Ansari, Vibration of plates of variable thickness, PhD Thesis, University of Roorkee, 2000.

11. S. Azimi, Free vibration of circular plates with elastic edge supports using the receptence method., J. Sound Vib, 120(1) (1988) 19-36.

12. A. W. Leissa, Vibration of Plates, NASA-SP, 1969.

13. R. Lal, Vibrations of elastic plates of variable thickness, PhD Thesis, University of Roorkee, 1979.

14. R. H. Gutierrez, E. Romanelli and P. A. A. Laura, Vibration and elastic stability of thin circular plates with variable profile, J. Sound Vib., 195(3)(1996) 391-399.