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Degree of Approximation of Lipschitz Function By (C, 1) (e, c) Means of its Fourier Series*

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Abstract: Shyam Lal and Prem Narain Singh[1] defined (C, 1) (E, 1) Summability of Fourier series and obtained approximation of Lip(ξ(t),p) function using it. Extending the above result. Shyam Lal and J. K. Kushwaha, obtained the degree of approximation of function of Lip α class by product summability mean of the form (C, 1) (E, q). It is known that (e, c) mean includes (E, 1) and (E, q) mean. In the present paper, we have defined (C, 1) (e, c) mean of Fourier series and generalizing the above two results, obtained the degree of approximation of function of Lipα class by (C, 1) (e, c) means.

Keywords: Lipschitz class, Fourier series, Degree of approximation, Product sum ability method, (C, 1) (e, c) mean.
Mathematics Subject Classification: 42B05, 42B08

1. Definition and Notations

Let f(t) be a periodic function with period - 2π and integrable in the sense of Lebesgue over $\left[-\pi,\pi\right]$. Let the Fourier series of f(t) be given by

$$f\left(x\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nt+b_n\sin nt\right).$$

Let

(1.1) $$\Phi_x(t)=\tfrac{1}{2}\left\{f\left(x+t\right)+f\left(x-t\right)-2f\left(x\right)\right\},$$

$$\mathrm{S_k}\left(f;x\right)-f\left(x\right)=\frac{1}{\pi}\int_0^{\pi}\frac{\Phi_x\left(t\right)}{\sin t/2}\sin\left(k+\frac{1}{2}\right)tdt$$

and

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$$\lim_{n\to\infty} e_n^c = \lim_{n\to\infty} \sqrt{\frac{c}{\pi n}} \sum_{r=-\infty}^{\infty} \exp\left(\frac{-cr^2}{k}\right) S_{k+r}$$

exists, where it is to be understood that $S_{k+r} = 0$, when $k + r < 0$.
A function $f \in Lip\,\alpha$ if

$$f(x+t) - f(x) = O\left(|t|^{\alpha}\right) \text{ for } 0<\alpha \le 1.$$

We shall write

$$\left\|e_n^c - f\right\| = \sup_{-\pi\le x\le\pi} \left|e_n^c(f;x) - f(x)\right|,$$

where $e_n^c(f;x)$ is n^{th} (e, c) mean of the Fourier series of f at *x*.

2. Inequalities

In the proof of our theorem, we shall use the following inequalities:

(2.1) $$\sum_{r=k+1}^{\infty} r \exp\left(\frac{-cr^2}{k}\right) \le \frac{k}{2c} \exp(-ck),$$

(2.2) $$\left|\sum_{r=k+1}^{\infty} \exp\left(\frac{-cr^2}{k}\right) \sin\left(k + r + \frac{1}{2}\right)t\right| \le \frac{kt}{2c} \exp(-ck),$$

(2.3) $$\sum_{r=k+1}^{\infty} \exp\left(\frac{-cr^2}{k}\right) \cos rt = O\left\{\frac{\exp(-ck)}{t}\right\},$$

(2.4) $$1 + 2\sum_{r=1}^{\infty} \exp\left(\frac{-cr^2}{k}\right) \cos rt = \sqrt{\frac{\pi k}{c}} \left\{\exp\left(\frac{-kt^2}{4c}\right) + O\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\}.$$

The inequality (2.2) follows from (2.1) which is due to Shrivastava and Verma[3], (2.3) may be obtained by using Abel's Lemma and (2.4) may be obtained by the classical formula for theta function (see Siddiqui[2])
We have

(2.5) $$e_n^c(f;x) - f(x) = \frac{1}{\pi} \sqrt{\frac{c}{\pi n}} \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \left[\sum_{r=-k}^{\infty} \exp\left(\frac{-cr^2}{k}\right) \sin\left(k + r + \frac{1}{2}\right)t\right] dt$$

and the series $\sum_{k=0}^{\infty} u_k$ is said to be (C, 1) summable to s if

(2.6) $\quad (C,1)=\frac{1}{n+1}\sum_{k=0}^{n}s_k \to s$ as $n\to\infty$.

The (C, 1) transform of the (e, c) transform is defined as the (C, 1) (e,c) transform of the partial sum s_n, of the series $\sum_{k=0}^{\infty}u_k$. Thus if

(2.7) $\quad (C,e)_n^c=\frac{1}{n+1}\sum_{k=0}^{n}e_k^c \to s$ as $n\to\infty$,

where e_n^c denotes the (e, c) transform of s_n then the series $\sum_{k=0}^{\infty}u_k$ is said to be summable (C, 1) (e, c) means or simply summable (C, 1) (e, c) to s. Shyam Lal and Prem Narain Singh[4] obtained approximation of $Lip(\xi(t),p)$ function by (C, 1) (E, 1) means of its Fourier series. They proved the following theorem:

Theorem. 1: *If* $f:R\to R$ *is* 2π *–periodic and* $Lip(\xi(t),p)$ *function belonging to* $Lip(\xi(t),p)$, *then the degree of approximation of f by (C, 1) (E, 1) means of Fourier series satisfies*

(2.8) $\quad \left\|(CE)_n^1-f(x)\right\|_p=O\left(\xi\left(\frac{1}{n+1}\right)(n+1)^{1/p}\right),$

provided $\xi(t)$ *satisfy the following conditions*

(2.9) $\quad \left\{\int_0^{1/(n+1)}\left(\frac{t\phi(t)}{\xi(t)}\right)^p dt\right\}^{1/p}=O\left(\frac{1}{n+1}\right),$

(2.10) $\quad \left\{\int_{1/(n+1)}^{\pi}\left(\frac{t^{-\delta}\phi(t)}{\xi(t)}\right)^p dt\right\}^{1/p}=O\left((n+1)^{\delta}\right),$

where δ *is an arbitrary number such that* $q(1-\delta)-1>0$, *conditions (2.9) and (2.10) hold uniformly in x and* $(CE)_n^1$ *are (C, 1) (E, 1) means of Fourier series (1.1).*

Generalizing the above result Shyam Lal and J.K. Kushwaha[5] obtained the degree of approximation of function of $Lip\alpha$ class by product summability mean of the form (C, 1) (E, q). Their theorem is as follows:

Theorem. 2: *If* $f:\mathbb{R}\to\mathbb{R}$ *is* 2π*–periodic Lebesgue integrable on* $\left[-\pi,\pi\right]$ *and belonging to the Lipschitz class, then the degree of approximation of f by the (C, 1) (E, q) product means of its Fourier series satisfies for n =0, 1, 2,*

(2.11) $$\left\|\left(CE\right)_n^q\left(x\right)-f\left(x\right)\right\|_\infty=\mathrm{O}\left(\left(n+1\right)^{-\alpha}\right)\text{ for }0<\alpha<1.$$

Since (e, c) method includes (E, q) method, it is natural to ask what will be the result if we apply product summability mean of the form (C, 1) (e, c) to obtain the degree of approximation for Lipschitz class?
We shall prove the following theorem:

Theorem: *If* $f:\mathbb{R}\to\mathbb{R}$ *is* 2π*–periodic Lebesgue integrable on* $\left[-\pi,\pi\right]$ *and belonging to the Lipschitz class then the degree of approximation of f by the (C, 1) (e,c) product means of its Fourier series satisfies for n= 0, 1, 2, ……,*

(2.12) $$\left\|\left(C\ e\right)_n^c(x)-f(x)\right\|_\infty=O\left(\left(n+1\right)^{-\alpha}\right)\text{ for }0<\alpha<1.$$

For the proof of our theorem following lemmas are required:

Lemma1: *Let* $K_n\left(t\right)=\dfrac{1}{\left(n+1\right)\pi}\displaystyle\sum_{k=0}^{n}\left[\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin {t}/{2}}\right]$. *Then*

$$K_n\left(t\right)=\mathrm{O}\left(n+1\right),\quad for\ 0<t<\frac{\pi}{\left(n+1\right)}.$$

Proof: Using $\sin nt\le n\sin t$ for $0<t<\dfrac{\pi}{\left(n+1\right)}$. Then

$$K_n\left(t\right)\le\frac{1}{\left(n+1\right)\pi}\sum_{k=0}^{n}\left[\frac{\left(2k+1\right)\sin {t}/{2}}{\sin {t}/{2}}\right]$$

$$\le\frac{1}{\left(n+1\right)}\sum_{k=0}^{n}\left(2k+1\right)$$

$$=O\left(n+1\right).$$

Lemma2: $K_n(t) = O\left(\frac{1}{t}\right)$ *for* $\frac{\pi}{(n+1)} < t < \pi$.

Proof: $K_n(t) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\sin\left(k+\frac{1}{2}\right)t \Big/ \left(\sin\frac{t}{2}\right).$

Now applying Jordan's lemma $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$ and $\sin kt \leq 1$ for $\frac{\pi}{n+1} \leq t \leq \pi$, we have

$$K_n(t) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n} 1/(t/\pi) = \mathrm{O}\left(\frac{1}{t}\right).$$

3. Proof of the Theorem

Following Titchmarsh[1], we have

$$S_n(f;x) - f(x) = \frac{1}{\pi}\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\sin\left(n+\frac{1}{2}\right)t\,dt\ . \tag{3.1}$$

The (e, c) transform e_n^c of s_n is given by

$$e_n^c(f;x) - f(x) = \frac{1}{\pi}\sqrt{\frac{c}{\pi n}}\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\left[\sum_{r=-k}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\right]dt \tag{3.2}$$

$$= \frac{1}{\pi}\sqrt{\frac{c}{\pi n}}\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\left[\left\{1+2\sum_{r=1}^{k}\exp\left(\frac{-cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t\right.$$
$$\left.+\sum_{r=k+1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\right]dt$$

$$= \frac{1}{\pi}\sqrt{\frac{c}{\pi n}}\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\left[\left\{1+2\sum_{r=1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t\right.$$
$$-2\sum_{r=k+1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\cos rt\sin\left(k+\frac{1}{2}\right)t$$
$$\left.+\sum_{r=k+1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\right]dt,$$

$$(3.3)\quad \left(C\,e\right)_n^c - f(x) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\sqrt{\frac{c}{\pi k}}\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\left[\left\{1+2\sum_{r=1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\cos rt\right\}\sin\left(k+\tfrac{1}{2}\right)t\right.$$

$$\left. -2\sum_{r=k+1}^{\infty}\left(\frac{-cr^2}{k}\right)\cos rt\sin\left(k+\frac{1}{2}\right)t + \sum_{r=k+1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\right]dt$$

$$= I_1 + I_2 + I_3.$$

$$(3.4)\quad I_1 \le \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\left[\sqrt{\frac{c}{\pi k}}\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\left\{\left\{1+2\sum_{r=1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t\right\}dt\right]$$

$$= \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\left[\sqrt{\frac{c}{\pi k}}\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\sqrt{\frac{\pi k}{c}}\left\{\exp\left(\frac{-kt^2}{4c}\right)+O\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\}\sin\left(k+\frac{1}{2}\right)tdt\right]$$

$$= \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\left[\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\sin\left(k+\frac{1}{2}\right)t\exp\left(\frac{-kt^2}{4c}\right)dt\right.$$

$$\left.+\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\sin\left(k+\frac{1}{2}\right)tO\left(\exp\left(\frac{-k\pi}{4c}\right)\right)dt\right]$$

$$= I_{1.1} + I_{1.2}.$$

$$I_{1.1} = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\left[\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\sin\left(k+\frac{1}{2}\right)t\exp\left(\frac{-kt^2}{4c}\right)dt\right]$$

$$= O\left(\exp\left(\frac{-nt^2}{4c}\right)\right)\int_0^{\pi}\Phi_x(t)K_n(t)dt,$$

where $K_n(t) = \dfrac{1}{(n+1)\pi}\displaystyle\sum_{k=0}^{n}\left\{\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin t/2}\right\}.$

(3.5) $$I_{1.1} = \mathrm{O}(1)\left[\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\right]\Phi_x(t)K_n(t)\,dt$$

$I_{1.1} = I_{1.11} + I_{1.12}.$

(3.6) $$I_{1.11} = \mathrm{O}(1)\int_0^{\pi/n+1}\Phi_x(t)K_n(t)\,dt$$

$$\leq \int_0^{\pi/n+1}\Phi_x(t)\mathrm{O}(n+1)\,dt \qquad \text{Using lemma (1)}$$

$$= \mathrm{O}(n+1)\int_0^{\pi/n+1}|t|^{\alpha}\,dt$$

$$= \mathrm{O}(n+1)\left\{\frac{t^{\alpha+1}}{\alpha+1}\right\}_0^{\pi/n+1}$$

$$= \mathrm{O}(n+1)\frac{1}{\alpha+1}\frac{\pi^{\alpha+1}}{(n+1)^{\alpha+1}}$$

$$= \mathrm{O}(n+1)^{-\alpha}$$

(3.7) $$I_{1.12} = \int_{\pi/n+1}^{\pi}\Phi_x(t)K_n(t)\,dt$$

$$= \int_{\pi/n+1}^{\pi}\Phi_x(t)\mathrm{O}\left(\frac{1}{t}\right)dt, \qquad \text{by lemma (2)}$$

$$= \int_{\pi/n+1}^{\pi}|t|^{\alpha}\,\mathrm{O}\left(\frac{1}{t}\right)dt$$

$$= \int_{\pi/n+1}^{\pi}t^{\alpha-1}\,dt = \left\{\frac{t^{\alpha}}{\alpha}\right\}_{\pi/n+1}^{\pi}$$

$$= \frac{1}{\alpha}\left[\pi^{\alpha} - \frac{\pi^{\alpha}}{(n+1)^{\alpha}}\right]$$

$$= O(n+1)^{-\alpha}$$

$$I_{1.2} = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\left[\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\sin\left(k+\frac{1}{2}\right)t\,O\left(\exp\left(\frac{-k\pi}{4c}\right)\right)dt\right]$$

$$= O\left(\exp\left(\frac{-n\pi}{4c}\right)\right)\int_0^{\pi}\Phi_x(t)K_n(t)\,dt,$$

where $$K_n(t) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\left[\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin t/2}\right].$$

(3.8) $$I_{1.2} = \left[\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\right]\Phi_x(t)K_n(t)\,dt$$

$$= I_{1.21} + I_{1.22}.$$

(3.9) $$I_{1.21} = \int_0^{\pi/n+1}\Phi_x(t)K_n(t)\,dt.$$

Using Lemma (1)

$$|I_{1.21}| \le \int_0^{\pi/n+1}\Phi_x(t)\,O(n+1)\,dt$$

$$= O(n+1)\int_0^{\pi/n+1}\Phi_x(t)\,dt$$

$$= O(n+1)\int_0^{\pi/n+1}|t|^{\alpha}\,dt$$

$$= O(n+1)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\pi/n+1}$$

$$= O(n+1)\frac{1}{(\alpha+1)}\frac{\pi^{\alpha+1}}{(n+1)^{\alpha+1}}$$

$$= O(n+1)^{-\alpha}.$$

$$I_{1.22} = \int_{\pi/n+1}^{\pi} \Phi_x(t) K_n(t)\,dt.$$

Using Lemma (2)

(3.10) $$|I_{1.22}| \le \int_{\pi/n+1}^{\pi} \Phi_x(t) O\left(\frac{1}{t}\right) dt$$

$$= \int_{\pi/n+1}^{\pi} |t|^{\alpha} O\left(\frac{1}{t}\right) dt = \int_{\pi/n+1}^{\pi} t^{\alpha-1} dt = \left[\frac{t^{\alpha}}{\alpha}\right]_{\pi/n+1}^{\pi}$$

$$= \frac{1}{\alpha}\left[\pi^{\alpha} - \frac{\pi^{\alpha}}{(n+1)^{\alpha}}\right]$$

$$= O(n+1)^{-\alpha}.$$

$$I_2 \le \frac{1}{(n+1)\pi} \sum_{k=0}^{n} \left[\sqrt{\frac{c}{\pi k}} \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \left\{\left\{-2\sum_{r=k+1}^{\infty} \exp\left(\frac{-cr^2}{k}\right) \cos rt\right\} \sin\left(k+\frac{1}{2}\right)t\right\} dt\right]$$

$$= \frac{-2}{(n+1)\pi} \sum_{k=0}^{n} \left[\sqrt{\frac{c}{\pi k}} \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sin\left(k+\frac{1}{2}\right) t O\left(\frac{\exp(-ck)}{t}\right) dt\right].$$

Using inequality (2.3)

$$= O\left(n^{-1/2} \exp(-cn)\right) \int_0^{\pi} \frac{\Phi_x(t)}{t} K_n(t)\,dt,$$

where

$$K_n(t) = \frac{1}{(n+1)\pi} \sum_{k=0}^{n} \left[\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin t/2}\right].$$

(3.11) $$I_2 = \mathrm{O}(1)\left[\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\right]\frac{\Phi_x(t)}{t}K_n(t)\,dt$$

$$= \quad \mathrm{I}_{2.1} + \mathrm{I}_{2.2.}$$

Now, using Lemma (1)

(3.12) $$|I_{2.1}| \le \int_0^{\pi/n+1}\frac{\Phi_x(t)}{t}\mathrm{O}(n+1)\,dt$$

$$= \mathrm{O}(n+1)\int_0^{\pi/n+1}\frac{|t|^\alpha}{t}dt = \mathrm{O}(n+1)\int_0^{\pi/n+1}t^{\alpha-1}dt$$

$$= \mathrm{O}(n+1)\left[\frac{t^\alpha}{\alpha}\right]_0^{\pi/n+1} = \mathrm{O}(n+1)\frac{1}{\alpha}\left[\frac{\pi^\alpha}{(n+1)^\alpha}\right]$$

$$= \mathrm{O}(n+1)^{-\alpha}\,.$$

(3.13) $$I_{2.2} \le \int_{\pi/n+1}^{\pi}\frac{\Phi_x(t)}{t}\mathrm{O}\left(\frac{1}{t}\right)dt\,,$$ by Lemma (2)

$$= \int_{\pi/n+1}^{\pi}\frac{|t|^\alpha}{t}\mathrm{O}\left(\frac{1}{t}\right)dt$$

$$= \int_{\pi/n+1}^{\pi}t^{\alpha-2}dt = \left[\frac{t^{\alpha-1}}{\alpha-1}\right]_{\pi/n+1}^{\pi}$$

$$= \frac{1}{\alpha-1}\left[\pi^{\alpha-1} - \frac{\pi^{\alpha-1}}{(n+1)^{\alpha-1}}\right]$$

$$= \mathrm{O}(n+1)^{-\alpha}\,.$$

$$I_3 = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\left[\sqrt{\frac{c}{\pi k}}\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\left\{\sum_{r=k+1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\right\}dt\right]$$

$$I_3 = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\left[\sqrt{\frac{c}{\pi k}}\int_0^{\pi}\frac{\Phi_x(t)}{\sin {t/2}}.\frac{kt}{2c}\exp(-ck)\,dt\right].$$

Using inequality (2.2)

$$\leq \frac{\sqrt{n}}{(n+1)}\sum_{k=0}^{n}\left[\int_0^{\pi}\frac{\Phi_x(t)}{\sin {t/2}}.\frac{t}{2c}\exp(-ck)\,dt\right] \tag{3.14}$$

$$|I_3| = O\left(n^{-1/2}\exp(-cn)\right)\int_0^{\pi}\Phi_x(t)\,dt, \qquad \text{since } \left|\sin {t/2}\right| \leq \frac{t}{2}$$

$$I_3 = O(1)\int_0^{\pi}\Phi_x(t)\,dt$$

$$= O(1)\left[\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\right]\Phi_x(t)\,dt$$

$$I_3 = I_{3.1} + I_{3.2}.$$

$$I_{3.1} = O(1)\int_0^{\pi/n+1}\Phi_x(t)\,dt \tag{3.15}$$

$$= O(1)\int_0^{\pi/n+1}|t|^{\alpha}\,dt$$

$$= O(1)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\pi/n+1}$$

$$= \frac{1}{(\alpha+1)}.\frac{\pi^{\alpha+1}}{(n+1)^{\alpha+1}}$$

$$= O(n+1)^{-\alpha}.$$

$$I_{3.2} = O(1)\int_{\pi/n+1}^{\pi}\Phi_x(t)\,dt \tag{3.16}$$

$$= O(1)\int_{\pi/n+1}^{\pi} |t|^{\alpha}\, dt$$

$$= O(1)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_{\pi/n+1}^{\pi}$$

$$= \frac{1}{(\alpha+1)}\left[\pi^{\alpha+1} - \frac{\pi^{\alpha+1}}{(n+1)^{\alpha+1}}\right]$$

$$= O(n+1)^{-\alpha}.$$

Now combining (3.3) to (3.16), we have

$$\left|(Ce)_n^c(x) - f(x)\right| = O\left((n+1)^{-\alpha}\right), \text{ for } 0<\alpha<1$$

So $$\left\|(Ce)_n^c(x) - f(x)\right\|_{\infty} = \sup_{-\pi\le x\le\pi}\left|(Ce)_n^c(x) - f(x)\right|$$

$$= O\left((n+1)^{-\alpha}\right) \text{for } 0<\alpha<1.$$

This completes the proof of the theorem.

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