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Properties of Q* Sets[ŧ]

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Abstract: In this paper, we study the properties of Q* open sets. In particular, we investigate the properties and theorems in affine spaces and irreducible spaces. Also we define Gd set, contra Q* closure and study some of these properties.

1. Introduction

We defined Q* closed sets and Q* open sets in an affine space[1] in the year 2010. Affine space is a topological space which characterizes most of the geometrical objects.

We need the following definitions:

Definition 1.1. Let (X, τ) be a topological space. Let $A \subset X$. A is said to be Q*closed if A is closed and int $A = \Phi$. Then the complement of Q* closed set is Q* open.

Definition 1.2[2]. Let C^n be a complex n – space. Let I be a collection of some complex polynomials of C^n. Let $V_I = \{x \in C^n / f(x) = 0 \text{ for all } f \in I\}$. That is common zero set of I. Then V_I is called affine algebraic variety.

Definition 1.3[2]. The set of all complements of affine algebraic varieties satisfies the four axioms defining a topology on C^n. This topology is called Zariski topology on C^n.

Definition 1.4[2]. The set C^n considered as a topological space with its Zariski topology is called affine n–space. We denote this affine n- space by A^n.

2. Q* Open Sets in Various Spaces

In this section we discuss the properties of Q* open sets in some particular spaces namely affine spaces and irreducible spaces.

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Theorem 2.1. *In A^n, every non-empty open set is Q* open set.*

Proof. Let U be any non-empty open set with respect to Zariski topology. Let $U \neq X$. The $U = V_{I1}^C$, for some I_1, where I_1 has at least one non-zero polynomial. Let G be any open set. Let $G \neq X$. Then $G = V_{I2}^C$, for some I_2 where I_2 has at least one nonzero polynomial.

Now $U \cap G = V_{I1}^C \cap V_{I2}^C$. Then $U \cap G = (V_{I1} \cup V_{I2})^C$. Therefore $U \cap G = (V_{I1.I2})^C$ SinceI_1I_2 has at least one nonzero polynomial, $(V_{I1.I2})^C \neq \Phi$. Therefore $U \cap G \neq \Phi$. U intersects every nonempty open set. Therefore U is dense. Hence every nonempty open set is Q* open set in A^n.

Definition 2.2[3]**.** A topological space X is called irreducible if for any decomposition $X = A_1 \cup A_2$ with closed subsets $A_i \subseteq X$ $(i = 1,2)$ then we have $X = A_1$ or $X = A_2$.
A subset X' of a topological space X is called irreducible if X' is irreducible as a subspace.

Example 1. Let $X = \{1, 2, 3\}$ and $\tau = \{\Phi, \{1\},\{1,2\}, X\}$.
Closed sets are Φ, $\{2,3\},\{3\}$, X. Then X is irreducible.

Example 2. Let X = N and $\tau = \{\Phi, \{1\}, \{1,2\},\ldots. X\}$
Then X is irreducible.

Example 3. Let $X = [1,100]$. Let $U_a = [1,a]$ and $\tau = \{ U_a / a \in X\}$.
Then X is irreducible.

Lemma 2.3. *The topological space X is irreducible if and only if every nonempty open set is Q* open.*

Proof. Let X be irreducible. Let U be any nonempty open set. If U = X then nothing to prove. Let $U \neq X$. Then cl $U \neq X$. If possible suppose that U is not Q* open. Then there exits an open set V such that $U \cap V = \Phi$. This implies $U^C \cup V^C = X$, where U^C and V^C are proper closed sets. This is a contradiction to X is irreducible.

Conversely, suppose that every open set is Q* open. We claim that X is irreducible. Suppose X is reducible. Then $X = A \cup B$, where A and B are proper nonempty closed sets. This implies $A^C \cap B^C = \Phi$. Then A^C is not dense. Then A^C is an open set but not Q* open, a contradiction. Hence X is irreducible.

Theorem 2.4. *Let (X,τ) be a topological space. Let $W \subset X$. Every nonempty open set of W is Q* open in W if and only if every nonempty open set of cl W is Q* open in cl W.*

Proof. Let every nonempty open set of W be Q* open set in W. We claim that every nonempty open set of cl W is Q* open in cl W.
Let A be any nonempty open set in cl W. Then A $\cap$W is open in W. By hypothesis A $\cap$W is dense in W. It is enough to prove that every open set intersects A. Let U be any open set of cl W. Take x $\in$ U. Now x$\in$ cl W. Therefore every open set of x intersects W.

Also U$\cap$W is a nonempty open set in W. Since A $\cap$W is dense in W, (A $\cap$W) $\cap$(U $\cap$W) $\neq \Phi$. This implies A $\cap$U $\neq \Phi$. Hence A is dense in cl W and hence A is Q* open in cl W.

Conversely, suppose that every nonempty open set of cl W is Q* open in cl W. We claim that every nonempty open set of W is Q* open in W.

Let U be any nonempty open set of W. Then there exists an open set G in cl W such that U = G$\cap$W. By hypothesis G is Q* open in W. That implies G$\cap$W is Q* open in W. Therefore U is Q* open in W. Hence the theorem.

Theorem 2.5. *Let f: $C^n \longrightarrow C$. Let A $\subset C^n$. Then $x_0 \in$ cl A, for any x_0 if and only if f is identically zero in A implies $f(x_o) = 0$.*

Proof. Let $x_0 \in$ cl A and f be identically zero in A.
Let I = {f}. Since f is identically zero in A, A $\subset V_I$. Since V_I is closed, cl A $\subset V_I$. Since $x_0 \in$ cl A $\subset V_I$, $f(x_0) = 0$. Conversely, let V_I be any closed set containing A. We claim that $x_0 \in V_I$. We have $V_I = \{ x \in C^n / f(x) = 0, \forall f \in I\}$. Since A $\subset V_I$, $f(x) = 0$ $\forall x \in A$, $\forall$ f $\in$I. Therefore $f(x_0) = 0$ $\forall$ f $\in$I. Hence $x_0 \in V_I$. But cl A is the smallest closed set containing A. Therefore $x_0 \in$cl A. Thus the Lemma.

Theorem 2.6. *If A is Q* open then there exists I such that $f(A^C) = 0$ $\forall$ f $\in$I and f (A) = 0 implies $f \equiv 0$.*

Proof. Let A be Q* open. Then A^C is closed and cl A = X. Then there exists I such that $V_I = A^C$ and cl A = X. If x $\in V_I = A^C$, then f(x) = 0 $\forall$ f $\in$I, x $\in A^C$. Therefore $f(A^C) = 0$ $\forall$ f $\in$I. Let us take f (A) = 0. We claim that $f \equiv 0$. Let $x_0 \in$cl A. Since f (A) = 0 $\forall$ x$\in$A and by Lemma (2.5), $f(x_0) = 0$. Therefore f(x) = 0 $\forall$ x $\in$ cl A. But cl A = X. Therefore f(x) = 0 $\forall$ x $\in$X. Hence $f \equiv 0$.

3. General Properties

Let (X, τ) be a topological space. We have proved that the collection of all Q* open sets together with Φ is a topology[1]. Let $\tau_1 = \tau_{Q*}$. We find $(\tau_1)_{Q*}$ which is denoted by τ_2 and so on.

Theorem 3.1. *Let (X,τ) be a topological space. Then the union of all proper open sets is Q* open.*

Proof. Let $A = \bigcup A_i$, where A_i is proper open set (with respect to τ). Clearly A is open. Always cl $A \subset X$. We claim that $X \subset$ cl A. Let $x_0 \in X$. Let U be any open set containing x_0. Therefore $U \subset A$. Then $U \bigcap A - \{x_0\} \neq \Phi$. Therefore $x_0 \in$ cl A. Then cl A =X. Hence A is Q*open.

Result 3.2. *Let (X,τ) be a topological space. Let A be union of all proper open subsets of X. (Let τ_{Q*} denote the collection of all Q*open sets with respect to τ). If $\tau_1 = \tau_{Q*}$, $\tau_2 = (\tau_1)_{Q*}$ ...etc, then $A \in \tau_i$, $\forall$ i = 1,2,...*

Proof. By Theorem 3.1, A is Q*open. Let $\tau_1 = \tau_{Q*}$. Clearly $A \in \tau_1$. If $B \in \tau_1$(B is Q*open with respect to τ) then B is open in τ. Then $B \subset A$. Then union of all proper open sets with respect to τ_1 is A. Therefore A is Q*open with respect to τ_1. Hence $A \in \tau_2$. Similarly $A \in \tau_I$, for all i.

Converse is not true. Consider the example

Let X= {a,b,c,d} and $\tau = \{\Phi,\{a,b\},\{a,b,c\}, X\}$. Also $\tau_{Q*} = \tau$. Let B = {a,b}. cl B = X. Also $\tau = \tau_{Q*}$. $B \in \tau_I$ for all i. But B $\neq$ union of all proper open subsets of X.

Result 3.3. *If A and B are open sets with $A \bigcap B = \Phi$, then A and B are not Q*open.*

Proof: Since $A \bigcap B = \Phi$, the points of B can't be limit points of A. Then cl $A \neq X$. Hence A is not Q*open. Similarly B is not Q*open.

Theorem 3.4. *Let (X,τ) be a topological space. If $\tau_1 = \tau_{Q*}$ and $\tau_2 = (\tau_1)_{Q*}$ then $\tau_1 = \tau_2$.*

Proof: Clearly τ_1 is finer than τ_2. We have to prove that τ_2 is finer than τ_1. Let $A \in \tau_1$. Since $\tau_1 \subset \tau$ and A is dense with respect to τ, A is dense with respect to τ_1. Then A is Q* open with respect to τ_1, that is, $A \in \tau_2$. Therefore τ_2 is finer than τ_1.Hence $\tau_1 = \tau_2$.

Theorem 3.5. *Let (X,τ) be a topological space. (Let (τ_A) denote the subspace topology on A). If $B \subset A \subset X$, where A is open with respect to τ and B is Q*open in X then B is Q*open in A.*

Proof: Given B is open in X and cl B =X. Then $B \bigcap A$ is open in A. Also $B \bigcap A = B$ is open in A. Claim cl B with respect to τ_A is A. Let U be open in

A. Since A is open in X, U is open in X. Then $U \in \tau$. Since cl B = X, $U \cap B \neq \Phi$. Hence every open set U in A intersects B. Therefore B is Q*open in A.

Theorem 3.6. *Let (X,τ) be a topological space. If $B \subset A \subset X$, where A is Q* open and B is Q*open in A then B is Q*open in X.*

Proof: Since B is open in A and A is open in X, B is open in X. We claim cl B with respect to τ is X. Let U be any open set with respect to τ. Since cl A = X, $U \cap A \neq \Phi$. Therefore $U \cap A$ is an open set with respect to τ_A. Since cl B with respect to τ_A is A, $(U \cap A) \cap B \neq \Phi$. Then $U \cap (A \cap B) \neq \Phi$. Hence cl B with respect to τ = X and hence B is Q*open in X.

Definition 3.7. Let (X,τ) be a topological space. Contra Q*cl A is defined by the intersection of all Q*open sets containing A.

Theorem 3.8. *Let (X,τ) be a topological space.Then contra cl $A \subset$ contra Q* cl A.*

Proof: Let $A \subset X$. We have contra cl A = $\bigcap\{B/ B$ is open, $A \subset B\}$ and contra Q*cl A= $\bigcap\{ B/ B$ is Q*open, $A \subset B\}$. Since every Q* open set is open, contra cl A $\subset$ contra Q*cl A.

Example. Let X = { a,b,c}. Let T = { Φ, {a,b},{c},X}. Also τ_{Q*} = {X}. Let A= {a,b}. Then contra cl A = {a,b}; contra Q*cl A = X.
Therefore contra cl A $\neq$ contra Q*cl A.

Remark. *It directly follows from definitions that if every Q*open set is Q*closed then Q*cl A= contra Q*cl A.*

Definition 3.9. Let (X,τ) be a topological space. Let $A \subset X$. A is said to be Gd set if $A = U \cap V$, where U is open and V is proper Q*open.

Example 3.10. Let X= [1,100]. Let U_a = [1,a] and $\tau = \{ U_a / a \in X\}$. Also $\tau_{Q*} = \tau$. Then every set $U_a \neq X$ is a Gd set.

Remark 3.10. *Every Gd set is Q*open but its converse is not true as is evident from the following example.*

Let X = {a,b,c,} and τ = {Φ, {a,b}, X}. Then τ_{Q*} = {X,{a,b}}. Here {b,c} is Q* open but not Gd set.

Definition 3.11. Let (X,τ) be a topological space. Let D be any directed set. Let $<x_\alpha>$, $\alpha \in D$ be a net in X. We say that $<x_\alpha>$ Q* converges to x_0 if given any Q* open set U containing x_0 there exists $\alpha_0 \in D$ such that $x_\alpha \in U$ $\forall \alpha \geq \alpha_0$.

Theorem 3.12. *Let (X, τ) be a topological space. Let $A \subset X$. $x_0 \in Q^*$ cl A if and only if there exists a net $<x_\alpha>$ in A, such that $<x_\alpha>$ Q* converges to x_0.*

Proof. Let $<x_\alpha>$ be a net in A such that $<x_\alpha>$ Q* converges to x_0. We claim that $x_0 \in$ Q* cl A. Let U be any Q* open set containing x_0. Since $<x_\alpha>$Q* converges to x_0 there exists $\alpha_0 \in D$ such that $x_\alpha \in U$, $\forall\ \alpha \geq \alpha_0$. Also $<x_\alpha>$ is a net in A. Then $x_{\alpha_0} \in U \cap A$. Therefore $U \cap A \neq \Phi$. Since U is arbitary, every Q* open set containing x_0 intersects A. Hence $x_0 \in$ Q* cl A. Conversely suppose $x_0 \in$ Q* cl A. We claim that there exists a net in A such that the net Q* converges to x_0. Let D = {U / U is Q* open set containing x_0}. Define $\leq$ in D as follows: $U_1 \leq U_2$ if $U_2 \subset U_1$. Clearly $(D, \leq)$ is a directed set. Let U be any Q* open set containing x_0. Since $x_0 \in$ Q* cl A, there is a point x_U in $U \cap A$. Therefore $<x_U>$ is a net in A. Given a Q* open set containing x_0, $U \geq G$ implies $U \subset G$. Since $x_U \in U \subset G$, $x_U \in G$. Therefore $<x_U>$ is a net in A such that $<x_U>$ Q* converges to x_0. Hence the Theorem.

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