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The Completion of n-Kernels of the Skeletal Congruences on a Distributive Lattice*

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Abstract: The Skeleton SC(L) = { $\theta \in$(L) : $\theta = \varphi^*$ for some $\varphi \in$ C(L)} = {$\theta \in$C(L) : $\theta = \varphi^{**}$} is a complete Boolean lattice. The meet of the set { θi } $\subseteq$ SC(L) is $\cap$ θI while the join is $\vee\theta$i =($\vee\theta$i)** = ($\cap\theta$i)* and the complement of $\theta \in$SC(L) is θ^*. For any n$\in$L, the set K_n SC(L) = { Ker_n θ : $\theta \in$ SC(L)} which is also a complete lattice, where $Ker_n\theta$ = { x : x $\equiv$ n (θ)}, is an n-ideal. In this paper, we have studied the n-kernels of the Skeletal congruence K_n SC(L) on a distributive lattice L, and generalized many results on the completions of special classes of lattices. We have also shown that the set K_n SC(L) of all n-kernels forms an upper continuous distributive lattice and the map a$\rightarrow$ $\langle a\rangle_n$ ={ x $\in$ L : a $\wedge$ x $\leq$ x $\leq$ a $\vee$ n } is a lower join – dense embedding of L into K_n SC(L).

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1. Introduction

Skeletal congruences of a distributive lattice have been studied by several authors including Cornish[1]. For any $\theta \in$C (L), θ^* denotes the pseudo complement of θ. By its very definition $\theta \cap \varphi = \omega$ (the smallest congruence) iff $\varphi \in$C (L). The existence of θ^* is guaranteed by the fact that C (L) is a distributive algebraic lattice. The skeleton of L is defined by SC(L) ={ $\theta \in$ C(L) : $\theta = \varphi^*$ for some $\varphi \in$C(L) }.={ $\theta \in$ C(L) :$\theta = \theta^{**}$ }. The meet of the set {θi} $\subseteq$ SC(L) is $\cap$ θi while the join is $\vee\theta$i = ($\vee$ θi)** =($\cap$ θi)* and the complement of $\theta \in$ SC(L) is θ^*. For any n$\in$ L, the set k_n SC(L) = { ker_n θ ; θ $\in$SC(L)} which is also a complete lattice, where the n-kernel of θ is defined

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by $\ker_n\theta = \{ x \in L : x \equiv n\ \theta \}$, which is clearly an n-ideal.

The idea of n-ideals in a lattice was first introduced by Cornish and Noor in several papers[2,3,4]. For a fixed element n of a lattice L, a convex sublattice containing n is called an n-ideal. If L has a '0', then replacing n by o, an n-ideal becomes an ideal. Moreover, if L has a '1', an n-ideal becomes a filter by replacing n by '1'. Thus the idea of n-ideals is a kind of generalization of both ideals and filters of a lattice. So any results involving n-ideals will give a generalization of the results on ideals and filters with o and 1 respectively in a lattice. The set of all n-ideals of a lattice L is denoted by I_n (L), which is an algebraic lattice under set-inclusion. Moreover, {n} and L are respectively the smallest and largest elements of I_n (L), while the set –theoretic intersection is the infimum.

For any two n-ideals I and J of L, it is easy to check that $I \wedge J = I \cap J = \{ x : x = m(i, n, j)$ for some $i \in I$, $j \in J$, where the median operation $m(x, y, z) = (x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$ is very well known in lattice theory. This has been used by several authors including Birkhoff and Kiss[5] for bounded distributive lattices, Jalubik and Kalibiar[6] for distributive lattices and Sholander[7] for median algebra and $I\vee J = \{x: i_1\wedge j_1 \leq x \leq i_2\vee j_2$, for some $i_1, i_2 \in I$ and $j_1, j_2 \in J$. The n-ideal generated by $a_1, a_2,,,,,,, a_m$ is denoted by $<a_1,a_2,,,,,,,a_m>_n$. Clearly $<a_1, a_2,,,,,,, a_m>_n = <a_1>_n \vee,,,,,,,,,\vee <a_m>_n$. The n-ideal generated by a finite number of elements is called a finitely generated n-ideal. The set of all finitely generated n-ideals is denoted by $F_n(L)$. Also the n-ideal generated by a single element is called a principal n-ideal. The set of all principal n-ideals of L is denoted by $P_n(L)$. An n-ideal P of a lattice L is called prime if $m(x, n, y) \in P$, $x, y \in L$ implies either $x \in P$ or $y \in P$. A subset T of a lattice L is called join-dense if each $z \in L$ is the join of its predesessors in T, while a meet-dense subset of T is defined dually. The congruence θ is called dense if $\theta^* = \omega$ .For a distributive lattice L with 0, the pseudo complement J* of an ideal J is the annihilator ideal, where $J^* = \{x \in L: x \wedge j = o$ for all $j \in J\}$. For any n-ideal J of L, we define $J+ = \{x \in L: m(x,n,j) = n$ for all $j \in J\}$. Obviously J+ is an n-ideal and $J \cap J+ = \{n\}$. We call J+, the annihilator n-ideal of J.

For a, b ∈ L, <a, b> denotes the relative annihilator. That is $<a, b> = \{x \in L; x \wedge a \leq b\}$. In presence of distributivity, it is easy to show that each relative annihilator is an ideal. Also note that $<a, b> = <a, a \wedge b>$. Dual relative annihilator ideal can be defined dually. For details on relative annihilator ideal, we refer the reader to Mandelker[7]. Relative annihilator n-ideal is denoted by < a, b >n where, $<a, b>n = \{x \in L: a \wedge b \wedge n \leq x \leq a \vee b \vee n\}$.

2. The completion of $K_nSC(L)$

A lattice–embedding of one lattice into another is said to be an upper regular (lower regular) if it preserves all existent joins (meets) and it is called regular if it is both upper regular and lower regular. The concept of a meet-dense subset of a lattice is defined in a manner which is dual to the definition of a join-dense subset. In a lattice-embedding embeds a lattice as a join-dense (meet-dense) sub lattice of another lattice. Thus, it is easy to verify that the embedding is lower (upper) regular. A lattice is said to be conditionally upper continuous if for any $x \in L$ and any directed subset $[x_i\}$ such that $\vee x_i$ exists, it follows that $\vee (x \wedge x_i)$ exists and $x \wedge \vee x_i = \vee (x \wedge x_i)$. According to Cornish[1], we have the following results;

Theorem1.1. *Let $J \in KSC(L)$. Then, $\theta(J)^{**}$ is the smallest skeletal congruence having J as a its kernel. Consequently, for any subset $\{J_i\} \subseteq KSC(L)$, where $\vee J_i$, the join of $\{J_i\}$ in KSC(L) is Ker($\theta(\vee J_i)^{**}$).*

According to Gratzar[8] [Theorem1(vi), p-166], $\theta(J)^{**} \cap \theta(K)^{**} = \theta(J \cap K)^{**}$ for any ideals J and K of a distributive lattice. Also, $\vee\theta(J_i)^{**} = (\vee\theta(J_i))^{**}$ holds for any set $\{J_i\}$ of ideals . Hence we obtain,

Theorem1.2. *The map $J \to \theta(J)$ is an upper regular embedding of KSC(L) into the complete Boolean lattice SC(L). Thus, KSC(L) is an upper continuous distributive lattice and the map $a \to (a]$ is a lower regular embedding of L into a join-dense sub lattice of $K_nSC(L)$.*

The following theorem is a generalization of the above theorem.

Theorem1.3. *Let L be a distributive lattice and J be an n-ideal of L. then the map $J \to \theta(J)^{**}$ is an upper regular embedding of $K_n SC(L)$ into the complete Boolean lattice SC(L). Thus $K_nSC(L)$ is an upper continuous distributive lattice and the map $a \to \langle a \rangle_n$ is a lower regular embedding of L into a join-dense sub lattice of $K_nSC(L)$.*

An n-ideal of a lattice L is called complete if it is closed under the formations of existent joins. The set of all complete n-ideals of a lattice L is denoted by $K_n(L)$, which is also a complete lattice, which is also the completion of L. The lattice L is conditionally upper continuous if and only if $K_n(L)$ is upper continuous. $\langle a,b \rangle$ denotes the relative annihilator ideal which is equal to $\{ x \in L: x \wedge a \leq b \}$.

The following theorem is a generalization of Theorem 3.3 of A. S. A. Noor[4].

Theorem.1.4. *Let L be a distributive lattice with n, then the following conditions are equivalent;*

(i) *Each relative annihilator n-ideal $<r,s>_n$, $r,s \in L$ is a complete n-ideal;*

(ii) *Each n-ideal $J \in K_nSC(L)$ is a complete n-ideal;*

(iii) *The embedding $a \rightarrow <a>_n$ of L into $K_nSC(L)$ is regular;*

(iv) *L is conditionally upper continuous;*

(v) *$K_n SC(L)$ is isomorphic to $K_n(L)$.*

Proof: (i) $\Rightarrow$ (ii) follows from Theorem 4 of Latif and Noor[9],

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Since each n-ideal J of K_n SC(L) is a complete n-ideal, so it is easily verified that the map $a \rightarrow <a>_n$ is an upper regular. Then by Theorem 1.3, the embedding $a \rightarrow <a>_n$ is regular. Hence (iii) holds.

(iii) $\Rightarrow$ (iv) . Suppose (iii) holds. Then by Th. 1.3, L possesses an upper regular embedding into a continuous lattice. Hence (iv) holds.

(iv) $\Rightarrow$ (v). Since (iii) $\Rightarrow$ (iv) holds so (iii) and (iv) shows that K_n SC(L) is isomorphic to $K_n(L)$.hence (v) holds.

(v)$\Rightarrow$(iv). Suppose (v) holds.Then by Th. 1.3, $K_n(L)$ is continuous and so L is conditionally upper continuous. Hence (iv) holds.

(iv) $\Rightarrow$ (i). Suppose (iv) holds. That is Lis conditionally upper continuous. Let $r,s \in L$ and$\{x_i\} \subseteq <r,s>_n$ with $x = \vee x_i$,existing in L. then we have $x_i \wedge r \wedge n \leq s \vee n$ .Then $x \wedge r \wedge n \wedge s = \vee x_i (\wedge r \wedge n \wedge s) = \vee (x_i \wedge r \wedge n \wedge s) \leq s \vee n \vee r$, which shows that $x \in <r,s>_n$ and so each relative annihilator is a complete n-ideal. Hence (i) holds.

Hence the theorem.

An n-ideal of a lattice L with n is called normal if it is an intersection of principal n-ideals. The set of all normal n-ideals of a lattice L with n is denoted by $N_n(L)$, which is also a complete lattice and which is its so-called normal or Dedekind- Macneille completion.

In view of Theorem 4 of Latif and Noor[9], and Theorem1.3 and Theorem 1.4 above, we have the following interesting results.

Theorem.1.5. *Let L be a distributive lattice with n, then the following conditions are equivalent:*

(i) Each relative annihilator n-ideal $<r, s>_n$ *, r, s* $\in$ *L is a normal n-ideal,*

(ii) Each n-ideal J $\in K_nSC(L)$ *is a normal n-ideal,*

(iii) $K_nSC(L)$ *is equal to* $N_n(L)$*,*

(iv) L is meet-dense in $K_nSC(L)$*,*

(v) $K_nSC(L)$*, is isomorphic to* $N_n(L)$*,*

(vi) L has an upper continuous,

(vii) $K_nSC(L)$, $N_n(L)$ *and* $K_n(L)$ *are all isomorphic.*

The following corollary follows from the above results, which is a characterization of upper continuous distributive lattices, which was first formulated and established by Latif and Noor[10] [Theorem 3.9].

Corollary. 1.6. *A distributive lattice L with n is upper continuous if and only if the n-kernel of each skeletal congruence is a principal n-ideal.*

In the spirit of this corollary, we have the following Theorem, which is a nice generalization of Cornish[1] [Theorem 3.6].

Theorem. 1.7. *If L is a distributive lattice with n which satisfies the descending chain condition then each n-ideal is the n-kernel of a skeletal congruence. Moreover, the converse is true when l is a bounded chain.*

Proof: Since L satisfies the descending chain condition, so each interval [a,b] of L is finite and so C(L) is Boolean. The complement of $\theta \in C(L)$ is $\vee \{ \theta (c,d) : c \equiv d (\theta) \text{ and } c \leq d \}$, since $\theta = \vee \{ \theta (a, b) : a \equiv b (\theta) \text{ and } a \leq b \}$. Also, the complement of $\theta \in SC(L)$ is θ^***.** Now , we are to show that for any n-ideal J of L , both $\theta(J)^*$ and $\theta(J+)$ have J+ as their n-kernel. So let $x \in Ker_n (\theta (J^*))$, then $x \equiv n\ \theta (J^*)$. Then we have $<x>_n \cap J = <n>_n \cap J$ if and only if $m(x,n,j) = m (n,n,j) = n$ for all $i \in J$ and $x \in J+$ and thus both $\theta(J^*)$ and $\theta (J+)$ have J+ as their n-kernel. So, an n-ideal J is the n-kernel of the skeletal congruence $\theta(J+)$.

Conversely, suppose that L is a bounded chain such that each n-ideal is the n-kernel of a skeletal congruence. Let $n <c_1 <c_\vee < \ldots\ldots < c_n <\ldots\ldots <$ 1 be a sub chain of L. Let $J =\cup <c_i>_n$. Let $x,y \in L$ such that $x \wedge c_i \wedge n = y \wedge c_i \wedge n$ for all i. If this sub chain is infinite then we have $m(x,n,c_i) = m(y,n,c_i)$ for all $c_i\ \theta$ J . Then by Theorem 4 of Latif and Noor[9], we have $x \equiv y\ \theta (J)^* = \omega$. Hence x = y . Again if $x \vee n \vee c_i = y \vee n \vee c_i$ for all $c_i \in J$, then $m(x,n,c_i) = (x \vee n) \wedge (x \vee c_i) \wedge (y \vee c_i) = (y \vee n) \wedge (n \vee c_i) \wedge (y \vee c_i)$, as $n \in J$,

$= m\,(y, n, c_i)$ for all $c_i \in J$.

Thus by Theorem 4 of Latif and Noor[9], we have $x \equiv y\ \theta\,(J)^* = \omega$ and hence x = y, which shows that J is both meet and join – dense. Then by Theorem 7 of Latif and Noor[9], we have $\theta\,(J)^* = \omega$ which implies that $\theta\,(J)^{**} = \iota$, the largest congruence. Now, suppose that each n-ideal is the n-kernel of a skeletal congruence. But by Th. 1.3, above we have J = L which gives a contradiction. Hence L must be finite.

A lattice L is called implicative or relatively pseudo complemented lattice if each relative annihilator is a principal n-ideal. The generator of $< r,s >_n$ is denoted by $r \wedge s \wedge n \to r \vee s \vee n$. An implicative lattice is necessarily distributive with largest element $L = r \to r$ for any r. Both Smith[5] [Th. 4.1], and Shmuely[11] [corollary 5.2] have independently and by different techniques, shown that the N_n (L), the completion of an implicative lattice is implicative and that the canonical embedding preserve the operation $\to$. Now we use K_n SC(L) to give a different approach. Before we give the result we recall that a continuous distributive lattice is isomorphic.

The following theorem is a generalization of Theorem 3.7 of Cornish[1].

Theorem. 1.8. *Let L be an implicative lattice with n. Then $K_nSC(L)$, $K_n(L)$ and $N_n(L)$ are all isomorphic and the embedding $a \to \langle a\rangle_n$ of L into $K_nSC(L)$ preserves the operation $\to$.*

Proof: In an implicative lattice, each relative annihilator is a principal n-ideal and hence a normal n-ideal. Thus by Theorem 1.5, we have $K_nSC(L)$, K_n (L) and $N_n(L)$ are all isomorphic . Now let $j \in K_nSC(L)$ be such that $J \cap \langle r\rangle_n \subseteq \langle s\rangle_n$ for given $r,s \in L$. Due to Theorem 1.3, the inequality $\vee \{ \langle j\rangle_n \cap \langle r\rangle_n : j \in J \} \leq \langle s\rangle_n$ holds in $K_nSC(L)$. Hence $\vee \{ [\, j \wedge n, j \vee n\,] \cap [\, r \wedge n, r \vee n\,] : j \in J \} \subseteq [\, s \wedge n, s \vee n\,]$ which implies that $\vee \{ [\, (j \wedge n) \vee (r \wedge n), (j \vee n) \wedge (r \vee n)\,] : j \in J \} \subseteq [\, s \wedge n, s \vee n\,] \Rightarrow \vee \{ [\, n \wedge (r \vee j), n \vee (r \wedge j)\,] : j \in J \} \subseteq [\, s \wedge n, s \vee n\,]$, which shows that $r \vee j \leq s$ and $r \wedge j \leq s$. That is $r \wedge j \leq s$ for all $j \in J$ and so $J \subseteq \langle r \to s\rangle_n$, which shows that $\langle a\rangle_n \to \langle s\rangle_n = \langle r \to s\rangle_n$, which is the required result .

A distributive lattice L with 1 is called dual disjunctive if $y < x \leq 1$ implies that there exists $a \in L$ such that $y < a < 1$ and $x \vee a = 1$.

We call that a lattice $F_n(L)$ is disjunctive if for $\{n\} \leq [a,b] \subset [c,d]$, then there exists $[e,f] \neq \{n\} \in F_n(L)$ such that $[a,b] \cap [e,f] = \{n\}$. Equivalently, we call that a lattice $F_n(L)$ is dual disjunctive if for $[c,d] < [a,b] \leq L$, then there exists $[e,f] \neq L \in F_n(L)$ such that $[a,b] \vee [e,f] = L$. Janowitz[6] [Th. 3.11] proved that N(L) of a bounded distributive lattice L is Boolean if and only if L is both disjunctive dual disjunctive. Shmuely[11] [Th.4.2 and Th.4.3] gave another prove of this theorem and also presented a class of disjunctive, dual disjunctive lattice whose members are not Boolean lattice. W. H. Cornish[1] in his paper used KSC(L) to give another proof of Janowitz's Theorem. He also showed that a bounded disjunctive distributive lattice is dual disjunctive if and only if it is conditionally upper continuous.

According to Cornish[1] [Th. 3.8], we have the following theorem;

Theorem. 2.1. *Let L be a bounded dual disjunctive lattice. Then each relative annihilator is normal.*

Now, we extend the above theorem.

Theorem.2.2. *Let F_n (L) be a bounded dual disjunctive lattice . Then each relative annihilator n-ideal is normal.*

Proof: Consider a relative annihilator n-ideal $<r ,s>_n$ with r>s. As $F_n(L)$ is a dual disjunctive , so the set $A = \{ <x>_n \in F_n(L) : <r>_n \vee <x>_n = L$ where $<r>_n < <x>_n < L \}$is non-empty, and if $<x>_n \in A$ and $[a ,b] \in <r ,s>_n$, then $[a ,b] \wedge <r>_n \leq <s>_n < <x>_n \quad < L$ and $<r>_n \vee <x>_n = L$, Then $[a ,b]$ $[a ,b] \wedge L = [a ,b] \wedge \{ <s>_n \vee <x>_n \} = \{ [a ,b] \wedge <r>_n \} \vee \{ [a ,b] \wedge <x>_n \} = <s>_n \vee <x>_n = <x>_n$. Thus, $<r, s>_n \subseteq \cap \{ <x>_n : <x>_n \in A \}$. Suppose $[a, b] \in \cap\{<x>_n : <x>_n \in A\}$. Assume that $[a, b] \wedge <r>_n \not\subset <s>_n$ so that $<s>_n \subset <s>_n \vee ([a ,b] \wedge <r>_n)$. Thus, $<s>_n < <u>_n < L$ and $<s>_n \vee ([a, b] \wedge <r>_n) \vee <u>_n = L$ for a suitable $<u>_n \in F_n(L)$. Since $<r>_n > <s>_n$ so $<r>_n \vee <u>_n = L$. Thus , $[a, b] \leq <u>_n$, $[a ,b] \wedge <r>_n \leq <u>_n$ and $<s>_n \vee <u>_n = L$. Then $<r>_n = <s>_n$, which contradicts our assumption. Thus $[a, b] \wedge <r>_n \subseteq <s>_n$ and $<r, s>_n = \cap\{ <x>_n : <x>_n \in A \}$, which shows that $<r, s>_n$ is a normal n-ideal.

We conclude this paper with the following theorem which is a nice generalization of Cornish[1] [Th. 3.9].

Theorem.2.3. *Let L be a bounded distributive lattice with n. Then the following conditions are equivalent:*

(i) $F_n(L)$ is disjunctive and dual disjunctive
(ii) $F_n(L)$ is disjunctive and conditionally upper continuous,
(iii) $N_n(L)$ is a Boolean lattice.

Proof: (i) ⇒ (ii) immediately follows from Th. 2, 2.
(ii) ⇒ (i) . Suppose (ii) holds. Then according to Latif and Noor[12] [Th. 2.3] and Th. 1.4, we have $F_n(L)$ is regularly embedded in the complete Boolean lattice $K_nSC(L)$. Let $<r>_n, <s>_n \in F_n(L)$. When $<s>_n < <r>_n$ and since $K_nSC(L)$ is Boolean then it is dual disjunctive and so there exists J ∈

$K_nSC(L)$ such that $<s>_n \subset J \subset F_n(L)$ and $<r>_n \vee J = L$. Since by Th. 1.4 , J is a complete n-ideal , so it is not possible for L to be the join in $F_n(L)$ of all the members of J. this means that there exists $<c>_n \in F_n(L)$ such that $<j>_n \leq <c>_n$ for all $j \in J$ and $<c>_n \neq L$. Then we have $<s>_n < <c>_n < L$ and $<r>_n \vee <c>_n = L$. Thus L is dual disjunctive, which is (i).

(i) $\Rightarrow$ (iii) immediately follows from Th. 1,5 and Th. 2.2.

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