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An SIR Model to Study the Effects of Ecological Factors on the Spread of Carrier Dependent Infectious Diseases*

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Abstract: In this paper, an SIR model to study the spread of carrier dependent infectious diseases in a population with constant immigration is proposed and analyzed by considering ecological effects. It is assumed that the density of carrier population is governed by a generalized logistic model, the growth rate of which increases as the cumulative density of ecological factors increases. The cumulative density of ecological factors is also assumed to be governed by the population density dependent logistic model. The model is analyzed by using qualitative theory of differential equations and by computer simulation.

It is shown that as the density of the carrier population, caused by ecological factors increases, the infectious disease spreads faster and it becomes more endemic. The growth of human population due to immigration further enhances the spread of infectious diseases.

Keywords: Infectious diseases; Carrier; Ecological factors; Modified carrying capacity; Immigration; Simulation.

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1. Introduction

There are many carrier dependent infectious diseases which afflict human population around the world. In particular, the regions in developing countries, which are situated in equatorial zones, are most affected by such diseases[1,2,3,4,5,6]. In such regions, the villages, towns and cities lack sanitation in general, which results in the presence of various kinds of carriers, such as flies, ticks, mites and others in the habitat. These carriers grow and survive in plant vegetation, bushes in residential areas near open drainage, garbage storages, parks, water storage tanks and ponds, etc.[3,7]. It may be pointed out that their population grow in the mounds formed by the roots of the bushes and plants as well as on the branches and the leaves[8,9,10].

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The carrier population increases further as the bushes and plants become denser and denser. It is also noted that densities of these ecological factors (such as biomass of leaves in bushes and plants etc.) may change due to the growth of human population in the area because of human interactions with these factors. Therefore in this paper, an SIR model is proposed for the spread of carrier dependent infectious diseases by considering explicitly the effect of ecological factors which may depend upon human population. To be specific in the modelling process, we consider that the cumulative density of ecological factors is governed by a logistic type equation which is modified by human interactions.

2. SIR Model with Constant Immigration

Let the total human population density $N(t)$ is divided into three classes: the susceptible density $X(t)$, the infective density $Y(t)$ and the density of removed class $Z(t)$, *i.e.* $N = X + Y + Z$. If $B(t)$ is the cumulative density of ecological factors as biomass of leaves in bushes and plants etc., favourable to the growth of carrier population then it is assumed that the cumulative density of ecological factors $B(t)$ is governed by a generalized logistic equation. Further, let $C(t)$ be the carrier population density whose logistic growth rate coefficient $s(B)$ and modified carrying capacity $L(B)$ depend upon the cumulative density of ecological factors $B(t)$.

Keeping in view the above factors and by assuming simple mass action interaction, an SIR model is proposed as follows:

$$\frac{dX}{dt} = A - \beta XY - \lambda XC - dX ,$$

$$\frac{dY}{dt} = \beta XY + \lambda XC - (\nu_1 + \alpha + d)Y ,$$

$$\frac{dZ}{dt} = \nu_1 Y - dZ ,$$

(1) $$\frac{dN}{dt} = A - dN - \alpha Y ,$$

$$\frac{dC}{dt} = s(B)C - \frac{s_0 C^2}{L(B)} - s_1 C = \{s(B) - s_1\}\left[C - \frac{C^2}{\{L(B)/s_0\}\{s(B) - s_1\}}\right],$$

$$\frac{dB}{dt} = r_0 B - \frac{r_0 B^2}{K} - r_1 B + r_2 BN = rB - (r_0 / K)B^2 + r_2 BN \text{, where } r_0 - r_1 = r$$

$$X + Y + Z = N,$$

$$X(0) > 0, Y(0) \geq 0, Z(0) > 0, C(0) \geq 0, B(0) > 0.$$

For feasibility of the model (1), we must have $s(B) - s_1 > 0$ for $B \geq 0$. In the above model, A is constant immigration rate of human population, d is natural death rate constant, β and λ are the transmission coefficients due to infective and carrier population respectively, α is the disease related death rate constant, s_1 is the death rate coefficient of carriers due to natural factors as well as by control measures. Further, r_0 is the natural growth rate coefficient of $B(t)$, r_1 is the natural depletion rate coefficient of $B(t)$, r_2 is the growth rate coefficient of $B(t)$ due to human population density related factors and K is the carrying capacity of $B(t)$ which is assumed to be a constant, ν_1 is recovery rate constant *i.e.* the rate at which individuals recover and transfer to the removed class from the infective class. In (1), $s(B)$ is the growth rate per capita of the carrier population density such that $s(B) - s_1$ is its intrinsic growth rate as compared to the usual logistic model. In view of the assumption that the growth rate per capita $s(B)$ increases as the cumulative density of ecological factors $B(t)$ increases, so we have,

(2) $$s(0) = s_0 > 0 \text{ and } \frac{ds(B)}{d(B)} \geq 0,$$

where s_0 is the value of $s(B)$ when $B = 0$. It may be pointed out here that, when $s(B)$ is independent of B, it takes constant value and that value is assumed to be s_0. Further, $s_0 > s_1$ as growth rate coefficient of carrier must be greater than it's control rate coefficient for it's existence.

It is assumed that the modified carrying capacity $L(B)$ increases as $B(t)$ increases, so we have

(3) $$L(0) = L_0 > 0 \quad \text{and} \quad \frac{dL(B)}{dB} \geq 0\ ,$$

where L_0 is the value of $L(B)$ when $B = 0$. It is to be noted here that, when $L(B)$ is independent of B, it takes constant value and that value is assumed to be L_0.

3. Equilibrium Analysis

To analyze the model (1), we consider the following reduced system (since $X + Y + Z = N$):

(4)
$$\begin{aligned}
\frac{dY}{dt} &= \beta(N - Y - Z)Y + \lambda(N - Y - Z)C - (\nu_1 + \alpha + d)Y\ , \\
\frac{dZ}{dt} &= \nu_1 Y - dZ\ , \\
\frac{dN}{dt} &= A - dN - \alpha Y\ , \\
\frac{dC}{dt} &= s(B)C - \frac{s_0 C^2}{L(B)} - s_1 C\ , \\
\frac{dB}{dt} &= rB - \frac{r_0 B^2}{K} + r_2 BN\ .
\end{aligned}$$

The following lemma is needed for analysis of the model (4) which is stated without proof.

Lemma: *The set* $W = \left\{ \begin{matrix} (Y, Z, N, C, B) : 0 \leq Y \leq N \leq A/d, \\ 0 \leq Z \leq N \leq A/d, 0 \leq C \leq C_m, 0 < B \leq B_m \end{matrix} \right\}$ *attracts all solutions initiating in the positive orthant, where* $B_m = (K/r_o)\{r + (r_2 A/d)\}$ *and* $C_m = \{L(B_m)/s_0\}\{s(B_m) - s_1\}$.

Now, we give the result of equilibrium analysis in the following theorem in terms of the basic reproduction number $R_0 = \dfrac{\beta A}{d(\nu_1 + \alpha + d)}$; which determines biologically whether the disease dies out or spreads under the given conditions.

Theorem 1: *There exist following six equilibria of the system (4)*

(i) $P_0 = (0,0,A/d,0,0)$*, which is a disease free equilibrium;*

(ii) $P_1 = (0,0,A/d,0,B_m)$*, where* $B_m = \{K/r_0\}\{r + (r_2 A/d)\}$

(iii) $P_2 = (\bar{Y}_1, \bar{Z}_1, \bar{N}_1, 0,0)$ *which exists provided the basic reproduction number* $R_0 = \dfrac{\beta A}{d(\nu_1 + \alpha + d)} > 1$ *;where* $\bar{Y}_1 = \dfrac{\beta A - d(\nu_1 + \alpha + d)}{\beta[\alpha + d(1 + (\nu_1/d))]}$, $\bar{Z}_1 = (\nu_1/d)\bar{Y}_1$,

$$\bar{N}_1 = \frac{1}{\alpha + d(1 + (\nu_1/d))}\{A(1 + (\nu_1/d)) + (\alpha/\beta)(\nu_1 + \alpha + d)\},$$

(iv) $P_3 = (\bar{Y}_1, \bar{Z}_1, \bar{N}_1, 0, \bar{B}_1)$ *which exists if the basic reproduction number* $R_0 > 1$*, where* $\bar{B}_1 = \{K/r_0\}\{r + r_2 \bar{N}_1\}$*,* $\bar{Y}_1, \bar{Z}_1$ *and* $\bar{N}_1$ *are defined above,*

(v) $P_4 = (\tilde{Y}_1, \tilde{Z}_1, \tilde{N}_1, \tilde{C}_1, 0)$*, where*

$$\tilde{Y}_1 = \left[V + \sqrt{V^2 + 4\beta(1 + (\alpha/d) + (\nu_1/d))(\lambda A/d)\tilde{C}_1}\right] / 2\beta(1 + (\alpha/d) + (\nu_1/d))$$

where $V = (\beta A/d) - (\nu_1 + \alpha + d) - \lambda(1 + (\alpha/d) + (\nu_1/d))\tilde{C}_1$, $\tilde{Z}_1 = (\nu_1/d)\tilde{Y}_1$,

$\tilde{N}_1 = (A - \alpha\tilde{Y}_1)/d$ *and* $\tilde{C}_1 = (L_0/s_0)(s_0 - s_1)$,

(vi) $P_5 = (\hat{Y}, \hat{Z}, \hat{N}, \hat{C}, \hat{B})$ *exists in the subregion*

$$W_s = \{(Y,Z,N,C,B) : A/(\alpha + d) \le Y \le N \le A/d,\ 0 \le Z \le N \le A/d,$$
$$0 \le C \le C_m,\ 0 < B \le B_m\}$$

of W *provided the basic reproduction number* $R_0 = \dfrac{\beta A}{d(\nu_1 + \alpha + d)} < 1$*. Here* P_5 *corresponds to persistence of disease.*

Proof: The existence of equilibrium points P_0, P_1, P_2 ,P_3 or P_4 is easy to prove. In the following we prove the existence of P_5. Existence of Equilibrium Point $P_5 = (\hat{Y}, \hat{Z}, \hat{N}, \hat{C}, \hat{B})$:

We prove the existence of P_5 by the isocline method .The equilibrium point P_5 is obtained from the following equations, by putting left hand sides of (4) equal to zero:

(5) $\beta Y^2 + Y[(\nu_1 + \alpha + d) - \beta(N - Z) + \lambda C] - \lambda(N - Z)C = 0$

(6) $Z = \nu_1 Y/d$

(7) $Y = (A - dN)/\alpha$

(8) $C = \{s(B) - s_1\}\{L(B)/s_0\}$,

(9) $B = (K / r_0)(r + r_2 N)$.

On substituting the value of Z from (6) into (5), we get

(10) $\beta\left(1+(\nu_1/d)\right)Y^2 + Y\left[\lambda\left(1+(\nu_1/d)\right)C + (\nu_1+\alpha+d) - \beta N\right] - \lambda NC = 0$.

Eliminating Y from (7) and (10), we get following relation

$$F_1(N) \equiv (\beta/\alpha^2)\left(1+(\nu_1/d)\right)(A-dN)^2$$
$$+\{(A-dN)/\alpha\}\left[\lambda\left(1+(\nu_1/d)\right)C + (\nu_1+\alpha+d) - \beta N\right] - \lambda NC = 0,$$

where C and B are given by (8) and (9) respectively.
Now, it is noted here that

$$F_1\left(A/(\alpha+d)\right) \equiv (\beta A^2/(\alpha+d)^2)\left(1+(\nu_1/d)\right) + (\lambda\nu_1 A/s_0 d(\alpha+d)).$$

(11) $\{s((rK/r_0)+(r_2KA/r_0(\alpha+d))) - s_1\}L((rK/r_0)+(r_2KA/r_0(\alpha+d)))$

$$+(A/(\alpha+d))\left[\left((\nu_1+\alpha+d\right) - (\beta A/(\alpha+d))\right] > 0,$$

provided the basic reproduction number $R_0 = \dfrac{\beta A}{d(\nu_1+\alpha+d)} < 1$.

Further

(12) $F_1\left(A/d\right) = -\lambda A C_m / d < 0$,

where $C_m = \{s(B_m) - s_1\}\{L(B_m)/s_0\}$ and $B_m = (K/r_0)\left(r + (r_2 A/d)\right)$.

Thus it is concluded from (11) and (12) that there exists a root $\hat{N}$ of $F_1(N) = 0$ in $A/(\alpha+d) < N < A/d$ provided $R_0 < 1$ and it is unique provided

(13) $F_1'(N) < 0$ for $A/(\alpha+d) < N < A/d$.

Thus, knowing the value of $\hat{N}$ we can compute the values of $\hat{Y}, \hat{Z}, \hat{C}$ and $\hat{B}$ from equations (6) to (9). Hence the equilibrium P_5 exists and is unique provided (13) is satisfied.

Remarks

(a) We note that

$$\left.\frac{dF_1(N)}{dN}\right|_{P_5} = -(d/\alpha)\left\{\beta\hat{Y}(1+(\nu_1/d)+(\lambda\hat{N}\hat{C}/\hat{Y})\right\} - \beta\hat{Y} - \lambda\hat{C}$$

$$-(\lambda K r_2/s_0 r_0)\left[s'(\hat{B})L(\hat{B}) + \left\{s(\hat{B}) - s_1\right\}L'(\hat{B})\right](\hat{N}-\hat{Y}) < 0,$$

Thus, it is clear that the above condition (13) is automatically satisfied at the equilibrium point P_5.

(b) We note that $\left.\frac{dY}{dr}\right|_{P_5} > 0$ and $\left.\frac{dC}{dr}\right|_{P_5} > 0$.

Since $\frac{dY}{dr}\left[\beta\left(1+\frac{\nu_1}{d}\right)Y + \left(\frac{\alpha\beta}{d}\right)Y + \frac{\lambda NC}{Y} + \frac{\lambda\alpha C}{d}\right] = \lambda\left\{N - Y\left(1+\frac{\nu_1}{d}\right)\right\}\frac{dC}{dr}$

and

$$\left.\frac{dY}{dr}\right|_{P_5} = \frac{\lambda\left\{\hat{N}-\hat{Y}\left(1+\frac{\nu_1}{d}\right)\right\}(K/s_0r_0)I(\hat{B})}{\beta\left(1+\frac{\nu_1}{d}\right)\hat{Y} + \left(\lambda\hat{N}\hat{C}/\hat{Y}\right) + (\alpha\beta/d)\hat{Y} + \left(\lambda\hat{C}\alpha/d\right) + \lambda\left\{\hat{N}-\hat{Y}\left(1+\frac{\nu_1}{d}\right)\right\}\frac{K\alpha r_2}{s_0 r_0 d}I(\hat{B})},$$

which is positive. This shows that as the growth rate coefficient of cumulative density of ecological factors increases, the infective human population density increases at the equilibrium point P_5. Further, from above it is clear that $\left.\frac{dC}{dr}\right|_{P_5} > 0$ whenever $\left.\frac{dY}{dr}\right|_{P_5} > 0$.

This shows that the carrier population density increases as the growth rate coefficient of cumulative density of ecological factors increases at the equilibrium point P_5.

(c) we also note that

(i) $$\left.\frac{dY}{dr_2}\right|_{P_5} = \frac{\lambda\left\{\hat{N}-\hat{Y}\left(1+\frac{\nu_1}{d}\right)\right\}\left(K\hat{N}/s_0r_0\right)I(\hat{B})}{\beta\left(1+\frac{\nu_1}{d}\right)\hat{Y} + \left(\lambda\hat{N}\hat{C}/\hat{Y}\right) + (\alpha\beta/d)\hat{Y} + \left(\lambda\hat{C}\alpha/d\right) + \lambda\left\{\hat{N}-\hat{Y}\left(1+\frac{\nu_1}{d}\right)\right\}\frac{K\alpha r_2}{s_0 r_0 d}I(\hat{B})} > 0,$$

which shows that as the growth rate coefficient of cumulative density of ecological factors due to human population density related factors increases, the infective human population density increases at the equilibrium point P_5.

(ii) $$\left.\frac{dC}{dr_2}\right|_{P_5} = \frac{K\hat{N}\left[\beta\left(1+\frac{\nu_1}{d}\right)\hat{Y}+\left(\lambda\hat{N}\hat{C}/\hat{Y}\right)+(\alpha\beta/d)\hat{Y}+\left(\lambda\hat{C}\alpha/d\right)\right]I(\hat{B})}{s_0r_0\left[\beta\left(1+\frac{\nu_1}{d}\right)\hat{Y}+\left(\lambda\hat{N}\hat{C}/\hat{Y}\right)+(\alpha\beta/d)\hat{Y}+\left(\lambda\hat{C}\alpha/d\right)+\lambda\left\{\hat{N}-\hat{Y}\left(1+\frac{\nu_1}{d}\right)\right\}\frac{K\alpha r_2}{s_0r_0d}I(\hat{B})\right]} > 0,$$

Thus it is clear that as the growth rate coefficient of cumulative density of ecological factors due to human population density related factors increases, the carrier population density increases at the equilibrium point P_5.

4. Stability Analysis

The local stability results of these equilibria are stated in the following theorem:

Theorem 2: *The equilibria* P_0, P_1, P_2, P_3 *and* P_4 *are locally unstable and the equilibrium* P_5 *is locally asymptotically stable provided the following conditions are satisfied,*

(14) $$\alpha\lambda^2\hat{C}^2 < 2d\beta^2\hat{Y}^2/3,$$

(15) $$3\alpha\lambda^2(\hat{N}-\hat{Y}-\hat{Z})^2L^2(\hat{B})\left\{s'(\hat{B})+\frac{s_0\hat{C}}{L^2(\hat{B})}L'(\hat{B})\right\}^2K^2r_2^2 < r_0^2d\beta^2s_0^2\hat{Y}^2,$$

(16) $$\nu_1\lambda^2\hat{C}^2 < 4d\beta^2\hat{Y}^2/3.$$

Proof: In the following, we study the local stability behavior of P_0, P_1, P_2, P_3 and P_4 by the method of variational matrix and the sixth equilibrium point P_5 is studied by using Lyapunov's theory.

The variational matrix T_i corresponding to the equilibrium points $P_i = 0, 1, 2, 3, 4, 5$ is given by:

$$T_i = \begin{bmatrix} G(Y,Z,N,C) & -\beta Y-\lambda C & \beta Y+\lambda C & \lambda(N-Y-Z) & 0 \\ \nu_1 & -d & 0 & 0 & 0 \\ -\alpha & 0 & -d & 0 & 0 \\ 0 & 0 & 0 & s(B)-\frac{2s_0C}{L(B)}-s_1 & s'(B)C+\frac{s_0C^2L'(B)}{L^2(B)} \\ 0 & 0 & r_2B & 0 & r-\frac{2r_0B}{K}+r_2N \end{bmatrix},$$

where $G(Y,Z,N,C)=\beta(N-2Y-Z)-\lambda C-(\nu_1+\alpha+d)$.

Stability Behavior of $P_0=(0,0,A/d,0,0)$:

For equilibrium point P_0, we find that one eigenvalue of T_0 is $r+r_2(A/d)$, which is positive. Thus, P_0 is unstable.

Stability Behavior of $P_1=(0,0,A/d,0,B_m)$:

For equilibrium point P_1, we note that one of the eigenvalues of T_1 is $s(B_m)-s_1$, which is positive. Thus P_1 is unstable.

Stability Behavior of $P_2=(\overline{Y}_1,\overline{Z}_1,\overline{N}_1,0,0)$:

For equilibrium point P_2, we note that one of the eigenvalues of T_2 is $r+r_2\overline{N}_1$, which is positive. Thus P_2 is unstable.

Stability Behavior of $P_3=(\overline{Y}_1,\overline{Z}_1,\overline{N}_1,0,\overline{B}_1)$:

For equilibrium point P_3, we note that one of the eigenvalues of T_3 is $s(\overline{B}_1)-s_1$, which is positive. Thus P_3 is unstable.

Stability Behavior of $P_4=(\tilde{Y}_1,\tilde{Z}_1,\tilde{N}_1,\tilde{C}_1,0)$:

For equilibrium point P_4, we find that one of the eigenvalues of T_4 is $r+r_2\tilde{N}_1$, which is positive. Thus P_4 is unstable.

Stability Behavior of $P_5=(\hat{Y},\hat{Z},\hat{N},\hat{C},\hat{B})$:

Since this equilibrium is very important and the nature of P_5 cannot be stated easily from the variational matrix. Thus, we prove the local stability result of P_5 using Lyapunov's theory by considering the positive definite function:

$$\text{(17)} \qquad \mathrm{V}=\frac{\mathrm{k}_0}{2}\mathrm{y}^2+\frac{\mathrm{k}}{2}\mathrm{z}^2+\frac{\mathrm{k}_1}{2}\mathrm{n}^2+\frac{\mathrm{k}_2}{2}\mathrm{c}^2+\frac{\mathrm{k}_3}{2}\mathrm{b}^2,$$

(where $y=Y-\hat{Y}$, $z=Z-\hat{Z}$, $n=N-\hat{N}$, $c=C-\hat{C}$, $b=B-\hat{B}$ and k_0, k, k_1, k_2 and k_3 are positive constants to be chosen appropriately.)

Thus, $\dfrac{dV}{dt}$ along the linearised system (4) can be written after rearrangement of terms as:

$$\text{(18)} \quad \frac{dV}{dt}=(k_0\beta\hat{Y}-k_1\alpha)yn+(k\nu_1-k_0\beta\hat{Y})yz-\{k_0\lambda(\hat{N}-\hat{Z})\hat{C}/\hat{Y}\}y^2$$
$$-\left[(k_0\beta\hat{Y}/3)y^2-(k_0\lambda\hat{C})yn+(k_1d/2)n^2\right]$$

$$-\left[(k_0\beta\hat{Y}/3)y^2+(k_0\lambda\hat{C})yz+(kd)z^2\right]$$
$$-\left[(k_0\beta\hat{Y}/3)y^2-\left\{k_0\lambda(\hat{N}-\hat{Y}-\hat{Z})\right\}yc+\{k_2s_0\hat{C}/2L(\hat{B})\}c^2\right]$$
$$-\left[\{k_2s_0\hat{C}/2L(\hat{B})\}c^2-k_2\hat{C}\left\{s'(\hat{B})+s_0\hat{C}L'(\hat{B})/L^2(\hat{B})\right\}cb+(k_3r_0\hat{B}/2K)b^2\right]$$
$$-\left[(k_1d/2)n^2-(k_3r_2\hat{B})nb+(k_3r_0\hat{B}/2K)b^2\right].$$

Choosing $k_1=1$, $k_0=\alpha/\beta\hat{Y}$ and $k=\alpha/\nu_1$, we have the following inequalities for $\frac{dV}{dt}$ to be negative definite,

(19) $$\alpha\lambda^2\hat{C}^2<2d\beta^2\hat{Y}^2/3,$$

(20) $$\nu_1\lambda^2\hat{C}^2<4d\beta^2\hat{Y}^2/3,$$

(21) $$\frac{3}{2}\frac{\alpha\lambda^2L(\hat{B})(\hat{N}-\hat{Y}-\hat{Z})^2}{\beta^2s_0\hat{Y}^2\hat{C}}<k_2<\frac{s_0r_0\hat{B}}{KL(\hat{B})\hat{C}\left\{s'(\hat{B})+\frac{s_0\hat{C}}{L^2(\hat{B})}L'(\hat{B})\right\}^2}k_3,$$

(22) $$k_3<\frac{r_0d}{Kr_2^2\hat{B}}.$$

Choosing $k_3=(r_0d)/2Kr_2^2\hat{B}$, the inequality (22) is automatically satisfied. Thus we can choose k_2 satisfying (21) provided

(23) $$3\alpha\lambda^2(\hat{N}-\hat{Y}-\hat{Z})^2L^2(\hat{B})\left\{s'(\hat{B})+\frac{s_0\hat{C}}{L^2(\hat{B})}L'(\hat{B})\right\}^2K^2r_2^2<r_0{}^2d\beta^2s_0^2\hat{Y}^2,$$

is satisfied.

Hence P_5 is locally stable if (19), (20) and (23) are satisfied. Hence the result. In the following theorem, we will show that P_5 is nonlinearly asymptotically stable under certain conditions:

Theorem 3: *In addition to assumptions (2) and (3), let $s(B)$ and $L(B)$ satisfy $0\le s'(B)\le p$ and $0\le L'(B)\le q$ for some positive constants p and q in W_s, then P_5 is nonlinearly asymptotically stable in W_s provided the following inequalities are satisfied:*

(24) $$\alpha\lambda^2 C_m^2 < 2d\beta^2\hat{Y}^2/3,$$

(25) $$3\alpha\lambda^2(\hat{N}-\hat{Y}-\hat{Z})^2 L^2(\hat{B})\left\{p+s_0 C_m \frac{q}{L_0^2}\right\}^2 K^2 r_2^2 < r_0{}^2 d\beta^2 s_0^2\hat{Y}^2,$$

(26) $$\nu_1\lambda^2\hat{C}_m^2 < 4d\beta^2\hat{Y}^2/3.$$

We can prove the above theorem by using the following positive definite function:

$$V = k_0\left(Y-\hat{Y}-\hat{Y}\ln\frac{Y}{\hat{Y}}\right)+\frac{k}{2}(Z-\hat{Z})+\frac{k_1}{2}(N-\hat{N})^2+k_2\left(C-\hat{C}-\hat{C}\ln\frac{C}{\hat{C}}\right)$$
$$+k_3\left(B-\hat{B}-\hat{B}\ln\frac{B}{\hat{B}}\right),$$

(where k_0, k, k_1, k_2 and k_3 are positive constants to be chosen appropriately.)

5. Simulation for Nonlinear Stability Analysis

In this section we analyze the model (4) by using computer simulation for appropriate values of parameters to show the nonlinear stability behavior of P_5. For numerical simulation, we choose $s(B)$ and $L(B)$ as follows:

$$s(B) = s_0 + s_{11}B \text{ and } L(B) = L_0 + L_{11}B.$$

Firstly, we shall show the existence of the nontrivial equilibrium point and then its nonlinear stability behavior by taking the following values of the various parameters:

$\beta = 5.3\times10^{-7}$, $\alpha = 0.0005$, $d = 0.0004$, $\lambda = 2.1\times10^{-8}$, $A = 10$, $s_0 = 0.9$, $s_1 = 0.6$, $s_{11} = 2.0\times10^{-6}$, $L_0 = 100000$, $L_{11} = 4\times10^{-6}$, $\nu_1 = 0.002$, $r_0 = 0.9$, $r = 0.7$, $r_2 = 0.225\times10^{-5}$, $K = 26000$.

The values of $\hat{Y}$ and $\hat{N}$ are determined as the point of intersection of the following equations (which are obtained by eliminating Z, C and B from (5) to (9)):

(27)
$$\beta(1+(\nu_1/d))Y^2+Y\left[(\nu_1+\alpha+d)-\beta N+\lambda(1+(\nu_1/d))\times\right.$$
$$\left.\{(s_0-s_1+s_{11}(K/r_0)(r+r_2N))(L_0+L_{11}(K/r_0)(r+r_2N))/s_0\}\right]$$
$$-\lambda N\{(s_0-s_1+s_{11}(K/r_0)(r+r_2N))(L_0+L_{11}(K/r_0)(r+r_2N))/s_0\}=0.$$

and

(28) $$Y=(A-dN)/\alpha\,.$$

Here the point of intersection (using MATLAB) gives $\hat{Y}=2949.23$ and $\hat{N}=21313.45$ and the values of other variables $\hat{Z}$, $\hat{C}$ and $\hat{B}$ can be obtained from the corresponding expressions. Further, while solving with the help of MAPLE, the equations (5) to (9) the numerical values of various parameters as given above, we obtain the nontrivial equilibrium point P_5 as follows:

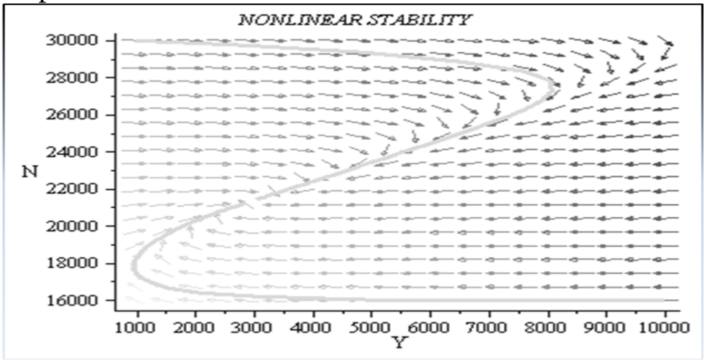
$\hat{Y}=2949.236608$, $\hat{N}=21313.45424$, $\hat{Z}=14746.18304$, $\hat{C}=38135.05446$, $\hat{B}=21607.59675$.

For equilibrium point P_5, the eigenvalues of matrix T_5 are: $-0.001849998837+0.001949242259i$, $-0.001849998837-0.001949242259i$, -0.7479999981, -0.3432200029, -0.00040.

Thus it has 3 real eigenvalues which are negative and 2 complex eigenvalues, which have negative real parts. Hence P_5 is locally stable.

It is also pointed out here that for the above set of parameters, the conditions for local stability and nonlinear stability have been checked and they are satisfied.

For the above set of parameters, a computer generated graph of Y vs N is shown below which indicates the nonlinear stability of the point $(\hat{Y},\hat{N})$ in the $Y-N$ plane.



6. Conclusions

In this paper, I have proposed and analyzed an SIR model for carrier dependent infectious diseases by considering ecological effects. I have assumed that the cumulative density of ecological factors, which are conducive to the growth of density of the carrier population, is governed by population density dependent growth rate equation. The equation governing the carrier population has been assumed to be a generalized logistic model with specific growth rate and carrying capacity which are functions of cumulative density of ecological factors. The model has been analyzed analytically and by computer simulation. The effects of parameters governing the ecological factors conducive to the growth of carrier population have been found to increase the density of carrier population, leading to fast spread of infectious diseases.It is noted that the ecological factors have destabilizing effects on the system. Further it has been found that an infectious disease becomes more endemic due to immigration.

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