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The Fractional Kinetic Equation*

Manoj Sharma and Renu Jain
Department of Mathematics, RJIT, BSF Academy, Tekanpur
School of Mathematics and Allied Sciences, Jiwaji University, Gwalior

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Abstract: In Astrophysics, kinetic equations designate a system of differential equations describing the rate of change of chemical composition of a star for each species in terms of the reaction rates for destruction and production of that species. Methods for modeling processes of destruction and production of stars have been developed for bio-chemical reactions and their unstable equilibrium states and for chemical reaction networks with unstable states, oscillations and hysteresis. The present paper aims at extending the solution of fractional kinetic equation obtained by Haubold and Mathai[2] neglecting spatial fluctuations in any arbitrary reaction to a case incorporating such inhomogenieties.
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1. Introduction

Let us consider an arbitrary reaction characterized by a time dependent quantity N=N(t). It is possible to equate the rate of change dN/dt to a balance between the destruction rate d and the production rate p of N, that is dN/dt=-d + p. It is generally presumed that destruction rate d and production rate p depend directly on the quantity N on account of interaction mechanisms. Thus d = d (N) and p = p (N). This dependence is complicated since the destruction or production at time t depends not only on N(t) but also on the past history N(t1), t1<t, of the variable N. This may be formally represented by

$$dN / dt = -d\,(N_t) + p\,(N_t), \tag{1.1}$$

where Nt denotes the function defined by $N_t(t^*) = N(t - t^*)$, $t^*>0$. Here d and p are functionals and. Equation (1.1) represents a functional-differential equation. Haubold and Mathai[1] studied a special case of equation (1.1) which is given by

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(1.2) $$d N / dt = -\alpha N(t),$$

with a constant $\alpha > 0$. In the present work we consider the general fractional - differential equation as follows:

(1.3) $$dN / dt = - c N (t - t^*).$$

Here $c > 0$ is a constant and $t^* > 0$.

2. Standard Kinetic Equation

The production and destruction of species is described by kinetic equations governing the change of the number density N_i of species i over time, i.e.

(2.1) $$\frac{dN_i}{dt} = -\sum_j N_i N_j \langle \sigma v \rangle_{ij} + \sum_{k,l\neq i} N_k N_l \langle \sigma v \rangle_{kl},$$

where $\langle \sigma v \rangle_{mn}$ denotes the reaction probability for an interaction involving species m and n, and the summation is taken over all reactions which either produce or destroy the species i (Haubold and Mathai[2]). Proceeding with equation (2.1), the first sum in equation (2.1) can also be written as

(2.2) $$-\sum_j N_i N_j \langle \sigma v \rangle_{ij} = -N_i \left(\sum_j N_j \langle \sigma v \rangle_{ij} \right) = N_i a_i,$$

where a_i is the statistically expected number of reactions per unit volume per unit time destroying the species i. It is also a measure of the speed in which the reaction proceeds. In the following we are assuming that there are N_j (j = 1, ..., i, ...) species j per unit volume and that for a fixed N_i the number of other reacting species that interact with the i-th species is constant in a unit volume. Following the same argument for the second sum in equation (2.1), we get

(2.3) $$+\sum_{k,l\neq i} N_k N_l \langle \sigma v \rangle_{kl} = +N_i b_i.$$

Here $N_i b_i$ is the statistically expected number of the i-th species produced per unit volume per unit time for a fixed N_i. The number density of species i, $N_i = N_i(t)$, is a function of time while $\langle \sigma v \rangle_{mn}$ containing the thermonuclear functions. The equation (1.1) implies that

(2.4) $$\frac{dN_i(t)}{dt} = -(a_i - b_i)\, N_i(t).$$

Above equation (2.3) has three distinct cases. $c_i = a_i - b_i > 0$. $c_i < 0$ and $c_i = 0$ of which the last case means that N_i does not vary over time, which means that the forward and reverse reactions involving species i are in equilibrium, such a value for N_i is called a fixed point and corresponds to a steady-state behavior. The first two cases exhibit that either the destruction ($c_i > 0$) of species i or production ($c_i < 0$) of species i dominates.
For the case $c_i > 0$ equation (2.1) becomes

$$\frac{dN_i(t)}{dt} = -c_i N_i(t), \tag{2.5}$$

with the initial condition that $N_i(t = 0) = N_0$ is the number density of species i at time t = 0, and it follows that

$$N_i(t)\, dt = N_0 e^{-c_i t}. \tag{2.6}$$

The exponential function in equation (2.6) represents the solution of the linear one-dimensional differential equation (2.5) in which the rate of destruction of the variable is proportional to the value of the variable[2].

3. Mathematical Prerequisites

The Mellin transform[3] f(s) of f(x) is defined as follows:

$$M\{f(x);s\} = f^*(s) = \int_0^\infty x^{s-1} f(x)dx,\ s \in C, \tag{3.1}$$

provided the integral exists. Then, under suitable conditions

$$M\left[R^{-\alpha} f(x)\right] = \frac{\Gamma(1-\alpha-s)}{\Gamma(1-s)} F(s+\alpha). \tag{3.2}$$

The inverse Mellin transform f(x) of f(s) defined as follows.

$$M^{-1}\{f^*(s);x\} = f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} x^{-s} f^*(s)ds,\ \ x \in C, \tag{3.3}$$

where $i = \sqrt{-1}$ and γ is a real number in the strip of analyticity of f*(s). When f*(s) is analytic in the relevant strip, f(x) is uniquely determined by f*(s) by the formula (3.3). The convolution for Mellin transform[3] is given by the relation.

(3.4) $$h(x)=\int_0^\infty f(\frac{x}{\xi})g(\xi)\frac{d\xi}{\xi} \leftrightarrow \mathrm{f}^*(\mathrm{s})\mathrm{g}^*(\mathrm{s})=\mathrm{h}^*(\mathrm{s}).$$

The I-function[7] can also be represented as

(3.5) $$I[z]=I_{P_i,Q_i:R}^{M,N}\left[z \Big/ \begin{matrix} \{(a_j,\alpha_j)_{1,n}\}, & \{(a_{ji},\alpha_{ji})_{n+1,p_i}\} \\ \{(b_j,\beta_j)_{1,m}\}, & \{(b_{ji},\beta_{ji})_{m+1,q_i}\} \end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_L \phi(s)z^{-s}ds,$$

where $i=\sqrt{-1,}$ $z\neq 0$ is a complex variable and $z^s=e^{[s(\log|z|)+i\arg z\ ]}$, in which log $|z|$ denotes the natural logarithm and arg z is not necessarily principal value. An empty product is interpreted as unity. Also

(3.6) $$\phi(s)=\frac{\prod_{j=1}^{M}\Gamma(b_j+\beta_j s)\prod_{j=1}^{N}\Gamma(1-a_j-\alpha_j s)}{\sum_{i=1}^{R}\left[\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}+\alpha_{ji}s)\right]},$$

where P_i (i= 1, 2, …, R) , Q_i (i= 1, 2, …, R), M, N are integers satisfying $0\le N\le P_i$, $1\le M\le Q_i$ $(i=1,2,...,R)$, R is finite, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$, are real and positive numbers and a_j, a_{ji}, b_{ji} are complex numbers such that none of the points $s=(b_j+\nu)/\beta_j,(j=1,2,...,M;\ \nu=0,1,2,...)$, which are the poles of $\Gamma(b_j-\beta_j s)$, (j= 1, 2, …, M) and the points $s=(a_j-\nu-1)/\alpha_j$, $(j=1,2,...,N;\ \nu=0,1,2,...)$, which are the poles of $\Gamma(1-a_j+\alpha_j s)$, coincide with one another i.e. $\alpha_j(b_h+\nu)\neq\beta_h(a_j-1-k)$ for $\nu,k=0,1,2,...$ h=1, 2,…M, J= 1,2,…R.

4. Fractional Kinetic Equation

If we integrate the kinetic equation (1.3), with replacing t* > 0 by α >0, we obtain

(4.1) $$N_i(t)-N_0=-c_i\ {}_0D_t^{-1}N_i(t-\alpha),\ (c_i>0,\ \alpha>0).$$

Here ${}_0D_t^{-1}$ is the standard Riemann integral operator. Haubold and Mathai[1] described as the number density of the species i, $N_i = N_i$ (t) is a function of time and N_i (t = 0) = N_0 is the number density of species i at time t = 0. If the index i is dropped and ${}_0D_t^{-1}$ is replaced by ${}_0D_t^{-\alpha}$, then we can written as follows

$$\mathrm{N}(t) - \mathrm{N}_0 = - c\ {}_0D_t^{-\alpha}\ \mathrm{N}\ (t-\alpha).$$

Now we prove the following theorem .

Theorem 1. *If c > 0,* $\alpha > 0$, *then the solution of the integral equation*

$$N(t) - N_0 f(t) = -c\ {}_0D_t^{-\alpha}\ \mathrm{N}\ (t-\alpha), \tag{4.2}$$

where f(t) is any integrable function on the finite interval [0, b], is given by the formula in the form of I-function.

$$N(t) = N_0 \int_0^\infty f(\frac{t}{\xi})\ I_{3,4:1}^{2,1}\left[\xi \left| \begin{matrix} \{(0,1)\},\{(c,0),(c+1,0)\} \\ \{(0,1),(c,0)\},\{(0,1),(\alpha,1)\} \end{matrix} \right.\right] \frac{d\xi}{\xi}, \tag{4.3}$$

where I is the I-function defined in (3.5).

Proof . Applying Mellin Transform defined in (3.1) to the equation (4.2) we obtain

$$\mathrm{N}(s) - \mathrm{N}_0 f^*(s) = - c\ \mathrm{N}(s)\ \frac{\Gamma(1-\alpha-s)}{\Gamma(1-s)}, \tag{4.4}$$

where f*(s) is the Mellin transform of f(s). This gives

$$N(s)\left[1+\frac{\Gamma(c+1)\Gamma(1-\alpha-s)}{\Gamma(c)\ \Gamma(1-s)}\right] = \mathrm{N}_0 f^*(s), \quad \text{i.e.} \tag{4.5}$$

$$N(s) = \frac{\mathrm{N}_0 f^*(s)\Gamma(1-s)\Gamma(c)}{\Gamma(c)\Gamma(1-s)+\Gamma(c+1)\Gamma(1-\alpha-s)}. \tag{4.6}$$

We have

$$\mathrm{M}^{-1}\left\{\frac{\Gamma(1-\mathrm{s})\Gamma(\mathrm{c})}{\Gamma(\mathrm{c})\Gamma(1-\mathrm{s})+\Gamma(\mathrm{c}+1)\Gamma(1-\alpha-\mathrm{s})}\right\} = \mathrm{I}_{3,4:1}^{2,1}\left[\mathrm{t}\left| \begin{matrix} \{(0,1)\},\{(\mathrm{c},0),(\mathrm{c}+1,0)\} \\ \{(0,1),(\mathrm{c},0)\},\{(0,1),(\alpha,1)\} \end{matrix} \right.\right], \tag{4.7}$$

$$M^{-1}\{f^*(s)\} = f(t). \tag{4.8}$$

Applying the inverse Mellin transform defined by (3.3), with equation (4.7) and (4.8) and also using convolution theorem (3.4) for Mellin transform in equation (4.6), we obtain

$$N(t) = N_0 \int_0^\infty f(\frac{t}{\xi})\ I^{2,1}_{3,4:1}\left[\xi \left| \begin{matrix} \{(0,1)\},\{(c,0),(c+1,0)\} \\ \{(0,1),(c,0)\},\{(0,1),(\alpha,1)\} \end{matrix} \right.\right] \frac{d\xi}{\xi}.$$

Corollary 1. *If c >0,* $\alpha > 0$ *and f(t) =* E_1 *(-t) then for the solution of the integral equation*

(4.9) $$N(t) - N_0 E_1(-t) = -cN(t-\alpha),$$

there holds the formula

(4.10) $$N(t) = N_0\ \mathrm{I}^{2,1}_{3,4:1}\left[\ Z \left| \begin{matrix} \{(0,1)\},\{(c,0),(c+1,0)\} \\ \{(0,1),(c,0)\},\{(0,1),(\alpha,1)\} \end{matrix} \right.\right].$$

Proof: Applying Mellin Transform defined in (3.2) to the equation (4.9), we obtain

(4.11) $$\mathrm{N(s)} - \mathrm{N_o}\ \Gamma(s) = \frac{-c\Gamma(1-\alpha-s)\ \mathrm{N(s)}}{\Gamma(1-s)},$$

(4.12) $$N(s)\left[1+\frac{c\Gamma(1-\alpha-s)}{\Gamma(1-s)}\right] = \mathrm{N}_0\Gamma(s),$$

(4.13) $$N(s)\left[1+\frac{\Gamma(c+1)\Gamma(1-\alpha-s)}{\Gamma(c)\ \Gamma(1-s)}\right] = \mathrm{N}_0\Gamma(s),$$

(4.14) $$N(s) = \frac{N_0\Gamma(s)\Gamma(c)\Gamma(1-s)}{\Gamma(c)\Gamma(1-s)+\Gamma(c+1)\Gamma(1-\alpha-s)}.$$

Applying the inverse Mellin transform defined by (3.3) in equation (4.14), we obtain

(4.15) $$N(t) = N_0 \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(c)\Gamma(1-s)}{\Gamma(c)\Gamma(1-s)+\Gamma(c+1)\Gamma(1-\alpha-s)} t^{-s} ds.$$

From above equation we obtain desired result (4.10).

Corollary 2. *If* **c** *>0,* $\alpha > 0$ *and f (t) =* e^{-t^2} *then for the solution of the integral equation*

(4.16) $$N(t) - N_0 e^{-t^2} = -c\,N(t-\alpha),$$

there holds the formula

(4.17) $$N(t)=\frac{N_0}{2}\ \mathrm{I}_{3,4:1}^{2,1}\left[\ Z\left|\begin{matrix}\{(0,1)\},\{(c,0),\ (\mathrm{c}+1,0)\}\\ \{(0,\frac{1}{2}),(c,0)\},\{(0,1),(\alpha,1)\}\end{matrix}\right.\right].$$

Proof: Applying Mellin Transform defined in (3.2) to the equation (4.16) we obtain

(4.18) $$N(s)-\frac{N_0}{2}\Gamma(\frac{s}{2})=-c\frac{\Gamma(1-\alpha-s)}{\Gamma(1-s)}N(s),$$

(4.19) $$N(s)\left[1+\frac{\Gamma(c+1)\Gamma(1-\alpha-s)}{\Gamma(c)\ \Gamma(1-s)}\right]\ =\ \frac{\mathrm{N}_0\Gamma(\frac{s}{2})}{2},$$

(4.20) $$N(s)=\frac{N_0}{2}\left[\ \frac{\Gamma(\frac{s}{2})\ \Gamma(c)\,\Gamma(1-s)}{\Gamma(c)\Gamma(1-s)+\Gamma(c+1)\Gamma(1-\alpha-s)}\ \right].$$

Applying the inverse Mellin transform defined by (3.3) to the equation (4.20). We obtain the desired result of (4.17).

This paper underlines the significance of fractional integral operators. Since the delay is arbitrary, we can use the fractional integral operator of arbitrary order to arrive at the solution of given integral equation. This completes the analysis.

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