

On Hypersurfaces of a Recurrent Finsler Space

P. N. Pandey and Suresh K. Shukla

Department of Mathematics

University of Allahabad, Allahabad-211 002, INDIA

E-mail: pnpiaps@rediffmail.com; shuklasureshk@gmail.com

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Abstract: The present paper is devoted to the study of hypersurfaces immersed in a recurrent Finsler space. U. P. Singh and G. C. Chaubey¹ obtained some results on hypersurfaces of a recurrent Finsler space under certain conditions. In this paper, a more general definition of a recurrent space has been adopted and several properties of an umbilical hypersurface immersed in such space have been investigated. Results of Singh and Chaubey¹ have been generalized and extended to a larger class of recurrent spaces.

Keywords: Landsberg spaces, Recurrent spaces, Umbilical hypersurfaces.

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1. Introduction

U. P. Singh and G. C. Chaubey¹ obtained Gauss characteristic equations of an umbilical hypersurface immersed in a Finsler space from the standpoint of Berwald connection. They defined a recurrent Finsler space which is not equivalent to the definition of several authors such as Kumar², Misra³, Pandey⁴ and others. Singh and Chaubey¹ considered two cases: (i) The recurrence vector field of the embedding space is not normal to a non-totally geodesic hypersurface, (ii) The recurrence vector field is normal to the hypersurface. For the first case, they investigated the conditions for the hypersurface immersed in a recurrent space to be recurrent in accordance with their definition of a recurrent space. For the second case, with an additional condition that the embedding space is affinely connected, they derived several properties of the hypersurface. It has been shown by Pandey and Dikshit⁵ that a recurrent Finsler space is a Landsberg space if and only if $\det(H_j^i) \neq 0$. In the present paper, we have adopted the definition of a recurrent space of Kumar², Misra³, Pandey⁴ and others. For case (i) discussed above, we have established some conditions for the hypersurface

immersed in a recurrent Finsler space satisfying $\det(H_j^i) \neq 0$ to be recurrent. The same conditions have been obtained for a hypersurface immersed in a recurrent Landsberg space to be recurrent, as corollaries. For case (ii), we have investigated several properties of a hypersurface immersed in a recurrent space satisfying $\det(H_j^i) \neq 0$. The same properties have been derived, as corollaries, for a Landsberg recurrent space whose class is larger than that of affinely connected recurrent spaces.

Let (F^n, L) be an n -dimensional Finsler space equipped with metric function $L(x, \dot{x})$ and corresponding metric tensor $g = (g_{ij}(x, \dot{x}))$. The relation between Berwald connection parameters and Cartan connection parameters is given by

$$(1.1) \quad G_{jk}^i = \Gamma_{jk}^{*i} + C_{jk|h}^i \dot{x}^h,$$

where $C_{jk}^i = g^{ip} C_{jpk}$ and $C_{jpk} = \frac{1}{2} \frac{\partial g_{jp}}{\partial \dot{x}^k}$ are symmetric tensors and the symbol ‘|’ stands for the Cartan process of covariant differentiation.

A Finsler space is said to be a Landsberg space if Berwald connection is metrical therein⁶, i.e. $g_{ij(k)} = 0$, $g_{ij(k)}$ denotes the Berwald covariant derivative of g_{ij} with respect to x^k . This space is characterized by⁶

$$(1.2) \quad C_{ijk|h} \dot{x}^h = 0,$$

or, alternatively by

$$(1.3) \quad y_r G_{jkh}^r = 0,$$

where $y_i = g_{ir} \dot{x}^r$.

Let F^{n-1} be a hypersurface of F^n , represented parametrically by the equations* $x^i = x^i(u^\alpha)$. The fundamental metric tensors of F^n and F^{n-1} are related by

$$(1.4) \quad g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x}) B_\alpha^i B_\beta^j,$$

where $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ are the projection factors.

*Throughout the discussion Latin indices run from 1 to n and Greek indices from 1 to $n-1$.

The unit normal N_i to the hypersurface F^{n-1} , at a point, satisfies the following:

$$(1.5) \quad \begin{cases} (a) & N_i B_\alpha^i = 0, \\ (b) & g_{ij} N^i N^j = 1, \end{cases}$$

where $N^i = g^{ij} N_j$.

Cartan's induced and intrinsic connection parameters $\Gamma_{\beta\gamma}^{*\alpha}$ and ${}^/\Gamma_{\beta\gamma}^{*\alpha}$ (vide Rund⁷) are related by⁸

$$(1.6) \quad {}^/\Gamma_{\beta\gamma}^{*\alpha} = \Lambda_{\beta\gamma}^\alpha + \Gamma_{\beta\gamma}^{*\alpha},$$

where

$$(1.7) \quad g_{\varepsilon\gamma} \Lambda_{\alpha\beta}^\varepsilon = \Lambda_{\alpha\gamma\beta} = (M_{\beta\gamma} \Omega_{\alpha\sigma} + M_{\alpha\gamma} \Omega_{\beta\sigma} - M_{\alpha\beta} \Omega_{\gamma\sigma}) \dot{u}^\sigma - (M_{\lambda\alpha} C_{\beta\gamma}^\lambda + M_{\lambda\beta} C_{\alpha\gamma}^\lambda - M_{\lambda\gamma} C_{\beta\alpha}^\lambda) \Omega_{\sigma\mu} \dot{u}^\sigma \dot{u}^\mu,$$

$M_{\alpha\beta} = M_{ij} B_{\alpha\beta}^{ij}$, $M_{ij} = C_{ijk} N^k$ and $\Omega_{\alpha\beta}$ are second fundamental quantities of F^{n-1} .

The normal curvature of F^{n-1} , in the direction of $\dot{u}^\sigma$, is given by

$$(1.8) \quad \kappa_n(u, \dot{u}) = (\Omega_{\sigma\lambda} \dot{u}^\sigma \dot{u}^\lambda) / F^2(u, \dot{u}),$$

which, in view of the normalizing condition $F^2(u, \dot{u}) = 1$, becomes

$$(1.9) \quad \kappa_n(u, \dot{u}) = \Omega_{\sigma\lambda} \dot{u}^\sigma \dot{u}^\lambda.$$

From (1.7) and (1.9) we obtain

$$(a) \quad \Lambda_{\alpha\beta}^\varepsilon \dot{u}^\alpha = \kappa_n M_\beta^\varepsilon,$$

$$(b) \quad g_{\varepsilon\gamma} \dot{u}^\gamma \Lambda_{\alpha\beta}^\varepsilon = -\kappa_n M_{\alpha\beta},$$

where

$$(1.10) \quad M_\beta^\varepsilon = g^{\varepsilon\gamma} M_{\gamma\beta}.$$

The mixed covariant derivative of an arbitrary vector field T_α^i is given by

$$T_{\alpha(\gamma)}^i = \partial_\gamma T_\alpha^i - (\dot{\partial}_\varepsilon T_\alpha^i) \dot{\partial}_\gamma G^\varepsilon - T_\varepsilon^i G_{\alpha\gamma}^\varepsilon + T_\alpha^p G_{ph}^i B_\gamma^h,$$

where $\partial_\gamma \equiv \frac{\partial}{\partial u^\gamma}$, $\dot{\partial}_\varepsilon \equiv \frac{\partial}{\partial \dot{u}^\varepsilon}$ and $G_{\alpha\gamma}^\varepsilon$ are Berwald's induced connection parameters. In view of the above equation, the mixed covariant derivative of projection factors is given by

$$(1.11) \quad B_{\alpha(\beta)}^i = V_{\alpha\beta}^i = B_{\alpha\beta}^i - B_\varepsilon^i G_{\alpha\beta}^\varepsilon + G_{hk}^i B_{\alpha\beta}^{hk}.$$

Equation (1.11) can be put in the following form (vide Sinha and Singh⁹)

$$(1.12) \quad V_{\alpha\beta}^i = N^i \Omega_{\alpha\beta} - B_\varepsilon^i (\Lambda_{\alpha\beta}^\varepsilon + C_{\alpha\beta|\sigma}^\varepsilon \dot{u}^\sigma) + C_{hk|r}^i \dot{x}^r B_{\alpha\beta}^{hk},$$

Sinha and Singh⁹ obtained the following form of Gauss equation:

$$(1.13) \quad \begin{aligned} H_{\varepsilon\delta\beta\gamma} &= H_{hlkj} B_{\varepsilon\delta\beta\gamma}^{hlkj} + (\Omega_{\varepsilon\beta} \Omega_{\delta\gamma} - \Omega_{\varepsilon\gamma} \Omega_{\delta\beta}) \\ &\quad + 2M_{lh} B_\delta^h (\Omega_{\varepsilon\beta} V_{\gamma\sigma}^l - \Omega_{\varepsilon\gamma} V_{\beta\sigma}^l) \dot{u}^\sigma \\ &\quad - 2N^h C_{hlk|p} \dot{x}^p \Omega_{\varepsilon[\beta} B_{\gamma]\delta}^{kl} + 2B_\delta^l g_{il} \Lambda_{\varepsilon[\beta}^\sigma V_{\gamma]\sigma}^i \\ &\quad + 2\Lambda_{\varepsilon[\beta(\gamma)}^\alpha g_{\alpha\delta} - 2C_{hlk|r(j)} \dot{x}^r B_{\varepsilon\delta[\beta\gamma]}^{hlkj} \\ &\quad - 2(\dot{\partial}_j C_{hlk|r}) \dot{x}^r B_{\varepsilon\delta[\beta}^{hlk} V_{\gamma]\sigma}^j \dot{u}^\sigma \\ &\quad - 2C_{hlk|r} B_{\varepsilon\delta[\beta}^{hlk} V_{\gamma]\sigma}^r \dot{u}^\sigma - 2C_{hlk|r} \dot{x}^r B_{\delta[\beta}^{lk} V_{\gamma]\varepsilon}^h, \end{aligned}$$

where H_{hlkj} and $H_{\varepsilon\delta\beta\gamma}$ are Berwald's associate curvature tensors in F^n and F^{n-1} , square brackets denote the skew-symmetric part with respect to the indices enclosed therein and $\dot{\partial}_j \equiv \frac{\partial}{\partial \dot{x}^j}$.

2. Umbilical Hypersurfaces

An umbilical hypersurface F^{n-1} , immersed in a Finsler space F^n , is characterized by

$$(2.1) \quad \Omega_{\varepsilon\beta}(u, \dot{u}) = \kappa_n g_{\varepsilon\beta}(u, \dot{u}).$$

The mean curvature vector M^{*i} of the hypersurface F^{n-1} is given by

$$(2.2) \quad M^{*i} = \frac{1}{n-1} g^{\alpha\beta} \Omega_{\alpha\beta} N^i.$$

From equations (2.1) and (2.2), we obtain

$$(2.3) \quad g_{ij} M^{*i} M^{*j} = M_j^* M^{*j} = \kappa_n^2.$$

In view of equations (2.1) and (2.3) and the facts

$$M_{lh} N^l B_\delta^h = M_\delta, \dot{x}_{(r)}^h = 0, C_{hr}^i \dot{x}^r = 0, u_{|\sigma}^\alpha = 0 \text{ and } C_{\alpha\beta}^\varepsilon \dot{u}^\alpha = 0,$$

equation (1.13) takes the form

$$(2.4) \quad \begin{aligned} H_{\varepsilon\delta\beta\gamma} &= H_{hlkj} B_{\varepsilon\delta\beta\gamma}^{hlkj} + (M_i^* M^{*i})(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\ &\quad + 2M_\delta \kappa_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}, \end{aligned}$$

where

$$(2.5) \quad \begin{aligned} P_{\varepsilon\delta\beta\gamma} &= 2M_{lh} B_{\delta\lambda}^{hl} (\Lambda_{\beta\sigma}^\lambda \Omega_{\varepsilon\gamma} - \Lambda_{\gamma\sigma}^\lambda \Omega_{\varepsilon\beta}) \dot{u}^\sigma - 2N^h C_{hlk|r} \dot{x}^r \Omega_{\varepsilon[\beta} B_{\gamma]\delta}^{kl} \\ &\quad + 2B_\delta^l g_{il} \Lambda_{\varepsilon[\beta}^\sigma V_{\gamma]\sigma}^i + 2\Lambda_{\varepsilon[\beta(\gamma)}^\alpha g_{\alpha\delta} - 2C_{hlk|r(j)} \dot{x}^r B_{\varepsilon\delta[\beta\gamma]}^{hlkj} \\ &\quad - 2(\dot{\partial}_j C_{hlk|r}) \dot{x}^r B_{\varepsilon\delta[\beta}^{hlkj} V_{\gamma]\sigma}^j \dot{u}^\sigma - 2C_{hlk|r} B_{\varepsilon\delta[\beta}^{hlkj} V_{\gamma]\sigma}^r \dot{u}^\sigma \\ &\quad - 2C_{hlk|r} \dot{x}^r B_{\delta[\beta}^{lk} V_{\gamma]\varepsilon}^h. \end{aligned}$$

Differentiating (2.4) covariantly with respect to u^θ and using (1.11), we have

$$(2.6) \quad \begin{aligned} H_{\varepsilon\delta\beta\gamma(\theta)} &= H_{hlkj(\theta)} B_{\varepsilon\delta\beta\gamma}^{hlkj} + H_{hlkj} B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h + H_{hlkj} B_{\varepsilon\beta\gamma}^{hkl} V_{\delta\theta}^l \\ &\quad + H_{hlkj} B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + H_{hlkj} B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j \\ &\quad + [(M_i^* M^{*i})(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\ &\quad + 2M_\delta \kappa_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}]_{(\theta)}. \end{aligned}$$

The mixed covariant derivative of H_{hlkj} with respect to u^θ is, by definition, given by

$$(2.7) \quad \begin{aligned} H_{hlkj(\theta)} &= \frac{\partial H_{hlkj}}{\partial u^\theta} - \frac{\partial H_{hlkj}}{\partial \dot{u}^\lambda} \frac{\partial G^\lambda}{\partial \dot{u}^\theta} \\ &\quad - (H_{rlkj} G_{hm}^r + H_{hrkj} G_{lm}^r + H_{hlrj} G_{km}^r + H_{hlkr} G_{jm}^r) B_\theta^m. \end{aligned}$$

The following is obvious

$$(2.8) \quad \frac{\partial H_{hlkj}}{\partial u^\theta} = \frac{\partial H_{hlkj}}{\partial x^m} B_\theta^m + \frac{\partial H_{hlkj}}{\partial \dot{x}^m} B_{\sigma\theta}^m \dot{u}^\sigma.$$

Using (2.8) in (2.7), we have

$$(2.9) \quad H_{h l k j(\theta)} = [\partial_m H_{h l k j} - (\dot{\partial}_i H_{h l k j}) G_m^i - (H_{r l k j} G_{h m}^r + H_{h r k j} G_{l m}^r + H_{h l r j} G_{k m}^r + H_{h l k r} G_{j m}^r) B_\theta^m + (\dot{\partial}_i H_{h l k j}) (G_m^i B_\theta^m + B_{\sigma \theta}^i \dot{u}^\sigma - G_\theta^\lambda B_\lambda^i),$$

where $\partial_j \equiv \frac{\partial}{\partial x^j}$ and $\dot{\partial}_j \equiv \frac{\partial}{\partial \dot{x}^j}$.

In view of (1.11), (2.9) takes the following form

$$(2.10) \quad H_{h l k j(\theta)} = H_{h l k j(m)} B_\theta^m + (\dot{\partial}_r H_{h l k j}) V_{\sigma \theta}^r \dot{u}^\sigma.$$

Using (2.10) in (2.6), we obtain the following form of Gauss characteristic equations of an umbilical hypersurface immersed in a Finsler space (vide U. P. Singh and G. C. Chaubey¹)

$$(2.11) \quad H_{\varepsilon \delta \beta \gamma(\theta)} = [H_{h l k j(m)} B_\theta^m + (\dot{\partial}_r H_{h l k j}) V_{\sigma \theta}^r \dot{u}^\sigma] B_{\varepsilon \delta \beta \gamma}^{h l k j} + [(M_i^* M^{*i}) (g_{\varepsilon \beta} g_{\delta \gamma} - g_{\varepsilon \gamma} g_{\delta \beta}) + 2M_{\delta} \kappa_n^2 (g_{\varepsilon \beta} g_{\sigma \gamma} - g_{\varepsilon \gamma} g_{\sigma \beta}) \dot{u}^\sigma + P_{\varepsilon \delta \beta \gamma} \rfloor_{(\theta)} + H_{h l k j} B_{\delta \beta \gamma}^{l k j} V_{\varepsilon \theta}^h + H_{h l k j} B_{\varepsilon \beta \gamma}^{h k j} V_{\delta \theta}^l + H_{h l k j} B_{\varepsilon \delta \gamma}^{h l j} V_{\beta \theta}^k + H_{h l k j} B_{\varepsilon \delta \beta}^{h l k} V_{\gamma \theta}^j.$$

3. Recurrent Finsler spaces

A non-flat Finsler space F^n is called a recurrent space if the curvature tensor satisfies the relation^{2,3,4}

$$(3.1) \quad H_{j k h(m)}^i = a_m H_{j k h}^i,$$

where a_m is a non-zero vector, called the recurrence vector. Pandey⁴ proved that the recurrence vector a_m is independent of directional arguments.

Using (3.1) in

$$H_{j l k h(m)}^i = g_{i l(m)} H_{j k h}^i + g_{i l} H_{j k h(m)}^i,$$

we have

$$(3.2) \quad H_{jlk h(m)} = g_{il(m)} H_{jkh}^i + a_m H_{jlk h}.$$

In view of (2.4) and (3.2), (2.11) becomes

$$(3.3) \quad \begin{aligned} H_{\varepsilon\delta\beta\gamma(\theta)} = & g_{pl(m)} H_{hkj}^p B_{\theta}^m B_{\varepsilon\delta\beta\gamma}^{hlkj} + a_m B_{\theta}^m [H_{\varepsilon\delta\beta\gamma} \\ & - (M_i^* M^{*i})(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\ & - 2M_{\delta} \kappa_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) - P_{\varepsilon\delta\beta\gamma}] \\ & + [(M_i^* M^{*i})(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\ & + 2M_{\delta} \kappa_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma} + P_{\varepsilon\delta\beta\gamma}]_{(\theta)} \\ & + (\dot{\partial}_r H_{hlkj}) V_{\sigma\theta}^r \dot{u}^{\sigma} B_{\varepsilon\delta\beta\gamma}^{hlkj} \\ & + H_{hlkj} (B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h + B_{\varepsilon\beta\gamma}^{hkl} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j). \end{aligned}$$

Pandey and Dikshit⁵ proved the following:

Theorem 3.1. (P. N. Pandey and Shalini Dikshit⁵): *A recurrent space F^n is a Landsberg space if and only if $\det(H_j^i) \neq 0$.*

Let us consider two cases:

- (a) The recurrence vector field a_m is not normal to the hypersurface F^{n-1} , i.e.

$$a_m B_{\theta}^m = a_{\theta} \neq 0,$$

- (b) The recurrence vector field a_m is normal to the hypersurface F^{n-1} , i.e.

$$a_m B_{\theta}^m = a_{\theta} = 0.$$

We assume that the hypersurface is non-totally geodesic.

Define

$$(3.4) \quad \begin{aligned} T_{\varepsilon\delta\beta\gamma} = & H_{\varepsilon\delta\beta\gamma} - (M_i^* M^{*i})(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\ & - 2M_{\delta} \kappa_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma} - P_{\varepsilon\delta\beta\gamma}. \end{aligned}$$

In view of case (a) and equation (3.4), (3.3) becomes

$$(3.5) \quad \begin{aligned} T_{\varepsilon\delta\beta\gamma(\theta)} = & g_{pl(m)} H_{hkj}^p B_{\theta}^m B_{\varepsilon\delta\beta\gamma}^{hlkj} + a_{\theta} T_{\varepsilon\delta\beta\gamma} + (\dot{\partial}_r H_{hlkj}) V_{\sigma\theta}^r \dot{u}^{\sigma} B_{\varepsilon\delta\beta\gamma}^{hlkj} \\ & + H_{hlkj} (B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h + B_{\varepsilon\beta\gamma}^{hkl} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j). \end{aligned}$$

Now, if

$$(3.6) \quad g_{pl(m)} H_{hkj}^p B_\theta^m B_{\varepsilon\delta\beta\gamma}^{hlkj} + (\partial_r H_{hlkj}) V_{\sigma\theta}^r \dot{u}^\sigma B_{\varepsilon\delta\beta\gamma}^{hlkj} \\ + H_{hlkj} (B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h + B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j) = 0,$$

then it follows from (3.5) that

$$T_{\varepsilon\delta\beta\gamma(\theta)} = a_\theta T_{\varepsilon\delta\beta\gamma}.$$

Thus, we have:

Theorem 3.2. *If F^{n-1} is a non-totally geodesic umbilical hypersurface immersed in a recurrent Finsler space F^n and the recurrence vector field a_m is not normal to the hypersurface, then the tensor $T_{\varepsilon\delta\beta\gamma}$, defined by (3.4), is recurrent with recurrence vector $a_\theta = a_m B_\theta^m$ if and only if (3.6) holds.*

Suppose that $\det(H_j^i) \neq 0$. Then the embedding space F^n is a Landsberg space by Theorem 3.1 and hence $g_{pl(m)} = 0$. In this case, from equation (3.2), we have

$$(3.7) \quad H_{jlkh(m)} = a_m H_{jlkh}.$$

Thus, in case of $\det(H_j^i) \neq 0$, our characterization of a recurrent space coincides with that of U. P. Singh and G. C. Chaubey¹.

From (3.6), we deduce the following:

Theorem 3.3. *Let F^{n-1} be a non-totally geodesic umbilical hypersurface immersed in a recurrent Finsler space F^n . If the recurrence vector field a_m is not normal to the hypersurface F^{n-1} and $\det(H_j^i) \neq 0$, then the tensor $T_{\varepsilon\delta\beta\gamma}$, defined by (3.4), is recurrent with recurrence vector $a_\theta = a_m B_\theta^m$ if and only if*

$$(3.8) \quad (\partial_r H_{hlkj}) V_{\sigma\theta}^r \dot{u}^\sigma B_{\varepsilon\delta\beta\gamma}^{hlkj} + H_{hlkj} (B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h + B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j) = 0.$$

From Theorem 3.1 and Theorem 3.3, we have the following result:

Corollary 3.4. *If F^{n-1} is a non-totally geodesic umbilical hypersurface immersed in a recurrent Landsberg space F^n and the recurrence vector field a_m is not normal to the hypersurface F^{n-1} , then the tensor $T_{\varepsilon\delta\beta\gamma}$, defined by (3.4), is recurrent with recurrence vector $a_\theta = a_m B_\theta^m$ if and only if (3.8) holds.*

If we define

$$(3.9) \quad \begin{aligned} J_{\varepsilon\delta\beta\gamma} = & (M_i^* M^{*i})(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\ & + 2M_{\delta} \kappa_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma} + P_{\varepsilon\delta\beta\gamma}, \end{aligned}$$

it follows from (3.4) that

$$(3.10) \quad H_{\varepsilon\delta\beta\gamma} = T_{\varepsilon\delta\beta\gamma} + J_{\varepsilon\delta\beta\gamma}.$$

Let $\det(H_j^i) \neq 0$. Then, in the light of equation (3.7), equation (3.10) implies that if $T_{\varepsilon\delta\beta\gamma}$ and $J_{\varepsilon\delta\beta\gamma}$ are recurrent with the recurrence vector a_{θ} , then $H_{\varepsilon\delta\beta\gamma}$ is also recurrent with the same recurrence vector.

In view of the above fact and Theorem 3.3, we have the following result:

Theorem 3.5. *Let F^{n-1} be a non-totally geodesic umbilical hypersurface immersed in a recurrent Finsler space F^n . If the recurrence vector field a_m is not normal to the hypersurface F^{n-1} and $\det(H_j^i) \neq 0$, then the hypersurface F^{n-1} is recurrent with recurrence vector a_{θ} if (3.8) holds and $J_{\varepsilon\delta\beta\gamma}$, defined by (3.9) is recurrent with recurrence vector a_{θ} .*

From Theorem 3.1 and Theorem 3.5, we may state:

Corollary 3.6. *If F^{n-1} is a non-totally geodesic umbilical hypersurface immersed in a recurrent Landsberg space F^n and the recurrence vector field a_m is not normal to the hypersurface F^{n-1} , then the hypersurface F^{n-1} is recurrent with recurrence vector a_{θ} if (3.8) holds and $J_{\varepsilon\delta\beta\gamma}$, defined by (3.9), is recurrent with recurrence vector a_{θ} .*

Case (b):

Let the recurrence vector field a_m is normal to the hypersurface F^{n-1} , i.e. $a_m B_{\theta}^m = a_{\theta} = 0$. The Bianchi identities for F^n are given by⁷

$$(3.11) \quad H_{hkj(r)}^i + H_{hrk(j)}^i + H_{hjr(k)}^i + H_{kj}^m G_{mhr}^i + H_{rk}^m G_{mhj}^i + H_{jr}^m G_{mhk}^i = 0.$$

For a recurrent space, Pandey⁴ proved that the Bianchi identities (3.11) split into the following two identities:

$$(3.12) \quad a_r H_{hkj}^i + a_j H_{hrk}^i + a_k H_{hjr}^i = 0,$$

and

$$(3.13) \quad H_{kj}^m G_{mhr}^i + H_{rk}^m G_{mhj}^i + H_{jr}^m G_{mhh}^i = 0.$$

Transvecting (3.12) by g_{il} , we have

$$(3.14) \quad a_r H_{hlkj} + a_j H_{hlrk} + a_k H_{hljr} = 0.$$

If we transvect (3.14) by $B_{\varepsilon\delta\beta\gamma}^{hlkj}$, it follows from case (b) that

$$(3.15) \quad H_{hlkj} B_{\varepsilon\delta\beta\gamma}^{hlkj} = 0.$$

Equation (2.4), in view of (3.15), takes the form

$$(3.16) \quad \begin{aligned} H_{\varepsilon\delta\beta\gamma} &= (M_i^* M^{*i})(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\ &\quad + 2M_{\delta} \kappa_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma} + P_{\varepsilon\delta\beta\gamma}. \end{aligned}$$

Using (3.16) in (3.4), we obtain

$$T_{\varepsilon\delta\beta\gamma} = 0.$$

Thus, we have the following result:

Theorem 3.7. *If F^{n-1} is an umbilical hypersurface immersed in a recurrent Finsler space F^n and the recurrence vector field a_m is normal to F^{n-1} , then the tensor $T_{\varepsilon\delta\beta\gamma}$, defined by (3.4), vanishes.*

For a Minkowskian hypersurface, $H_{\varepsilon\delta\beta\gamma} = 0$. Using this in (3.16), we get

$$(3.17) \quad \begin{aligned} &(M_i^* M^{*i})(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\ &\quad + 2M_{\delta} \kappa_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma} + P_{\varepsilon\delta\beta\gamma} = 0. \end{aligned}$$

Thus, we have the following result:

Theorem 3.8. *If F^{n-1} is an umbilical hypersurface immersed in a recurrent Finsler space F^n and the recurrence vector field a_m is normal to F^{n-1} , then the hypersurface F^{n-1} is Minkowskian if and only if (3.17) holds.*

If $\det(H_j^i) \neq 0$, then embedding space is a Landsberg space by Theorem 3.1 and hence $C_{ijk|r}\dot{x}^r = 0$. Equation (2.5), in view of this fact, takes the form

$$(3.18) \quad P_{\varepsilon\delta\beta\gamma} = 2M_{lh}B_{\delta\lambda}^{hl}(\Lambda_{\beta\sigma}^{\lambda}\Omega_{\varepsilon\gamma} - \Lambda_{\gamma\sigma}^{\lambda}\Omega_{\varepsilon\beta})\dot{u}^{\sigma} + 2\Lambda_{\varepsilon[\beta(\gamma)}^{\alpha}g_{\alpha\delta} \\ - 2g_{\delta\phi}\Lambda_{\varepsilon[\beta}^{\sigma}(\Lambda_{\gamma]\sigma}^{\phi} + C_{\gamma]\sigma|\xi}^{\phi}\dot{u}^{\xi}).$$

Thus, we have:

Theorem 3.9. *Let F^{n-1} be an umbilical hypersurface immersed in a recurrent Finsler space F^n . If the recurrence vector field a_m is normal to F^{n-1} and $\det(H_j^i) \neq 0$, then the hypersurface F^{n-1} is Minkowskian if and only if*

$$(3.19) \quad (M_i^*M^{*i})(g_{\varepsilon\beta}g_{\delta\gamma} - g_{\varepsilon\gamma}g_{\delta\beta}) + 2M_{\delta}\kappa_n^2(g_{\varepsilon\beta}g_{\sigma\gamma} - g_{\varepsilon\gamma}g_{\sigma\beta})\dot{u}^{\sigma} \\ + 2M_{lh}B_{\delta\lambda}^{hl}(\Lambda_{\beta\sigma}^{\lambda}\Omega_{\varepsilon\gamma} - \Lambda_{\gamma\sigma}^{\lambda}\Omega_{\varepsilon\beta})\dot{u}^{\sigma} + 2\Lambda_{\varepsilon[\beta(\gamma)}^{\alpha}g_{\alpha\delta} \\ - 2g_{\delta\phi}\Lambda_{\varepsilon[\beta}^{\sigma}(\Lambda_{\gamma]\sigma}^{\phi} + C_{\gamma]\sigma|\xi}^{\phi}\dot{u}^{\xi}) = 0.$$

From Theorem 3.1 and Theorem 3.9, we have:

Corollary 3.10. *If F^{n-1} is an umbilical hypersurface immersed in a recurrent Landsberg space F^n and the recurrence vector field a_m is normal to F^{n-1} , then the hypersurface F^{n-1} is Minkowskian if and only if (3.19) holds.*

If the hypersurface F^{n-1} is of minimal variety, then the mean curvature vector M^{*i} vanishes. By virtue of this fact and equations (2.1) and (2.3), it follows that

$$(3.20) \quad \kappa_n = 0, \quad \Omega_{\alpha\beta} = 0.$$

From equations (1.7) and (3.21), we have

$$(3.21) \quad \Lambda_{\beta\gamma}^{\varepsilon} = 0.$$

From Theorem 3.9, equations (3.20) and (3.21), we have the following result:

Theorem 3.11. *Let F^{n-1} be an umbilical hypersurface of minimal variety immersed in a recurrent Finsler space F^n . If the recurrence vector field a_m*

is normal to F^{n-1} and $\det(H_j^i) \neq 0$, then the hypersurface F^{n-1} is Minkowskian.

From Theorem 3.1 and Theorem 3.11, we deduce that:

Corollary 3.12. *If F^{n-1} is an umbilical hypersurface of minimal variety immersed in a recurrent Landsberg space and the recurrence vector field a_m is normal to F^{n-1} , then F^{n-1} is Minkowskian.*

Transvecting (3.16) by $\dot{u}^\delta$ and using (3.18) and the fact $M_{lh}\dot{x}^h = 0$, we obtain

$$(3.22) \quad H_{\varepsilon\delta\beta\gamma}\dot{u}^\delta = (M_i^* M^{*i})(g_{\varepsilon\beta}g_{\delta\gamma} - g_{\varepsilon\gamma}g_{\delta\beta})\dot{u}^\delta + 2\Lambda_{\varepsilon[\beta(\gamma)]}^\alpha g_{\alpha\delta}\dot{u}^\delta - 2g_{\delta\phi}\dot{u}^\delta \Lambda_{\varepsilon[\beta}^\sigma (\Lambda_{\gamma]\sigma}^\phi + C_{\gamma] \sigma|\xi}^\phi \dot{u}^\xi).$$

We characterize a hypersurface F^{n-1} of constant curvature by

$$(3.23) \quad H_{\varepsilon\delta\beta\gamma}\dot{u}^\delta = K(g_{\varepsilon\beta}g_{\delta\gamma} - g_{\varepsilon\gamma}g_{\delta\beta})\dot{u}^\delta.$$

Thus, we may state:

Theorem 3.13. *Let F^{n-1} be an umbilical hypersurface immersed in a recurrent Finsler space F^n . If the recurrence vector field a_m is normal to F^{n-1} and $\det(H_j^i) \neq 0$, then the hypersurface F^{n-1} is of constant curvature if and only if*

$$(3.24) \quad \Lambda_{\varepsilon[\beta(\gamma)]}^\alpha - \Lambda_{\varepsilon[\beta}^\sigma (\Lambda_{\gamma]\sigma}^\alpha + C_{\gamma] \sigma|\xi}^\alpha \dot{u}^\xi) = 0.$$

From Theorem 3.1 and Theorem 3.13, we conclude:

Corollary 3.14. *If F^{n-1} is an umbilical hypersurface immersed in a recurrent Landsberg Finsler space F^n and the recurrence vector field a_m is normal to F^{n-1} , then the hypersurface F^{n-1} is of constant curvature if and only if (3.24) holds.*

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